4 Basics of integration

Integral is a bridge between functions of point and functions of set.

4a Introduction

As already pointed out, many of the quantities of interest in continuum mechanics represent *extensive properties,* such as mass, momentum and energy. An extensive property assigns a value to *each part of the body.* From the mathematical point of view, an extensive property can be regarded as a *set function,* in the sense that it assigns a value to each subset of a given set. Consider, for example, the case of the mass property. Given a material body, this property assigns to each subbody its mass. Other examples of extensive properties are: volume, electric charge, internal energy, linear momentum. *Intensive properties,* on the other hand, are represented by *fields,* assigning to *each point of the body* a definite value. Examples of intensive properties are: temperature, displacement, strain.

As the example of mass clearly shows, very often the extensive properties of interest are *additive set functions,* namely, the value assigned to the union of two disjoint subsets is equal to the sum of the values assigned to each subset separately. Under suitable assumptions of continuity, it can be shown that an additive set function is expressible as the integral of a *density* function over the subset of interest. This density, measured in terms of property per unit size, is an ordinary pointwise function defined over the original set. In other words, the density associated with a continuous additive set function is an intensive property. Thus, for example, the mass density is a scalar field.

Marcelo Epstein[1]

We need a mathematical theory of the correspondence between set functions $\mathbb{R}^n \supset E \mapsto S(E) \in \mathbb{R}$ and (ordinary) functions $\mathbb{R}^n \ni x \mapsto f(x) \in \mathbb{R}$ via integration, $S(E) = \int_E f$. The theory should address (in particular) the following questions.

* What are admissible sets E and functions f? (Arbitrary sets are as useless here as arbitrary functions.)
* What is meant by "disjoint"?
* What is meant by integral?
* What are the general properties of the integral?
* How to calculate the integral explicitly for given f and E ?

Many approaches coexist. Some authors[2] start with Riemann sums (more natural for complex-valued and vector-valued integrands) and then proceed to Darboux sums. Other consider Darboux sums only; we do so, too.

Ultimately, all authors define $\int_E f$ as $\int_{\mathbb{R}^n} f_E$ where

(4a1)
$$f_E(x) = \begin{cases} f(x) & \text{for } x \in E, \\ 0 & \text{otherwise.} \end{cases}$$

(Note that f_E is generally discontinuous, even if f is continuous.) But initially one considers much simpler sets E. Most authors use products of intervals, called n-rectangles,[3] coordinate parallelepipeds,[4] compact boxes[5] etc., and for these simple E define $\int_E f$ before $\int_{\mathbb{R}^n} f$. But some authors[6] use dyadic cubes, called also pixels,[7] for defining $\int_{\mathbb{R}^n} f$ (before $\int_E f$) for bounded f with bounded support. We follow this way, thus avoiding partitions, common refinements, and simple but nasty technicalities that some authors treat in detail[8] and others leave to exercises.[9] The cost is that the shift invariance (change of origin) needs a proof,[10] similarly to rotation invariance (change of basis) that needs a proof in every approach.

[1] "The elements of continuum biomechanics", Wiley 2012. (See Sect. 2.2.1.)
[2] Zorich.
[3] Lang.
[4] Zorich.
[5] Shurman.
[6] Hubbard.
[7] Terry Tao.
[8] For instance, Lang, p. 570 and 573.
[9] For instance, Shifrin, p. 271.
[10] Hubbard, Prop. 4.1.21.

In the one-dimensional theory, seeing $\int_a^b f(x)\,\mathrm{d}x$, we do not ask, is this the integral over the open interval (a,b) or the closed interval $[a,b]$; we neglect the boundary $\{a,b\}$ of the interval. Similarly, in higher dimension we want to neglect the boundary of E.

Two notions of "small" sets are used. One notion is called "volume zero"[1] or "zero content";[2] the other notion is called "measure zero".[3] For compact sets these two notions coincide, but in general they are very different. Fortunately, the boundary $\partial E = \overline{E} \setminus E^\circ$ of a bounded set E is always compact; requiring it to be small (in either sense) we need not bother, whether the integral is taken over the open set E° or the closed set $\overline{E}$; and we may treat sets E, F as disjoint when they have no common *interior* points. In this case the equality

$$S(E \cup F) = S(E) + S(F) \tag{4a2}$$

is additivity of the set function S; and the inequality

$$\operatorname{vol}(E) \inf_{x\in E} f(x) \le S(E) \le \operatorname{vol}(E) \sup_{x\in E} f(x) \tag{4a3}$$

is the clue to the relation between f and S.

4b Darboux sums

We consider a function $f : \mathbb{R}^n \to \mathbb{R}$ satisfying two conditions:[4]

$$f \text{ is bounded;} \quad \text{that is,} \quad \sup_{x\in\mathbb{R}^n} |f(x)| < \infty\,, \tag{4b1}$$

$$f \text{ has bounded support;} \quad \text{that is,} \quad \sup_{x:f(x)\ne 0} |x| < \infty\,. \tag{4b2}$$

First, recall dimension one (that is, $n=1$). Assuming existence of the integral $\int_{-\infty}^{+\infty} f(x)\,\mathrm{d}x$ and denoting it just $\int_{\mathbb{R}} f$, we may sandwich it as follows ($\mathbb{Z}$ is the set of integers, from $-\infty$ till $+\infty$):

$$\sum_{k\in\mathbb{Z}} \inf_{x\in[k,k+1]} f(x) \le \int_{\mathbb{R}} f \le \sum_{k\in\mathbb{Z}} \sup_{x\in[k,k+1]} f(x)\,,$$

[1] Hubbard, Shifrin, Shurman; sometimes called "negligible" (Lang) which, however, could be confused with the other notion.

[2] Burkill.

[3] Hubbard, Zorich.

[4] If puzzled, why the bounded support, or why no continuity, look again at (4a1).

since $\int_{\mathbb{R}} f = \sum_{k\in\mathbb{Z}} \int_k^{k+1} f(x)\,\mathrm{d}x$ (additivity, see also (4a2)), and $\inf_{x\in[k,k+1]} f(x) \le \int_k^{k+1} f(x)\,\mathrm{d}x \le \sup_{x\in[k,k+1]} f(x)$ (see also (4a3)) for each k. We write the integral over the whole $\mathbb{R}$ and the sum over the whole $\mathbb{Z}$, but only a bounded region contributes due to (4b2).

