5 Iterated integral

Iterated integral is an indispensable tool for calculating multidimensional integrals (in particular, volumes).

5a Introduction

It is easy to see that

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \to \int_{\mathbb{R}^2} f \quad \text{as } \varepsilon \to 0$$

for every continuous $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support. The double summation is evidently equivalent to iterated summation,

$$\varepsilon^2 \sum_{k,l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) = \varepsilon \sum_{k\in\mathbb{Z}} \biggl(\varepsilon \sum_{l\in\mathbb{Z}} f(\varepsilon k, \varepsilon l) \biggr) ,$$

which suggests that

$$\int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(\int_{\mathbb{R}} f(x,y) \, \mathrm{d}y \biggr) \mathrm{d}x \, ,$$

(alternative notation: $\iint f(x,y)\,\mathrm{d}x\mathrm{d}y = \int \mathrm{d}x \int \mathrm{d}y\, f(x,y)$, and the like), that is,

$$\text{(5a1)} \qquad \int_{\mathbb{R}^2} f = \int_{\mathbb{R}} \biggl(x \mapsto \int_{\mathbb{R}} f(x,\cdot) \biggr) \, ,$$

where $f(x,\cdot) : \mathbb{R} \to \mathbb{R}$ is defined by $f(x,\cdot) : y \mapsto f(x,y)$.

It should be very useful, to integrate with respect to one variable at a time.

Related problems:

* $*$ does integrability of f imply integrability of $f(x,\cdot)$ for every x?
* $*$ is the function $x \mapsto \int_{\mathbb{R}} f(x,\cdot)$ integrable?
* $*$ is the two-dimensional integral equal to the iterated integral?
* $*$ if the iterated integral is well-defined, does it follow that f is integrable?

And, of course, we need a multidimensional theory; $\mathbb{R}^2$ is only the simplest case.

Some authors[1] impose on f additional requirements. Others[2] consider all integrable functions f; we do so, too, in Sect. 5d, but first we consider simpler cases (Sect. 5b) and counterexamples (Sect. 5c).

5b Simple cases

STEP FUNCTIONS

First we consider a step function $f : \mathbb{R}^2 \to \mathbb{R}$, treated as in Sect. 4g: a linear combination of indicator functions of boxes (and boxes of volume zero are allowed).

Given $B = [a_1, b_1] \times [a_2, b_2]$ and $f = 1\!\!1_B$, we have

$$f(x,\cdot) = 1\!\!1_{[a_1,b_1]}(x) 1\!\!1_{[a_2,b_2]} \, ; \qquad \int_{\mathbb{R}} f(x,\cdot) = (b_2 - a_2) 1\!\!1_{[a_1,b_1]}(x) \, ;$$
$$\int_{\mathbb{R}} \Big(x \mapsto \int_{\mathbb{R}} f(x,\cdot) \Big) = (b_1 - a_1)(b_2 - a_2) = v(B) = \int_{\mathbb{R}^2} f \, .$$

(Alternative notation: $\int \mathrm{d}y \, f(x,y) = (b_2 - a_2) 1\!\!1_{[a_1,b_1]}(x)$; $\int \mathrm{d}x \int \mathrm{d}y \, f(x,y) = v(B) = \iint f(x,y) \, \mathrm{d}x\mathrm{d}y$.)

Similarly, given a box $B \subset \mathbb{R}^{m+n}$, we have $B = B_1 \times B_2$ for some boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$; thus, $f(x,\cdot) = 1\!\!1_{B_1}(x) 1\!\!1_{B_2}$; $\int_{\mathbb{R}^n} f(x,\cdot) = v(B_2) 1\!\!1_{B_1}(x)$;

$$\int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x,\cdot) \Big) = v(B_1) v(B_2) = v(B) = \int_{\mathbb{R}^{m+n}} f \, .$$

By linearity,

$$\int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x,\cdot) \Big) = \int_{\mathbb{R}^{m+n}} f$$

for every step function $f : \mathbb{R}^{m+n} \to \mathbb{R}$; in this case, all sections $f(x,\cdot)$ are step functions, and the function $x \mapsto \int_{\mathbb{R}^n} f(x,\cdot)$ is also a step function. Similarly,

$$\text{(5b1)} \qquad \int_{\mathbb{R}^n} \Big(y \mapsto \int_{\mathbb{R}^m} f(\cdot,y) \Big) = \int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x,\cdot) \Big) \, .$$

[1] Lang, Shifrin, Shurman.

[2] Burkill, Hubbard, Zorich.

CONTINUOUS FUNCTIONS

Now we consider a continuous function $f : \mathbb{R}^{m+n} \to \mathbb{R}$ with bounded support. Integrability of f is ensured by 4f1(a), as well as integrability of $f(x, \cdot)$.