For a better sandwich we use a finer partition; here $N = 0, 1, 2, \dots$ (and for $N = 0$ we get the case above):

$$\frac{1}{2^N} \sum_{k\in\mathbb{Z}} \inf_{x\in[\frac{k}{2^N},\frac{k+1}{2^N}]} f(x) \le \int_{\mathbb{R}} f \le \frac{1}{2^N} \sum_{k\in\mathbb{Z}} \sup_{x\in[\frac{k}{2^N},\frac{k+1}{2^N}]} f(x)\,.$$

In dimension two (that is $n = 2$), paving the plane by squares, we hope to have, first,

$$\sum_{k,\ell\in\mathbb{Z}} \inf_{\substack{x\in[k,k+1]\\ y\in[\ell,\ell+1]}} f(x,y) \le \int_{\mathbb{R}^2} f \le \sum_{k,\ell\in\mathbb{Z}} \sup_{\substack{x\in[k,k+1]\\ y\in[\ell,\ell+1]}} f(x)\,,$$

that is (using two-dimensional x and k),

$$\sum_{k\in\mathbb{Z}^2} \inf_{x\in Q+k} f(x) \le \int_{\mathbb{R}^2} f \le \sum_{k\in\mathbb{Z}^2} \sup_{x\in Q+k} f(x)\,,$$

where $Q = [0,1]^2 = [0,1]\times[0,1]$ and $Q + k = \{x + k : x \in Q\}$; and more generally,

$$\sum_{k\in\mathbb{Z}^2} \underbrace{2^{-2N} \inf_{x\in 2^{-N}(Q+k)} f(x)}_{L_{N,k}(f)} \le \int_{\mathbb{R}^2} f \le \sum_{k\in\mathbb{Z}^2} \underbrace{2^{-2N} \sup_{x\in 2^{-N}(Q+k)} f(x)}_{U_{N,k}(f)}\,,$$

where $2^{-N}(Q + k) = \{2^{-N}(x + k) : x \in Q\}$. In arbitrary dimension n we hope to have

$$L_N(f) \le \int_{\mathbb{R}^n} f \le U_N(f)\,,$$

where $L_N(f)$ and $U_N(f)$ are the lower and upper Darboux sums defined by

$$\text{(4b3)} \qquad L_N(f) = \sum_{k\in\mathbb{Z}^n} L_{N,k}(f)\,, \quad L_{N,k}(f) = 2^{-nN} \inf_{x\in 2^{-N}(Q+k)} f(x)\,,$$

$$\text{(4b4)} \qquad U_N(f) = \sum_{k\in\mathbb{Z}^n} U_{N,k}(f)\,, \quad U_{N,k}(f) = 2^{-nN} \sup_{x\in 2^{-N}(Q+k)} f(x)\,;$$

here $Q = [0,1]^n$.

Clearly, $L_N(f) \le U_N(f)$ and $L_N(f) = -U_N(-f)$ (think, why).

4b5 Lemma. For every N,

$$L_{N+1}(f) \ge L_N(f)\,, \quad U_{N+1}(f) \le U_N(f)\,.$$

Proof. The cube $Q = [0,1]^n$ contains 2^n smaller cubes $2^{-1}(Q+\ell)$, $\ell \in \{0,1\}^n$. Accordingly, the cube $2^{-N}(Q+k)$ contains 2^n smaller cubes $2^{-(N+1)}(Q+2k+\ell)$, $\ell \in \{0,1\}^n$. Thus, $\sum_{\ell\in\{0,1\}^n} U_{N+1,2k+\ell}(f) \le U_{N,k}(f)$, whence

$$U_{N+1}(f) = \sum_{k\in\mathbb{Z}^n} U_{N+1,k}(f) = \\ = \sum_{k\in\mathbb{Z}^n} \sum_{\ell\in\{0,1\}^n} U_{N+1,2k+\ell}(f) \le \sum_{k\in\mathbb{Z}^n} U_{N,k}(f) = U_N(f)\,.$$

Finally, $L_{N+1}(f) = -U_{N+1}(-f) \ge -U_N(-f) = L_N(f)$. $\square$

It follows that both sequences $\big(L_N(f)\big)_N$, $\big(U_N(f)\big)_N$ converge.

4c Integral

4c1 Definition. *Lower and upper integrals* of f are

$$L(f) = \lim_{N\to\infty} L_N(f)\,, \quad U(f) = \lim_{N\to\infty} U_N(f)\,.$$

Clearly, $-\infty < L(f) \le U(f) < \infty$.

4c2 Definition. A bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is called *integrable,* if $L(f) = U(f)$. In this case their common value is the *integral* of f.

The integral is often denoted by[1]

$$\int_{\mathbb{R}^n} f = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}x = \int_{\mathbb{R}^n} f(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \\ \underset{\mathbb{R}^n}{\int\cdots\int} f(x_1,\dots,x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n\,,$$

and sometimes by[2] $\int_{\mathbb{R}^n} f\,\mathrm{d}V = \int_{\mathbb{R}^n} f(x)\,\mathrm{d}V_x$, or[3] $\int_{\mathbb{R}^n} f\,|\mathrm{d}^n x| = \int_{\mathbb{R}^n} f(x)\,|\mathrm{d}^n x|$, or[4] $I_{\mathbb{R}^n}(f)$.

[1]Burkill, Lang, Shurman, Zorich.
[2]Shifrin.
[3]Hubbard.
[4]Lang.

4c3 Exercise. Let

$$\begin{aligned} f(x) = 1\,, \quad g(x) = 0 \quad &\text{for all rational } x \in (0,1)\,, \\ f(x) = 0\,, \quad g(x) = 1 \quad &\text{for all irrational } x \in (0,1)\,, \\ f(x) = 0\,, \quad g(x) = 0 \quad &\text{for all } x \in \mathbb{R} \setminus (0,1)\,. \end{aligned}$$

Prove that

$$\begin{aligned} L(af + bg) &= \min(a,b)\,, \\ U(af + bg) &= \max(a,b) \end{aligned}$$

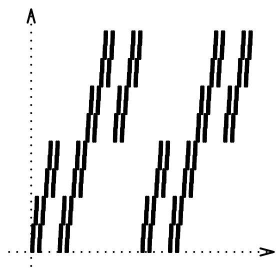
for all $a, b \in \mathbb{R}$.

4c4 Exercise. Find $\int_0^1 x \, \mathrm{d}x$ using only 4c1, 4c2. That is, $\int_{\mathbb{R}} f$ where $f(x) = x$ for $x \in (0,1)$, otherwise $f(x) = 0$.[1]

4c5 Exercise. Let $f : \mathbb{R} \to [0,1)$ be defined via binary digits, by

$$f(x) = \sum_{k=1}^{\infty} \frac{\beta_{2k}(x)}{2^k} \quad \text{for } x = \sum_{k=1}^{\infty} \frac{\beta_k(x)}{2^k}\,, \ \beta_k(x) \in \{0,1\}\,, \ \liminf_k \beta_k(x) = 0\,,$$

and $f(x) = 0$ for $x \in \mathbb{R} \setminus (0,1)$. Prove that f is integrable, and find $\int_{\mathbb{R}} f$.[2]



A wonder: this integrable function has no intervals of continuity!