If $x_n \to x$, then $f(x_n, \cdot) \to f(x, \cdot)$ uniformly (due to uniform continuity of f), whence by 4e6 $\int_{\mathbb{R}^n} f(x_n, \cdot) \to \int_{\mathbb{R}^n} f(x, \cdot)$; we see that the function $x \mapsto \int_{\mathbb{R}^n} f(x, \cdot)$ is continuous. Clearly it has a bounded support, and therefore is integrable.

Now we use the sandwich. Given $\varepsilon > 0$, by 4g8 there exist step functions g, h such that $g \le f \le h$ and $\int h - \int g \le \varepsilon$.[1] We have $g(x, \cdot) \le f(x, \cdot) \le h(x, \cdot)$ everywhere; $\int_{\mathbb{R}^n} g(x, \cdot) \le \int_{\mathbb{R}^n} f(x, \cdot) \le \int_{\mathbb{R}^n} h(x, \cdot)$ for all x. On one hand,

$$\int_{\mathbb{R}^{m+n}} g = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} g(x, \cdot) \Big) \le \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x, \cdot) \Big) \le \\ \le \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} h(x, \cdot) \Big) = \int_{\mathbb{R}^{m+n}} h \, ;$$

on the other hand, $\int_{\mathbb{R}^{m+n}} g \le \int_{\mathbb{R}^{m+n}} f \le \int_{\mathbb{R}^{m+n}} h$. We see that

$$\Big| \int_{\mathbb{R}^{m+n}} f - \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x, \cdot) \Big) \Big| \le \varepsilon \, ,$$

since both numbers belong to the interval $[\int g, \int h]$ of length $\le \varepsilon$. We conclude that (5b1) holds for every continuous f with bounded support.

5b2 Exercise. Prove that

$$\int_{\mathbb{R}^{m+n}} f(x_1, \dots, x_m) g(y_1, \dots, y_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \, \mathrm{d}y_1 \dots \mathrm{d}y_n = \\ = \Big(\int_{\mathbb{R}^m} f(x_1, \dots, x_m) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \Big) \Big(\int_{\mathbb{R}^n} g(y_1, \dots, y_n) \, \mathrm{d}y_1 \dots \mathrm{d}y_n \Big)$$

for continuous functions $f : \mathbb{R}^m \to \mathbb{R}$, $g : \mathbb{R}^n \to \mathbb{R}$ with bounded support.

5b3 Exercise. Calculate each integral in two ways:

(a) $\int_0^1 \mathrm{d}x \int_0^1 \mathrm{d}y \, \mathrm{e}^{x+y}$;

(b) $\int_0^1 \mathrm{d}y \int_0^{\pi/2} \mathrm{d}x \, xy \cos(x + y)$.

[1] This argument applies to all integrable f, of course; but (for now) the continuity ensures existence of the iterated integral.

5b4 Exercise. Calculate integrals

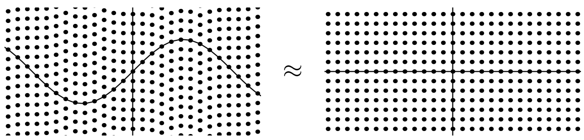
(a) $\int_{[0,1]^n} (x_1^2 + \dots + x_n^2) \, \mathrm{d}x_1 \dots \mathrm{d}x_n$;

(b) $\int_{[0,1]^n} (x_1 + \dots + x_n)^2 \, \mathrm{d}x_1 \dots \mathrm{d}x_n$.

5b5 Exercise. For every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support,

$$\iint_{\mathbb{R}^2} f(x, y + \sin x) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

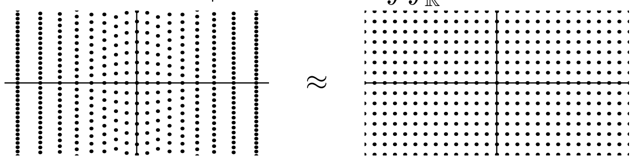
Prove it.



5b6 Exercise. For every continuous function $f : \mathbb{R}^2 \to \mathbb{R}$ with bounded support,

$$\iint_{\mathbb{R}^2} f\Big(x^3 + x, \frac{y}{3x^2+1}\Big) \, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

Prove it.



5c Some counterexamples

5c1 Example. [1] Integrability of f does not imply integrability of $f(x,\cdot)$ for every x.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1 & \text{if } x = 0 \text{ and } y \in [0,1] \text{ is rational,} \\ 0 & \text{otherwise.} \end{cases}$$

Then $f = 0$ outside a set $\{0\} \times [0,1]$ of area 0, therefore f (being negligible) is integrable. However, $f(0,\cdot)$ is not integrable. On the other hand, $f(\cdot,y)$ (being negligible) is integrable for every y, and $\int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}} \mathrm{d}x \, f(x,y) = 0 = \int_{\mathbb{R}^2} f$.

5c2 Example. Existence of the iterated integral[2] does not imply boundedness (the more so, integrability) of f, even if f is positive and symmetric in the sense that $f(x,y) = f(y,x)$ (and therefore the iterated integrals $\int \mathrm{d}x \int \mathrm{d}y \, f(x,y)$, $\int \mathrm{d}y \int \mathrm{d}x \, f(x,y)$ are both well-defined, and equal).

[1] Shifrin, Example 5 on p. 281.

[2] That is, integrability of $f(x,\cdot)$ for all x *and* integrability of the function $x \mapsto \int f(x,\cdot)$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} \frac{1}{\sqrt{x+y}} & \text{if } x,y \in (0,1), \\ 0 & \text{otherwise} \end{cases}$$

and observe that

$$\int_{\mathbb{R}} f(x,\cdot) = \int_0^1 \frac{\mathrm{d}y}{\sqrt{x+y}} = 2\sqrt{x+y}\Big|_{y=0}^{y=1} = 2\sqrt{x+1} - 2\sqrt{x}$$

for $x \in (0,1)$, evidently an integrable function.

5c3 Example. [1] Existence of both iterated integrals does not imply their equality, even if f is antisymmetric in the sense that $f(x,y) = -f(y,x)$.

Define $f : \mathbb{R}^2 \to \mathbb{R}$ by

$$f(x,y) = \begin{cases} 1/y^2 & \text{if } 0 < x < y < 1, \\ -1/x^2 & \text{if } 0 < y < x < 1, \\ 0 & \text{otherwise;} \end{cases}$$

then

$$\int_{\mathbb{R}} f(x,\cdot) = \int_0^x \Big(-\frac{1}{x^2}\Big)\,\mathrm{d}y + \int_x^1 \frac{1}{y^2}\,\mathrm{d}y = -\frac{1}{x^2}\cdot x + \Big(-\frac{1}{y}\Big)\Big|_{y=x}^{y=1} = -\frac{1}{x} - \Big(1 - \frac{1}{x}\Big) = -1$$

for all $x \in (0,1)$. Thus, one iterated integral is negative (-1). By the antisymmetry, the other iterated integral is positive $(+1)$.

Or, alternatively,

$$f(x,y) = \begin{cases} \frac{x-y}{(x+y)^3} & \text{if } x,y \in (0,1), \\ 0 & \text{otherwise;} \end{cases}$$

here

$$\int_{\mathbb{R}} f(x,\cdot) = \int_0^1 \frac{x-y}{(x+y)^3}\,\mathrm{d}y = \int_0^1 \frac{2x-(x+y)}{(x+y)^3}\,\mathrm{d}y =$$
$$= 2x\int_0^1 \frac{\mathrm{d}y}{(x+y)^3} - \int_0^1 \frac{\mathrm{d}y}{(x+y)^2} = 2x\cdot\Big(-\frac{1}{2}\Big)\frac{1}{(x+y)^2}\Big|_{y=0}^{y=1} - (-1)\cdot\frac{1}{x+y}\Big|_{y=0}^{y=1} =$$
$$= -x\Big(\frac{1}{(x+1)^2} - \frac{1}{x^2}\Big) + \Big(\frac{1}{x+1} - \frac{1}{x}\Big) = \frac{-x+(x+1)}{(x+1)^2} = \frac{1}{(x+1)^2}$$

for all $x \in (0,1)$. Thus, one iterated integral is positive (in fact, $1/2$). By the antisymmetry, the other iterated integral is negative $(-1/2)$.

[1] Burkill, Exercise 9 on p. 265.

5c4 Remark. One may wonder, does existence of both iterated integrals imply their equality if f is just bounded (but not necessarily integrable)? Surprisingly, the answer is affirmative.[1,2,3] It may be tempting to use this fact for enlarging the two-dimensional integral. However, what about change of variables then?

5c5 Example. Existence of the iterated integral does not imply integrability of f even if f is *bounded* and symmetric (and therefore both iterated integrals exist and are equal).

Here we use existence of a dense countable set $A \subset (0,1) \times (0,1)$, symmetric (in the sense that $(x,y) \in A \iff (y,x) \in A$) and such that $\{y : (x,y) \in A\}$ is finite for every x.

For instance,[4] the set of all $\big(\frac{i}{q}, \frac{j}{q}\big) \in (0,1) \times (0,1)$ for natural i,j and prime q.