4c6 Proposition (linearity)**.** All integrable functions $\mathbb{R}^n \to \mathbb{R}$ are a vector space, and the integral is a linear functional[3] on this space.

That is, if $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable and $a, b \in \mathbb{R}$, then $af + bg$ is integrable and $\int_{\mathbb{R}^n} (af + bg) = a \int_{\mathbb{R}^n} f + b \int_{\mathbb{R}^n} g$.

[1] Hint: calculate $L_N(f)$ and $U_N(f)$.

[2] Hint: calculate $L_{2N}(f)$ and $U_{2N}(f)$.

[3] Functions on infinite-dimensional spaces are often called functionals.

Proof. For $a \ge 0$ we have $L_N(af) = aL_N(f)$ and $U_N(af) = aU_N(f)$ (think, why), hence $L(af) = aL(f)$ and $U(af) = aU(f)$. Thus, $L(f) = U(f)$ implies $L(af) = U(af)$, and in this case $\int (af) = a \int f$.

For $a \le 0$ we have $L_N(af) = aU_N(f)$ and $U_N(af) = aL_N(f)$ (think, why), hence $L(af) = aU(f)$ and $U(af) = aL(f)$. Still, $L(f) = U(f)$ implies $L(af) = U(af)$, and in this case $\int (af) = a \int f$.

It remains to consider the sum $f+g$. We have $L_N(f+g) \ge L_N(f) + L_N(g)$ and $U_N(f+g) \le U_N(f) + U_N(g)$ (think, why), hence $L(f+g) \ge L(f) + L(g)$ and $U(f+g) \le U(f) + U(g)$. Thus, $L(f) = U(f)$ and $L(g) = U(g)$ imply $L(f+g) = U(f+g)$, and in this case $\int (f+g) = \int f + \int g$. $\square$

4c7 Remark. Denoting the lower and upper integral by $_*\!\int_{\mathbb{R}^n} f$ and $^*\!\int_{\mathbb{R}^n} f$ we note some properties.

Monotonicity:

$$\text{if } f(\cdot) \le g(\cdot) \quad \text{then} \quad {}_*\!\int f \le {}_*\!\int g \,, \quad {}^*\!\int f \le {}^*\!\int g \,,$$
$$\text{and for integrable } f, g, \quad \int f \le \int g \,.$$

(It can happen that $^*\!\int f > {}_*\!\int g$; find an example.)

Homogeneity:

$$\begin{gathered} {}_*\!\int cf = c \, {}_*\!\int f \,, \quad {}^*\!\int cf = c \, {}^*\!\int f \quad \text{for } c \ge 0 \,; \\ {}_*\!\int cf = c \, {}^*\!\int f \,, \quad {}^*\!\int cf = c \, {}_*\!\int f \quad \text{for } c \le 0 \,; \\ \text{if } f \text{ is integrable then } cf \text{ is, and} \quad \int cf = c \int f \quad \text{for all } c \in \mathbb{R} \,. \end{gathered}$$

(Sub-, super-) additivity:

$$\begin{gathered} {}^*\!\int (f+g) \le {}^*\!\int f + {}^*\!\int g \,; \\ {}_*\!\int (f+g) \ge {}_*\!\int f + {}_*\!\int g \,; \\ \text{if } f, g \text{ are integrable then } f+g \text{ is, and} \quad \int (f+g) = \int f + \int g \,. \end{gathered}$$

(It can happen that $^*\!\int (f+g) < {}^*\!\int f + {}^*\!\int g$; find an example.)

4d Volume

Given a set $E \subset \mathbb{R}^n$, its indicator (or characteristic) function, denoted $\mathbb{1}_E$ or χ_E, is defined by

$$\mathbb{1}_E(x) = \begin{cases} 1 & \text{for } x \in E, \\ 0 & \text{for } x \in \mathbb{R}^n \setminus E. \end{cases}$$

The integral of the indicator function (if exists) is called[1] the volume, or[2] n-dimensional volume, or[3] content, or[4] Jordan measure, and denoted $v(E)$, $\mathrm{vol}_n(E)$, $c(E)$. It exists if and only if $\mathbb{1}_E$ is integrable. In this case one says[5] that E is admissible, or[6] pavable, or[7] has content.

4d1 Definition. (a) A bounded set $E \subset \mathbb{R}^n$ is *admissible,* if $\mathbb{1}_E$ is integrable.

(b) The volume $v(E) = \mathrm{vol}(E) = \mathrm{vol}_n(E)$ of an admissible set E is $\int_{\mathbb{R}^n} \mathbb{1}_E$.

(c) For arbitrary bounded E, ${}^*\!\int_{\mathbb{R}^n} \mathbb{1}_E = v^*(E)$ is the *outer volume* of E, and ${}_*\!\int_{\mathbb{R}^n} \mathbb{1}_E = v_*(E)$ is the *inner volume* of E.[8]

Note that $v^*(E) = \lim_N U_N(\mathbb{1}_E)$, and $U_N(\mathbb{1}_E)$ is the total volume of all N-pixels that intersect E. Also, $v_*(E) = \lim_N L_N(\mathbb{1}_E)$, and $L_N(\mathbb{1}_E)$ is the total volume of all N-pixels contained in E. And finally, E is admissible if and only if $v_*(E) = v^*(E)$; and in this case $v_*(E) = v(E) = v^*(E)$, of course.

Later we'll see that a bounded E is admissible if and only if $v(\partial E) = 0$, but for now we do not need it. If $v^*(E) = 0$, then necessarily E (is admissible and) has volume zero. By monotonicity (recall 4c7), if E has volume zero, then every subset of E has volume zero. If E has volume zero, then $E^\circ = \emptyset$ (think, why); the converse does not hold (think, why).[9]

4d2 Exercise. The cube $[0,1]^n$ is admissible, and $v\big([0,1]^n\big) = 1$.

Prove it.[10]

Similarly, all dyadic cubes ("pixels") are admissible, and $v(Q) = 2^{-nN}$ for every N-pixel Q.

[1] Lang, Shurman.

[2] Hubbard.

[3] Burkill, Zorich.

[4] Zorich.

[5] Lang, Zorich.

[6] Hubbard.

[7] Burkill.

[8] Or, inner and outer Jordan content, according to Burkill, Sect. 6.8, p. 182.

[9] Moreover, a closed subset of $[0,1]$ with empty interior need not have volume zero ("fat Cantor set").

[10] Hint: $L_N(\mathbb{1}_{[0,1]^n}) = 1$ and $U_N(\mathbb{1}_{[0,1]^n}) = 2^{-nN}(2^N + 2)^n$.

4d3 Lemma (additivity of volume)**.** Let $E, F \subset \mathbb{R}^n$ be admissible, and $E \cap F$ have volume zero. Then $E \cup F$ is admissible, and $v(E \cup F) = v(E) + v(F)$.