Or the set of all $\big((2i-1)/2^n, (2j-1)/2^n\big) \in (0,1) \times (0,1)$.

Or the set of all $(x,y) \in (0,1) \times (0,1)$ such that $x\sqrt{2} + y$ and $x + y\sqrt{2}$ are (both) rational.

For every such A, its indicator function $f = 1\!\!1_A$ satisfies $0 = {}_*\!\int_{\mathbb{R}^2} f < {}^*\!\!\int_{\mathbb{R}^2} f = 1$ and $\int_{\mathbb{R}} f(x,\cdot) = 0$ for all x.

5c6 Exercise. [5] Consider a function $f : \mathbb{R}^2 \to \mathbb{R}$ of the form $f(x,y) = g(x)h(y)$ where $g,h : \mathbb{R} \to \mathbb{R}$ are bounded functions with bounded support.

(a) If g is negligible, then f is negligible. Prove it.[6]

(b) Integrability of f does not imply that the set $\{x : f(x,\cdot) \text{ is not integrable}\}$ is of volume zero. Find a counterexample.[7,8]

5d Integrable functions

Recall that every integrable function is bounded, with bounded support.

[1] In Riemann integration, of course. In Lebesgue integration the corresponding problem is more complicated.

[2] Lichtenstein 1911, Fichtenholz 1913; see Sect. 16.6 in book "An interactive introduction to mathematical analysis" by J.W. Lewin.

[3] Amazingly, such f need not be Lebesgue measurable. (Basically, Sierpiński 1920; see book "Measure theory" by V.I. Bogachev, vol. 1, Item 3.10.49 on page 232). I thank Yonatan Shelah for this note.

[4] Burkill, Exercise 8 on page 265; Shifrin, Example 7 on page 282.

[5] Burkill, Exercise 6 on page 264.

[6] Hint: $|g| \le \varphi$ (step function), $\int \varphi \le \varepsilon$; $|h| \le C \cdot 1\!\!1_{[-M,M]}$; then $\int |f| \le 2CM\varepsilon$.

[7] Hint: recall 4f12, use both cases ($c_k \to 0$, and $c_k = 1$); use (a).

[8] Contrary to: Hubbard, Corollary A16.3 on page 724. Do you see the error there in the proof?

5d1 Theorem. If a function $f : \mathbb{R}^{m+n} \to \mathbb{R}$ is integrable, then the iterated integrals

$$\int_{\mathbb{R}^m} \mathrm{d}x \, {}_*\!\!\int_{\mathbb{R}^n} \mathrm{d}y \, f(x,y) \, , \qquad \int_{\mathbb{R}^m} \mathrm{d}x \, {}^*\!\!\int_{\mathbb{R}^n} \mathrm{d}y \, f(x,y) \, ,$$
$$\int_{\mathbb{R}^n} \mathrm{d}y \, {}_*\!\!\int_{\mathbb{R}^m} \mathrm{d}x \, f(x,y) \, , \qquad \int_{\mathbb{R}^n} \mathrm{d}y \, {}^*\!\!\int_{\mathbb{R}^m} \mathrm{d}x \, f(x,y)$$

are well-defined and equal to

$$\iint_{\mathbb{R}^{m+n}} f(x,y) \, \mathrm{d}x\mathrm{d}y \, .$$

Clarification. The claim that $\int \mathrm{d}x \, {}_*\!\int \mathrm{d}y \, f(x,y)$ is well-defined means that the function $x \mapsto {}_*\!\int \mathrm{d}y \, f(x,y)$ is integrable.

The equality

$$\int \Big(x \mapsto {}_*\!\!\int f(x,\cdot) \Big) = \int \Big(x \mapsto {}^*\!\!\int f(x,\cdot) \Big)$$

implies integrability (with the same integral) of every function sandwiched between the lower and upper integrals.[1] It is convenient to interpret $x \mapsto \int f(x,\cdot)$ as *any* such function and write, as before,

$$\int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{\mathbb{R}^n} f(x,\cdot) \Big)$$

and

$$\int \mathrm{d}x \int \mathrm{d}y \, f(x,y) = \iint f(x,y) \, \mathrm{d}x\mathrm{d}y = \int \mathrm{d}y \int \mathrm{d}x \, f(x,y)$$

even though f_x may be non-integrable for some x.

Theorem 5d1 is proved via sandwiching (recall Sect. 4g), — either by step functions or by continuous functions. Let us use the former.

Proof. By (4g6), ${}^*\!\int_{\mathbb{R}^{m+n}} f = \inf_{h \ge f} \int_{\mathbb{R}^{m+n}} h$ where h runs over all step functions. For every such h, $\int_{\mathbb{R}^{m+n}} h = \int_{\mathbb{R}^m} \big(x \mapsto \int_{\mathbb{R}^n} h(x,\cdot) \big)$ by (5b1). We have $\int_{\mathbb{R}^n} h(x,\cdot) = {}^*\!\int_{\mathbb{R}^n} h(x,\cdot) \ge {}^*\!\int_{\mathbb{R}^n} f(x,\cdot)$ (since $h(x,\cdot) \ge f(x,\cdot)$), thus, $\int_{\mathbb{R}^{m+n}} h \ge {}^*\!\int_{\mathbb{R}^m} \big(x \mapsto {}^*\!\int_{\mathbb{R}^n} f(x,\cdot) \big)$ for all these h. Therefore

$${}^*\!\!\int_{\mathbb{R}^{m+n}} f \ge {}^*\!\!\int_{\mathbb{R}^m} \Big(x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \, .$$

[1] But not every bounded function that is equal to the integral whenever it exists! In contrast to Lebesgue integration, here we cannot take 0 whenever the integral does not exist; recall 5c6(b). See also Zorich, Sect. 11.4.3, Exercise 1(c).

Similarly (or via $(-f)$),

$$ {}_*\!\int_{\mathbb{R}^{m+n}} f \le {}_*\!\int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \, . $$

Using integrability of f,

$$ \int_{\mathbb{R}^{m+n}} f \le {}_*\!\int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \le {}^*\!\!\int_{\mathbb{R}^m} \Big(x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \le \int_{\mathbb{R}^{m+n}} f \, , $$

therefore

$$ \int_{\mathbb{R}^{m+n}} f = {}_*\!\int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) = {}^*\!\!\int_{\mathbb{R}^m} \Big(x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \, . $$

Integrability of the function $x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot)$ follows, since

$$ \int_{\mathbb{R}^{m+n}} f = {}_*\!\int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \le {}^*\!\!\int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \le $$
$$ \le {}^*\!\!\int_{\mathbb{R}^m} \Big(x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) = \int_{\mathbb{R}^{m+n}} f \, . $$

Similarly, the function $x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot)$ is also integrable. Thus,

$$ \int_{\mathbb{R}^{m+n}} f = \int_{\mathbb{R}^m} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) = \int_{\mathbb{R}^m} \Big(x \mapsto {}^*\!\!\int_{\mathbb{R}^n} f(x,\cdot) \Big) \, . $$

The other two iterated integrals are treated similarly (or via $\tilde f(y,x) = f(x,y)$). □

5d2 Exercise. Give another proof of 5d1, via sandwiching by continuous functions.

5d3 Exercise. Generalize 5b2 to integrable functions
(a) assuming integrability of the function $(x,y) \mapsto f(x)g(y)$,
(b) deducing integrability of the function $(x,y) \mapsto f(x)g(y)$ from integrability of f and g (via sandwich).

5d4 Exercise. For every integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ the function $x, y \mapsto f(x, y + \sin x)$ is also integrable, and

$$ \iint_{\mathbb{R}^2} f(x, y+\sin x) \, \mathrm{d}x \mathrm{d}y = \iint_{\mathbb{R}^2} f(x,y) \, \mathrm{d}x \mathrm{d}y \, . $$

Prove it.[1]

[1] Hint: use 5b5.

5d5 Exercise. For every integrable function $f : \mathbb{R}^2 \to \mathbb{R}$ the function $x, y \mapsto f\big(x^3 + x, \frac{y}{3x^2+1}\big)$ is also integrable, and

$$\iint_{\mathbb{R}^2} f\Big(x^3 + x, \frac{y}{3x^2 + 1}\Big)\, \mathrm{d}x\mathrm{d}y = \iint_{\mathbb{R}^2} f(x, y)\, \mathrm{d}x\mathrm{d}y\,.$$

Prove it.[1]

5e Cavalieri's principle

5e1 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ are admissible sets then the set $E = E_1 \times E_2 \subset \mathbb{R}^{m+n}$ is admissible.

Prove it.

Applying Theorem 5d1 to a function $f 1\!\!1_E$ and taking 4d5 into account we get the following.

5e2 Corollary. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be integrable on every box, and $E \subset \mathbb{R}^{m+n}$ an admissible set; then

$$\int_E f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{E_x} f_x \Big)$$

where $E_x = \{y : (x, y) \in E\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

Clarification. First, note that $\{x : E_x \ne \emptyset\}$ is bounded, and $\int_\emptyset f_x = 0$. Second: it may happen that $\int_{E_x} f_x$ is ill-defined for some x; then it is interpreted as anything between ${}_*\!\int f_x 1\!\!1_{E_x}$ and ${}^*\!\!\int f_x 1\!\!1_{E_x}$.