Proof. We have $1\!\!1_{E\cup F} = 1\!\!1_E + 1\!\!1_F - 1\!\!1_{E\cap F}$ (think, why). Also, $1\!\!1_{E\cap F}$ (is integrable and) has integral zero; by linearity (recall 4c6), $1\!\!1_{E\cup F}$ is integrable, and $\int 1\!\!1_{E\cup F} = \int 1\!\!1_E + \int 1\!\!1_F$. $\square$

A box[1] in $\mathbb{R}^n$ is the (Cartesian) product of intervals,

$$B = [a_1, b_1] \times \dots \times [a_n, b_n] \, .$$

4d4 Exercise. Every box B is admissible, its interior B° is also admissible, and

$$v(B) = (b_1 - a_1) \dots (b_n - a_n) = v(B^\circ) \, .$$

Prove it.[2]

Thus, every bounded "pixelated set", that is, finite union of pixels, is admissible, and we know its volume.

4d5 Definition. Let $E \subset \mathbb{R}^n$ be an admissible set. A bounded function $f : E \to \mathbb{R}$ is *integrable on* E, if the corresponding function $f_E : \mathbb{R}^n \to \mathbb{R}$ (see (4a1)) is integrable (on $\mathbb{R}^n$). In this case, $\int_E f = \int_{\mathbb{R}^n} f_E$.

It is usual and convenient to write $f \cdot 1\!\!1_E$ instead of f_E; accordingly,

$$\int_E f = \int_{\mathbb{R}^n} f \cdot 1\!\!1_E \, .$$

The same applies when f is defined on the whole $\mathbb{R}^n$, or on a set that contains E. Note that

$$\int_E 1 = v(E) \, ; \quad \int_E c = cv(E) \text{ for } c \in \mathbb{R} \, ; \tag{4d6}$$

$$v(E) \inf_{x\in E} f(x) \le \int_E f \le v(E) \sup_{x \in E} f(x) \, ; \tag{4d7}$$

$$v(E) = 0 \quad \Longrightarrow \quad \int_E f = 0 \, . \tag{4d8}$$

Assuming $v(E) \ne 0$ one defines the *mean value* of f on E as

$$\frac{1}{v(E)} \int_E f \, .$$

[1] See Sect. 4a for other names. Some authors allow the degenerate case $v(B) = 0$ (Lang, Shurman); others disallow it explicitly (Burkill) or implicitly (Shifrin, Zorich), or do not bother (Hubbard). For now we need not bother, too. But in Sect. 4g we'll allow degeneration.

[2] Hint: $U_N(1\!\!1_B) \le (b_1 - a_1 + 2 \cdot 2^{-N}) \dots (b_n - a_n + 2 \cdot 2^{-N})$ and $L_N(1\!\!1_{B^\circ}) \ge (b_1 - a_1 - 2 \cdot 2^{-N}) \dots (b_n - a_n - 2 \cdot 2^{-N})$.

4e Normed space of equivalence classes

All bounded functions $\mathbb{R}^n \to \mathbb{R}$ with bounded support[1] are a vector space. On this space, the functional

$$f \mapsto {}^{*}\!\!\int_{\mathbb{R}^n} |f|$$

is a *seminorm;* that is, satisfies the first two conditions (recall 1f13),

$$\begin{gathered} {}^{*}\!\!\int_{\mathbb{R}^n} |cf| = |c| \, {}^{*}\!\!\int_{\mathbb{R}^n} |f| \,, \\ {}^{*}\!\!\int_{\mathbb{R}^n} |f+g| \le {}^{*}\!\!\int_{\mathbb{R}^n} |f| + {}^{*}\!\!\int_{\mathbb{R}^n} |g| \end{gathered}$$

(think, why), but violates the third condition,

$${}^{*}\!\!\int_{\mathbb{R}^n} |f| > 0 \quad \text{whenever } f \ne 0 \,. \qquad \text{(Wrong!)}$$

Functions f such that ${}^{*}\!\!\int_{\mathbb{R}^n} |f| = 0$ will be called *negligible.* Functions f, g such that $f - g$ is negligible will be called *equivalent.* For example, for each pixel Q functions $1\!\!1_{Q^\circ}$ and $1\!\!1_Q$ are equivalent.[2] The equivalence class of f will be denoted $[f]$.[3]

4e1 Exercise. (a) Negligible functions are an infinite-dimensional vector space.

(b) Equivalence classes are an infinite-dimensional vector space;[4] the functional

$$[f] \mapsto {}^{*}\!\!\int_{B} |f|$$

is well-defined on this vector space,[5] and is a norm.[6]

Prove it.

Thus, equivalence classes are a normed space, therefore also a metric space:

$$\rho\big([f],[g]\big) = \| \, [f] - [g] \, \| = {}^{*}\!\!\int_{B} |f-g| \,;$$

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[1] Each functions separately.

[2] Indeed, the equality ${}_{*}\!\!\int 1\!\!1_{Q^\circ} = {}^{*}\!\!\int 1\!\!1_Q$ follows easily from 4d4.

[3] Zorich, Sect. 11.3.1.

[4] The linear operations are $c[f] = [cf]$ and $[f] + [g] = [f + g]$, of course.

[5] That is, insensitive to the choice of a function within the given equivalence class.

[6] In fact, every seminorm on a vector space leads to a normed space of equivalence classes.

this metric will be called the *integral metric,* and the corresponding convergence the *integral convergence.*

4e2 Exercise. Functionals

$$[f] \mapsto {}_*\!\!\int_{\mathbb{R}^n} f\,, \quad [f] \mapsto {}^*\!\!\int_{\mathbb{R}^n} f$$

on the normed space of equivalence classes are well-defined and continuous; moreover,

$$\Big| {}_*\!\!\int_{\mathbb{R}^n} f - {}_*\!\!\int_{\mathbb{R}^n} g \Big| \le \|f-g\|\,, \quad \Big| {}^*\!\!\int_{\mathbb{R}^n} f - {}^*\!\!\int_{\mathbb{R}^n} g \Big| \le \|f-g\|\,.$$

Prove it.[1]

Here and henceforth we often write $\|f\|$ instead of $\|\,[f]\,\|$.

4e3 Remark. A function equivalent to an integrable function is integrable.

Proof: if $[f]=[g]$ then ${}_*\!\int_{\mathbb{R}^n} f = {}_*\!\int_{\mathbb{R}^n} g$ and ${}^*\!\int_{\mathbb{R}^n} f = {}^*\!\int_{\mathbb{R}^n} g$ by 4e2, thus ${}_*\!\int_{\mathbb{R}^n} f = {}^*\!\int_{\mathbb{R}^n} f$ implies ${}_*\!\int_{\mathbb{R}^n} g = {}^*\!\int_{\mathbb{R}^n} g$.

4e4 Exercise. If bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support differ only on a set of volume zero then they are equivalent.

Prove it.[2,3,4]

> We may safely ignore values of integrands on sets of volume zero (as far as they are bounded). Likewise we may ignore sets of volume zero when dealing with volume.

4e5 Remark. If $f_1, f_2, \dots$ are integrable and $\|f_k - f\| \to 0$, then f is integrable. In other words:

> The set of all (equivalence classes of) integrable functions is closed (in the integral metric).

Proof: ${}_*\!\int_{\mathbb{R}^n} f_k \to {}_*\!\int_{\mathbb{R}^n} f$ and ${}^*\!\int_{\mathbb{R}^n} f_k \to {}^*\!\int_{\mathbb{R}^n} f$ by 4e2, thus ${}_*\!\int_{\mathbb{R}^n} f_k = {}^*\!\int_{\mathbb{R}^n} f_k$ implies ${}_*\!\int_{\mathbb{R}^n} f = {}^*\!\int_{\mathbb{R}^n} f$.

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[1] Hint: ${}_*\!\int_{\mathbb{R}^n} f = -{}^*\!\int_{\mathbb{R}^n} (-f)$.

[2] Hint: $|f-g| \le \text{const} \cdot \mathbb{1}_E$.

[3] "Sets of volume zero are small enough that they don't interfere with integration" (Shurman, p.272).

[4] The converse does not hold; see 4f12.

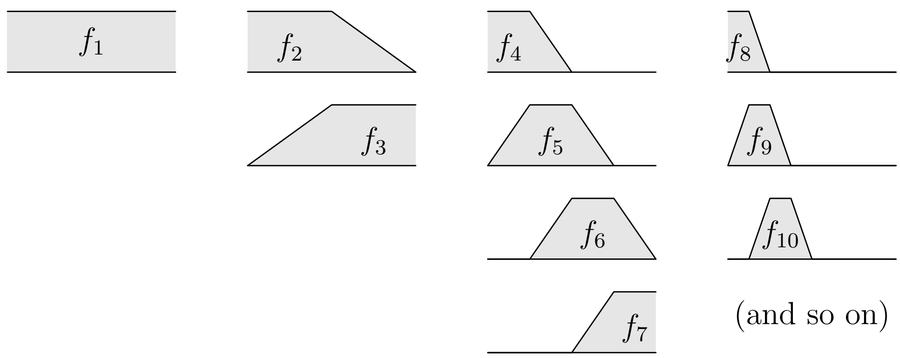
Any admissible set $E \subset \mathbb{R}^n$ may be used instead of the whole $\mathbb{R}^n$. Equivalence classes of bounded functions $E \to \mathbb{R}$ are a normed space (infinite-dimensional if $v(E) \ne 0$, but 0-dimensional if $v(E) = 0$).

4e6 Exercise. (a) Uniform convergence of bounded functions $E \to \mathbb{R}$ implies integral convergence; prove it;

(b) the converse is generally wrong; find a counterexample.

4e7 Remark. Pointwise convergence (on E) does not imply integral convergence, even if the functions are uniformly bounded.[1] Here is a counterexample. We take a sequence $(x_k)_k$ of pairwise different points $x_k \in (0,1)$ that is dense in $(0,1)$ and consider dense countable sets $A_k = \{x_{k+1}, x_{k+2}, \dots\}$. Clearly, $A_1 \supset A_2 \supset \dots$ and $\bigcap_k A_k = \emptyset$. Indicator functions $f_k = 1\!\!1_{A_k}$ converge to 0 pointwise (and monotonically). Nevertheless, ${}^*\!\!\int_{(0,1)} f_k = 1$ for all k.

4e8 Remark. Integral convergence (on E) does not imply pointwise convergence, even if the functions are continuous. Not even in "most" of the points. Here is a counterexample on $E = [0,1] \subset \mathbb{R}$:



4f Approximation

It is usual and convenient to treat functions as equivalent classes, when dealing with integrals of discontinuous functions.

A box B leads to the equivalence class $[1\!\!1_{B^\circ}] = [1\!\!1_B]$. Linear combinations[2] of these are called *step functions.* Dealing with a step function we ignore its values at discontinuity points (but still assume that the function is bounded). All step functions are integrable.

[1] It does, if the functions are integrable! But this fact is far beyond basis of integration.

[2] Finite, of course.

4f1 Exercise. (a) Every continuous $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support is integrable;

(b) every continuous function on a box is integrable on this box.
Prove it.[1]

4f2 Exercise. Let $f : (0,1)^n \to \mathbb{R}$ be continuous (on the open cube!) and bounded. Then f is integrable (on this open cube).

Prove it.[2]

For example, the function $f(x) = \sin \cot \pi x$ on $(0,1)$ is integrable.

4f3 Proposition. Step functions are dense among integrable functions. That is, for every integrable $f : \mathbb{R}^n \to \mathbb{R}$ and every $\varepsilon > 0$ there exists a step function g such that $\|f - g\| \le \varepsilon$.

Proof. We take N such that $U_N(f) - L_N(f) \le \varepsilon$ and introduce step functions g, h by

$$g(a) = \inf_{x \in Q} f(x)\,, \quad h(a) = \sup_{x \in Q} f(x) \qquad \text{for } a \in Q^\circ$$

where Q runs over all N-pixels. We have $\int_{\mathbb{R}^n} g = L_N(f)$, $\int_{\mathbb{R}^n} h = U_N(f)$ (think, why), and $g \le f \le h$ everywhere (except maybe a set of volume zero). Thus,

$$\|f - g\| = \int_{\mathbb{R}^n} |f - g| \le \int_{\mathbb{R}^n} (h - g) = U_N(f) - L_N(f) \le \varepsilon\,.$$

$\square$

4f4 Remark. In addition, g can be chosen such that

$$\inf f(\cdot) \le \inf g(\cdot) \le \sup g(\cdot) \le \sup f(\cdot) \quad \text{and} \quad \sup_{x: g(x) \ne 0} |x| \le \sup_{x: f(x) \ne 0} |x| + \varepsilon\,.$$

4f5 Remark. A function is integrable if and only if it is the limit of some sequence of step functions (in the integral convergence), which follows from 4f3 and 4e5. In other words:

> The set of all (equivalence classes of) integrable functions is the closure of the set of all (equivalence classes of) step functions (in the integral metric).

4f6 Exercise. There exist continuous $g_k : \mathbb{R}^n \to [0,1]$ with uniformly bounded support such that $\|g_k - \mathbb{1}_{[0,1]^n}\| \to 0$.

Prove it.

[1] Hint: uniform continuity, and approximation by step functions.

[2] Hint: approximation by $f \cdot \mathbb{1}_{[\varepsilon, 1-\varepsilon]^n}$.