In particular, taking $f(\cdot) = 1$ we get

$$v_{m+n}(E) = \int_{\mathbb{R}^m} v_n(E_x)\, \mathrm{d}x \qquad (5e3)$$

where v_k is the volume in $\mathbb{R}^k$. For instance, the volume of a 3-dimensional geometric body is the 1-dimensional integral of the area of the 2-dimensional section of the body.

5e4 Corollary. If admissible sets $E, F \subset \mathbb{R}^3$ satisfy $v_2(E_x) = v_2(F_x)$ for all x then $v_3(E) = v_3(F)$.[2]

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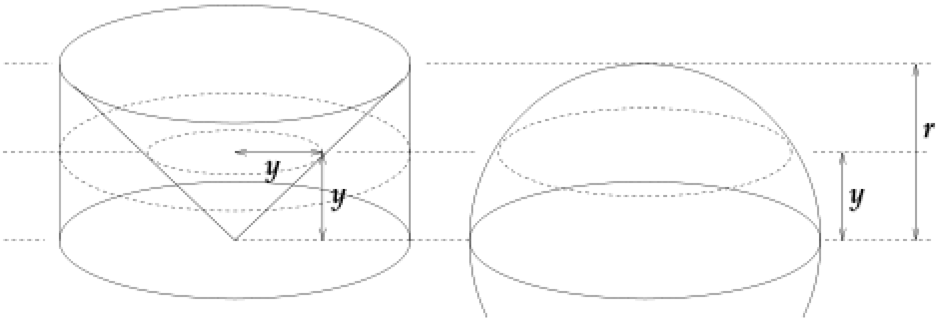
[1] Hint: use 5b6.

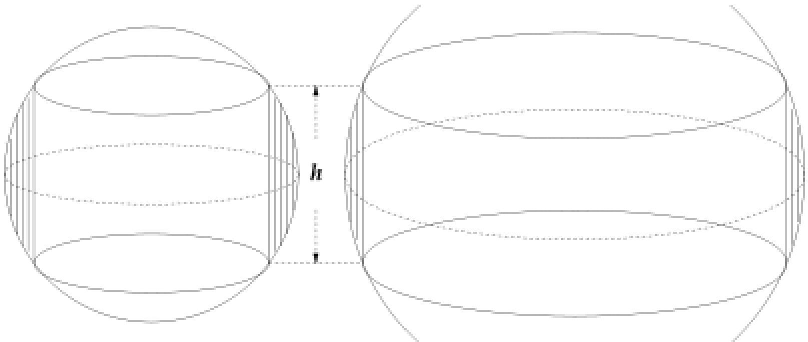
[2] It is sufficient to check the equality for all x of a dense subset of $\mathbb{R}$ (since two *Riemann integrable* functions equal on a dense set must have equal integrals by 4f13).

This is a modern formulation of Cavalieri's principle:[1,2]
Suppose two regions in three-space (solids) are included between two parallel planes. If every plane parallel to these two planes intersects both regions in cross-sections of equal area, then the two regions have equal volumes.

Before emergence of the integral calculus, Cavalieri was able to calculate some volumes by ingenious use of this principle. Here are two examples. First, the volume of the upper half of a sphere is equal to the volume of a cylinder minus volume of a cone:



Second, when a hole of length h is drilled straight through the center of a sphere, the volume of the remaining material surprisingly does not depend on the size of the sphere:

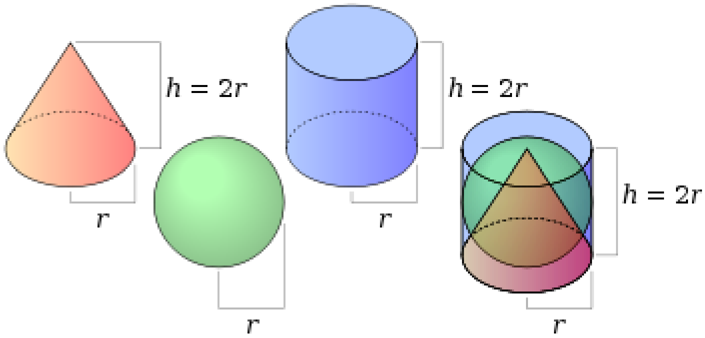


5e5 Exercise. Check the two results of Cavalieri noted above.

[1]Bonaventura Francesco Cavalieri (in Latin, Cavalerius) (1598–1647), Italian mathematician.

[2]Images (and some text) from Wikipedia, "Cavalieri's principle".

5e6 Exercise. Check a famous result of Archimedes:[1,2] a sphere inscribed within a cylinder has two thirds of the volume of the cylinder.