The same holds for every pixel; taking a linear combination and using 4f3 we get the following.

4f7 Corollary. For every integrable f there exist continuous $g_k : \mathbb{R}^n \to \mathbb{R}$ with uniformly bounded support such that $\|g_k - f\| \to 0$. Thus:

> The set of all (equivalence classes of) integrable functions is the closure of the set of all (equivalence classes of) continuous functions with bounded support (in the integral metric).

4f8 Lemma. If f is integrable, then $f^2 : x \mapsto \big(f(x)\big)^2$ is integrable.

Proof. Using 4f3 and 4f4 we take step functions[1] g_k and a number M such that $\|g_k - f\| \to 0$ and $|f(\cdot)| \le M$, $|g_k(\cdot)| \le M$. It remains to prove that $\|g_k^2 - f^2\| \to 0$ (since g_k^2 are step functions). We have

$$|g_k^2(x) - f^2(x)| = |g_k(x) + f(x)| \cdot |g_k(x) - f(x)| \le 2M|g_k(x) - f(x)|\,,$$

thus, $\|g_k^2 - f^2\| \le 2M\|g_k - f\| \to 0$. $\square$

4f9 Corollary. The (pointwise) product of two integrable functions is integrable.

Indeed, $fg = \frac14\big((f+g)^2 - (f-g)^2\big)$.

4f10 Exercise. If f is integrable, then $|f|$, $f^+ = \frac12\big(f + |f|\big)$, $\sin f$, $1 - \cos f$, and $\mathrm{e}^f - 1$ are integrable. If g is also integrable, then $\max(f,g)$ is integrable.

Prove it.[2]

4f11 Exercise. (a) If f and f_1 are equivalent, then f^2 and f_1^2 are equivalent; the same holds for $|f|$, f^+, $\sin f$, $1 - \cos f$, and $\mathrm{e}^f - 1$.

(b) If $[f] = [f_1]$ and $[g] = [g_1]$, then $[fg] = [f_1 g_1]$ and $[\max(f,g)] = [\max(f_1, g_1)]$.

Prove it.

4f12 Remark. It can happen that f and f_1 are equivalent, but $\operatorname{sgn} f$ and $\operatorname{sgn} f_1$ are not. Here is a counterexample. Let $(r_k)_k$ be an enumeration of all rational numbers on $[0,1]$; consider f such that $f(r_k) = c_k$ for all k, $f(x) = 0$ for irrational $x \in [0,1]$ and for all $x \in \mathbb{R} \setminus [0,1]$. If $c_k \to 0$, then $[f] = [0]$ (think, why); but if $c_k = \pm 1$, then $[f] \ne [0]$ (think, why).

If two continuous functions are equal on a dense set, then they are equal everywhere. This is not the case for integrable functions. But here is a surprise.

[1] Continuous functions may be used equally well.

[2] Hint: consider $(g - f)^+$.

4f13 Exercise. If two integrable functions are equal on a dense set, then they are equivalent.

Prove it.[1]

On the other hand, a function equal to an integrable function on a dense set need not be integrable (think, why).

4f14 Proposition. If $E, F \subset \mathbb{R}^n$ are admissible sets, then the sets $E \cap F$, $E \cup F$ and $E \setminus F$ are admissible.

Proof. First, $E \cap F$ is admissible since $1\!\!1_{E\cap F} = 1\!\!1_E \cdot 1\!\!1_F$ is integrable by 4f9. Further, $1\!\!1_{E\cup F} = 1\!\!1_E + 1\!\!1_F - 1\!\!1_{E\cap F}$ and $1\!\!1_{E\setminus F} = 1\!\!1_E - 1\!\!1_{E\cap F}$ are integrable. □

4f15 Exercise. Give another proof of 4f14 using $\max(f, g)$ (and $\min(f, g)$) rather than fg.

4f16 Proposition. (a) A function integrable on $\mathbb{R}^n$ is integrable on every admissible set;

(b) a function integrable on an admissible set is integrable on every admissible subset of the given set.

Proof. (a) $f \cdot 1\!\!1_E$ is integrable by 4f9.

(b) Given $E \subset F$, the function $f \cdot 1\!\!1_E = (f \cdot 1\!\!1_F) \cdot 1\!\!1_E$ is integrable by 4f9. □

4g Sandwich

The Darboux sums $L_N(f) = -U_N(-f)$ and $U_N(f)$ defined by (4b3), (4b4) may be thought of as integrals of step functions,

$$L_N(f) = \int_{\mathbb{R}^n} \ell_{N,f}\,, \quad U_N(f) = \int_{\mathbb{R}^n} u_{N,f}\,, \tag{4g1}$$

$$u_{N,f}(a) = \underbrace{\sup_{x \in 2^{-N}(Q+k)} f(x)}_{2^{nN} U_{N,k}(f)} \quad \text{for } a \in 2^{-N}(Q^\circ + k) \tag{4g2}$$

and $\ell_{N,f} = -u_{N,-f}$; here $Q = [0,1]^n$, again. In Sect. 4f we did not bother about values of step functions at points of discontinuity. But sometimes we need the inequality $\ell_{N,f} \le f \le u_{N,f}$ to hold everywhere (including pixel boundaries). We can ensure this by taking

$$2^{-nN} u_{N,f} = \sum_{k \in \mathbb{Z}^n : U_{N,k}(f) > 0} U_{N,k}(f) 1\!\!1_{2^{-N}(Q+k)} + \sum_{k \in \mathbb{Z}^n : U_{N,k}(f) < 0} U_{N,k}(f) 1\!\!1_{2^{-N}(Q^\circ+k)} \tag{4g3}$$

[1]Hint: $L_N(|f - g|) = 0$.

and $\ell_{N,f} = -u_{N,-f}$ (again). The values of these step functions on pixel boundaries are somewhat strange but harmless; we have (think, why)

$$\ell_{N,f} \le f \le u_{N,f}\,, \tag{4g4}$$

$$2^n \inf f(\cdot) \le \inf \ell_{N,f}(\cdot) \le \sup u_{N,f}(\cdot) \le 2^n \sup f(\cdot)\,; \tag{4g5}$$

the latter shows that $\ell_{N,f}$ and $u_{N,f}$ are bounded, uniformly in N.

A box was defined in Sect. 4d as $B = [a_1, b_1] \times \dots \times [a_n, b_n]$. Now we clarify that $-\infty < a_i \le b_i < +\infty$ for $i = 1, \dots, n$; the degenerate case $v(B) = 0$ is allowed. Further, we define a step function as a (finite) linear combination of indicator functions of boxes. (On the level of equivalence classes this definition conforms to Sect. 4f.)