Moreover, show that the volumes of a cone, sphere and cylinder of the same radius and height are in the ratio 1 : 2 : 3.

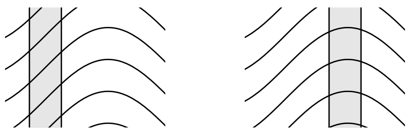
5e7 Exercise. For f, g and E as in 4i2 prove that

(a) $v_{n+1}(E) = \int_{\mathbb{R}^n} (g-f)^+$;

(b) $\int_E h = \int_{\mathbb{R}^n} \mathrm{d}x \, 1\!\!1_{f<g}(x) \int_{f(x)}^{g(x)} \mathrm{d}t \, h(x,t)$ for every $h : E \to \mathbb{R}$ integrable on E.

5e8 Remark. Here $1\!\!1_{f<g}$ is the indicator of the set $\{x : f(x) < g(x)\}$. This set need not be admissible (it can be a dense countable set, recall 4f12).[3] And nevertheless, the iterated integral is well-defined (according to the clarifications...).

5e9 Remark. Cavalieri's principle is about parallel planes. What about parallel surfaces or curves? Applying 5d4 to $f = 1\!\!1_E$ we get the following: if admissible sets $E, F \subset \mathbb{R}^2$ satisfy $v_1(E_y) = v_1(F_y)$ for all y then $v_2(E) = v_2(F)$; here $E_y = \{x : (x, y + \sin x) \in E\}$ (and the same for F_y). But do not think that $v_1(E_y)$ is the length of the sinusoid inside E; it is not.

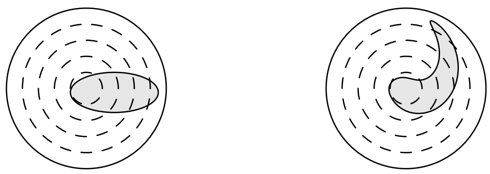


[1] Archimedes (≈ 287–212 BC), a Greek mathematician, generally considered to be the greatest mathematician of antiquity and one of the greatest of all time.
Cicero describes visiting the tomb of Archimedes, which was surmounted by a sphere inscribed within a cylinder. Archimedes ... regarded this as the greatest of his mathematical achievements.

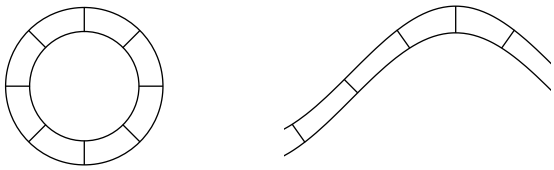
[2] Images (and some text) from Wikipedia, "Volume" (section "Volume ratios for a cone, sphere and cylinder of the same radius and height").

[3] And even if f and g are continuously differentiable, still, this set is just open (not necessarily admissible), see Sect. 2a, Footnote 2 on page 22.

Here is another case: $E_r = \{\theta \in [0, 2\pi) : (r\cos\theta, r\sin\theta) \in E\}$; now $v_1(E_r)$ is the length of the circle inside E, multiplied by r; and in fact, the equality $v_1(E_r) = v_1(F_r)$ for all r implies $v_2(E) = v_2(F)$.



Note that the parallel circles are equidistant; the parallel sinusoids are not.



However, curvilinear integration is postponed to Analysis 4.

5e10 Exercise. [1] Consider the set $E = \{(x, y, z) : 0 \le z \le 1-x^2-y^2\} \subset \mathbb{R}^3$.

(a) Find the volume of E via $\int v_2(E^z)\,\mathrm{d}z$.

(b) Using (a) and the equality $\int v_2(E^z)\,\mathrm{d}z = \int v_1(E_{x,y})\,\mathrm{d}x\mathrm{d}y$, find the mean[2] of the function $(x, y) \mapsto 1 - x^2 - y^2$ on the disk $\{(x, y) : x^2 + y^2 \le 1\} \subset \mathbb{R}^2$.

(c) Similarly to (a), (b), find the mean of the function $x \mapsto |x|^p$ on the ball $\{x : |x| \le 1\} \subset \mathbb{R}^n$ for $p \in (0, \infty)$.[3]

5e11 Exercise. Calculate the integral

$$\iiint_E (x_1^2 + x_2^2 + x_3^2)\, dx_1 dx_2 dx_3 \,,$$

where $E = \{(x_1, x_2, x_3) \in [0, \infty)^3 : x_1 + x_2 + x_3 \le a\} \subset \mathbb{R}^3$.

Answer: $a^5/20$.

5e12 Exercise. Find the volume of the intersection of two solid cylinders in $\mathbb{R}^3$: $\{x_1^2 + x_2^2 \le 1\}$ and $\{x_1^2 + x_3^2 \le 1\}$.