Note that $1\!\!1_{B^\circ}$ is a step function; for a proof, open the brackets in

$$\big(1\!\!1_{[a_1,b_1]}(x_1) - 1\!\!1_{\{a_1\}}(x_1) - 1\!\!1_{\{b_1\}}(x_1)\big) \dots \big(1\!\!1_{[a_n,b_n]}(x_n) - 1\!\!1_{\{a_n\}}(x_n) - 1\!\!1_{\{b_n\}}(x_n)\big)$$

(assuming $a_1 < b_1, \dots, a_n < b_n$, of course). It follows that $\ell_{N,f}$ and $u_{N,f}$ are step functions.

4g6 Proposition. For every bounded $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$\sideset{_*}{}\int_{\mathbb{R}^n} f = \sup\Big\{ \int_{\mathbb{R}^n} g \,\Big|\, \text{step } g \le f \Big\}, \quad \sideset{}{^*}\int_{\mathbb{R}^n} f = \inf\Big\{ \int_{\mathbb{R}^n} h \,\Big|\, \text{step } h \ge f \Big\}.$$

Proof. It is sufficient to prove the latter; the former follows via $(-f)$.

"$\le$": $\int_{\mathbb{R}^n} h = {}^*\!\!\int_{\mathbb{R}^n} h \ge {}^*\!\!\int_{\mathbb{R}^n} f$ by 4c7 (monotonicity).

"$\ge$": taking $h = u_{N,f}$ we see that the infimum $\le \int_{\mathbb{R}^n} u_{N,f} = U_N(f)$ for all N. $\square$

Clearly, we have an equivalent definition of integrability and integral.

4g7 Corollary. For every bounded $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$\sideset{_*}{}\int_{\mathbb{R}^n} f = \sup\Big\{ \int_{\mathbb{R}^n} g \,\Big|\, \text{integrable } g \le f \Big\}, \quad \sideset{}{^*}\int_{\mathbb{R}^n} f = \inf\Big\{ \int_{\mathbb{R}^n} h \,\Big|\, \text{integrable } h \ge f \Big\}.$$

4g8 Corollary. A function $f : \mathbb{R}^n \to \mathbb{R}$ is integrable if and only if for every $\varepsilon > 0$ there exist step functions g and h such that $g \le f \le h$ and $\int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} g \le \varepsilon$.

We see that an integrable function can be sandwiched between step functions. Or, alternatively, between continuous functions, see 4g9.

4g9 Exercise. (a) For every box $B \subset \mathbb{R}^n$ and $\varepsilon > 0$ there exist continuous functions $g, h : \mathbb{R}^n \to [0,1]$ with bounded support such that $g \le 1\!\!1_{B^\circ} \le 1\!\!1_B \le h$ and $\int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} g \le \varepsilon$;

(b) for every step function $f : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon > 0$ there exist continuous functions g and h with bounded support such that $g \le f \le h$ and $\int_{\mathbb{R}^n} h - \int_{\mathbb{R}^n} g \le \varepsilon$;

(c) the same holds for every integrable f.

Prove it.[1]

4g10 Exercise. (a) Define ${}_*\!\int_E f$ and ${}^*\!\int_E f$ similarly to 4d5;

(b) prove additivity of the upper integral: ${}^*\!\int_{E \uplus F} f = {}^*\!\int_E f + {}^*\!\int_F f$, and the same for the lower integral;[2]

(c) generalize (4d7) to lower and upper integrals.

Thus, if f is not integrable, then the corresponding set function satisfying (4a2) and (4a3) is not unique; we have at least two such set functions, $E \mapsto {}_*\!\int_E f$ and $E \mapsto {}^*\!\int_E f$.

4h Translation (shift) and scaling

As before, we assume that $f : \mathbb{R}^n \to \mathbb{R}$ is bounded, with bounded support.

Given a function f and a vector $a \in \mathbb{R}^n$, we consider the shifted function $f(\cdot + a) : x \mapsto f(x + a)$.

If $a \in \mathbb{Z}^n$, then $L_0\big(f(\cdot + a)\big) = L_0(f)$ and $U_0\big(f(\cdot + a)\big) = U_0(f)$ (think, why). Moreover, if $a \in 2^{-N}\mathbb{Z}^n$, then $L_{N+i}\big(f(\cdot+a)\big) = L_{N+i}(f)$ and $U_{N+i}\big(f(\cdot+a)\big) = U_{N+i}(f)$ for $i = 0, 1, 2, \dots$, whence ${}_*\!\int_{\mathbb{R}^n} f(\cdot + a) = {}_*\!\int_{\mathbb{R}^n} f$ and ${}^*\!\int_{\mathbb{R}^n} f(\cdot + a) = {}^*\!\int_{\mathbb{R}^n} f$. Our theory is invariant under binary-rational shifts. What about arbitrary shifts?

4h1 Proposition. $f(\cdot + a)$ is integrable if and only if f is integrable, and in this case $\int_{\mathbb{R}^n} f(\cdot + a) = \int_{\mathbb{R}^n} f$.

Proof. First, if f is the indicator function of a box, then the claim holds by 4d4.

Second, by linearity the claim holds for step functions.

We apply it to the step functions g and h of 4g6, note that $g \le f \iff g(\cdot + a) \le f(\cdot + a)$ and $h \ge f \iff h(\cdot + a) \ge f(\cdot + a)$, and conclude that

$$ {}_*\!\int_{\mathbb{R}^n} f(\cdot + a) = {}_*\!\int_{\mathbb{R}^n} f \,, \qquad {}^*\!\int_{\mathbb{R}^n} f(\cdot + a) = {}^*\!\int_{\mathbb{R}^n} f \,; $$

thus, ${}_*\!\int_{\mathbb{R}^n} f = {}^*\!\int_{\mathbb{R}^n} f \iff {}_*\!\int_{\mathbb{R}^n} f(\cdot + a) = {}^*\!\int_{\mathbb{R}^n} f(\cdot + a)$. □

[1] Hint: (a) product of n piecewise linear functions of one variable each; (a)$\Longrightarrow$(b)$\Longrightarrow$(c).

[2] Hint: use 4g7.

4h2 Corollary. For every set $E \subset \mathbb{R}^n$ and vector $a \in \mathbb{R}^n$, the shifted set $E+a$ is admissible if and only if E is admissible, and in this case $v(E+a) = v(E)$.

Consider now a linear operator $A : \mathbb{R}^n \to \mathbb{R}^n$ of the form $A(x_1, \dots, x_n) = (a_1 x_1, \dots, a_n x_n)$ (that is, diagonal matrix), and assume that $a_1 \ne 0, \dots, a_n \ne 0$ (that is, A is invertible).

4h3 Exercise. $f \circ A$ is integrable if and only if f is integrable, and in this case $|a_1 \dots a_n| \int_{\mathbb{R}^n} f \circ A = \int_{\mathbb{R}^n} f$.