Answer: $16/3$.

[1] Exam of 26.01.14, Question 4.

[2] Recall the end of Sect. 4d.

[3] Hint: you do not need the volume of the ball (nor the area of the disk)! And of course, $|x|^p$ stands for $(x_1^2 + \dots + x_n^2)^{p/2}$.

5e13 Exercise. Find the volume of the solid in $\mathbb{R}^3$ under the paraboloid $\{x_1^2 + x_2^2 = x_3\}$ and above the square $[0,1]^2 \times \{0\}$.
Answer: 2/3.

5e14 Exercise. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Prove that

$$\int_0^x dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n\, f(x_n) = \int_0^x f(t) \frac{(x-t)^{n-1}}{(n-1)!}\, dt\,.$$

5e15 Example. Let us calculate the integral

$$\int_{[0,1]^n} \max(x_1, \dots, x_n)\, dx_1 \dots dx_n\,.$$

First of all, by symmetry, we assume that $1 \ge x_1 \ge x_2 \ge \dots \ge x_n \ge 0$, and multiply the answer by $n!$. Then $\max(x_1, ..., x_n) = x_1$, and we get

$$n! \int_0^1 x_1\, dx_1 \int_0^{x_1} dx_2 \dots \int_0^{x_{n-1}} dx_n = n! \int_0^1 \frac{x_1^n\, dx_1}{(n-1)!} = \frac{n}{n+1}\,.$$

5e16 Exercise. Compute the integral $\int_{[0,1]^n} \min(x_1, \dots, x_n)\, dx_1 \dots dx_n$.
Answer: $\frac{1}{n+1}$.

5e17 Exercise. Find the volume of the n-dimensional simplex

$$\{x : x_1, ..., x_n \ge 0,\ x_1 + ... + x_n \le 1\}\,.$$

Answer: $\frac{1}{n!}$.

5e18 Exercise. Suppose the function f depends only on the first coordinate. Then

$$\int_V f(x_1)\, dx = v_{n-1} \int_{-1}^1 f(x_1)(1 - x_1^2)^{(n-1)/2}\, dx_1\,,$$

where V is the unit ball in $\mathbb{R}^n$, and v_{n-1} is the volume of the unit ball in $\mathbb{R}^{n-1}$.

The next exercises examine further a very interesting phenomenon of "concentration of high-dimensional volume" touched before, in 4i5(b); it was seen there that in high dimension the volume of a ball concentrates near the sphere,[1] and now we'll see that it also concentrates near a hyperplane![2]

5e19 Exercise. Let V be the unit ball in $\mathbb{R}^n$, and $P = \{x \in V : |x_1| < 0.01\}$. What is larger, $v_n(P)$ or $v_n(V \setminus P)$, if n is sufficiently large?

[1] See also 5e10(c).
[2] Do you see a contradiction in these claims?

5e20 Exercise. Given $\varepsilon > 0$, show that the quotient

$$\frac{v_n(\{x \in V : |x_1| > \varepsilon\})}{v_n(V)}$$

tends to zero as $n \to \infty$.[1]

Could you find the asymptotic behavior of the quotient above as $n \to \infty$?

Given an integrable $f : \mathbb{R}^n \to \mathbb{R}$ and a box $B \subset \mathbb{R}^n$ (of non-zero volume), we introduce $f_B : \mathbb{R}^n \to \mathbb{R}$ by

$$f_B(x) = \frac{1}{v(B)} \int_{B+x} f \, ;$$

that is, $f_B(x)$ is the mean value of f on the shifted box $B+x = \{b+x : b \in B\}$.

5e21 Exercise. Prove that f_B is a continuous function.

5e22 Exercise. (a) Let $n = 2$ and $B = [s_1, t_1] \times [s_2, t_2]$. For a continuous f with bounded support, prove that $f_B \in C^1(\mathbb{R}^n)$ and

$$\frac{\partial}{\partial x_1} f_B(x_1, x_2) = \frac{1}{t_2 - s_2} \int_{[s_2,t_2]} \frac{1}{t_1 - s_1} \big(f_{x_1+t_1} - f_{x_1+s_1} \big) \, ;$$

(b) generalize (a) to arbitrary n.

5e23 Exercise. Prove that every continuous f with bounded support is the limit of some uniformly convergent sequence of functions of $C^1(\mathbb{R}^n)$.[2]

[1] Hint: the quotient equals $\frac{\int_\varepsilon^1 (1-t^2)^{(n-1)/2} \, dt}{\int_0^1 (1-t^2)^{(n-1)/2} \, dt}$.

[2] Hint: consider f_B for a small B close to 0.