Prove it.[1]

4h4 Exercise. For every set $E \subset \mathbb{R}^n$, its image $A(E) = \{Ax : x \in E\}$ is admissible if and only if E is admissible, and in this case $v(A(E)) = |a_1 \dots a_n| v(E)$.

Prove it.

In particular,

$$|a|^n \int_{\mathbb{R}^n} f(ax) \, \mathrm{d}x = \int_{\mathbb{R}^n} f \, , \tag{4h5}$$

$$v(aE) = |a|^n v(E) \, . \tag{4h6}$$

The following fact is evident for continuous f but, surprisingly, does not require continuity.

4h7 Proposition. For every integrable $f : \mathbb{R}^n \to \mathbb{R}$ and $\varepsilon > 0$ there exists $\delta > 0$ such that for all $a \in \mathbb{R}^n$

$$|a| \le \delta \quad \Longrightarrow \quad \| f(\cdot + a) - f \| \le \varepsilon \, .$$

Proof. First, assume in addition that f is continuous. Then we take $M \in (0, \infty)$ such that $\{x : f(x) \ne 0\} \subset [-M, M]^n$, and then, using uniform continuity of f, we take δ such that $|a| \le \delta$ implies

$$\forall x \quad |f(x+a) - f(x)| \le \frac{\varepsilon}{2^n (M+\delta)^n} \, .$$

Then $\{x : f(x+a) - f(x) \ne 0\} \subset [-(M+\delta), M+\delta]^n$ (think, why), whence

$$\int_{\mathbb{R}^n} |f(\cdot + a) - f(\cdot)| \le \max_x |f(x+a) - f(x)| \cdot v\big([-(M+\delta), M+\delta]^n \big) \le \\ \le \frac{\varepsilon}{2^n (M+\delta)^n} \cdot \big(2(M+\delta) \big)^n = \varepsilon \, ,$$

[1] Hint: similarly to 4h1.

that is, $\|f(\cdot + a) - f\| \le \varepsilon$.

Second, given an integrable f, by 4f7 there exists a continuous $g : \mathbb{R}^n \to \mathbb{R}$ with bounded support such that $\|g - f\| \le \varepsilon/3$. We take δ such that $\|g(\cdot + a) - g\| \le \varepsilon/3$. Then, using the triangle inequality,

$$\|f(\cdot + a) - f\| \le \|f(\cdot + a) - g(\cdot + a)\| + \|g(\cdot + a) - g\| + \|g - f\| \le \\ \le \|f - g\| + \frac{\varepsilon}{3} + \|g - f\| \le \frac{\varepsilon}{3} + \frac{\varepsilon}{3} + \frac{\varepsilon}{3} = \varepsilon\,.$$

□

4i The volume under a graph

Here is a rich source of admissible sets.

4i1 Proposition. If a function $f : \mathbb{R}^n \to [0, \infty)$ is integrable, then the set

$$E = \{(x, t) : 0 < t < f(x)\} \subset \mathbb{R}^n \times \mathbb{R}$$

is admissible, and $v_{n+1}(E) = \int_{\mathbb{R}^n} f$.

Proof. For $N = 0, 1, 2, \dots$ and $k \in \mathbb{Z}^n$ we introduce such boxes in $\mathbb{R}^{n+1}$:

$$B_{N,k} = 2^{-N}(Q + k) \times [0, 2^{nN} U_{N,k}(f)]\,, \quad C_{N,k} = 2^{-N}(Q + k) \times [0, 2^{nN} L_{N,k}(f)]$$

(here $Q = [0, 1]^n$, as in Sect. 4b) and note that $\cup_k C^\circ_{N,k} \subset E \subset \cup_k B_{N,k}$, therefore (recall (4b3), (4b4) and (4d4))

$$L_N(f) = \sum_k \underbrace{v(C^\circ_{N,k})}_{L_{N,k}(f)} \le v_*(E) \le v^*(E) \le \sum_k \underbrace{v(B_{N,k})}_{U_{N,k}(f)} = U_N(f)$$

for all N. □

4i2 Corollary. If functions $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable, then the set

$$E = \{(x, t) : f(x) < t < g(x)\} \subset \mathbb{R}^n \times \mathbb{R}$$

is admissible.

Proof. We take a box $B \subset \mathbb{R}^n$ such that $f = g = 0$ on $\mathbb{R}^n \setminus B$, and a number M such that $|f| \le M$, $|g| \le M$ everywhere. Then

$$E = \{(x, t) : x \in B,\ -M < t < g(x)\} \cap \{(x, t) : x \in B,\ f(x) < t < M\}$$

(think, why). By 4f14 it is sufficient to prove that these two sets are admissible. The second set becomes similar to the first set after reflection $(x, t) \mapsto (x, -t)$ (recall 4h4). The first set is a shift (recall 4h2) by $(0, -M)$ of the set $\{(x, t) : x \in B,\ 0 < t < g(x) + M\}$ admissible by 4i1 applied to $(g + M)\mathbb{1}_B$. □

It is easy to guess that $v_{n+1}(E) = \int_{\mathbb{R}^n} (g-f)^+$. We could prove it now with some effort.[1] However, in the next section we'll get the same effortlessly.

4i3 Exercise. For f as in 4i1, the set

$$\{(x,t) : t = f(x) > 0\} \subset \mathbb{R}^n \times \mathbb{R}$$

is of volume zero.

Prove it.[2]

4i4 Exercise. Prove that

(a) the disk $\{x : |x| \le 1\} \subset \mathbb{R}^2$ is admissible;

(b) the ball $\{x : |x| \le 1\} \subset \mathbb{R}^n$ is admissible;

(c) for every $p > 0$ the set $E_p = \{(x_1, \dots, x_n) : |x_1|^p + \dots + |x_n|^p \le 1\} \subset \mathbb{R}^n$ is admissible;

(d) $v(E_p)$ is a strictly increasing function of p.

4i5 Exercise. For the balls $E_r = \{x : |x| \le r\} \subset \mathbb{R}^n$ prove that

(a) $v(E_r) = r^n v(E_1)$;

(b) $v(E_r) < \mathrm{e}^{-n(1-r)} v(E_1)$ for $r < 1$.

A wonder: in high dimension the volume of a ball is concentrated near the sphere!

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[1] Hint: $\sum_k (L_{N,k}(g) - U_{N,k}(f))^+ \le v_*(E) \le v^*(E) \le \sum_k (U_{N,k}(g) - L_{N,k}(f))^+$, and $(U_{N,k}(g) - L_{N,k}(f))^+ - (L_{N,k}(g) - U_{N,k}(f))^+ \le (U_{N,k}(g) - L_{N,k}(f)) - (L_{N,k}(g) - U_{N,k}(f)) = (U_{N,k}(g) - L_{N,k}(g)) + (U_{N,k}(f) - L_{N,k}(f))$.

[2] Hint: try $f(x) + \varepsilon$.

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