4 Extrema

4a Extrema of a random function

Let μ be a probability measure on the space $C^2[a,b]$ of twice continuously differentiable functions. Two assumptions on μ are introduced below (similarly to 3c).

The first assumption: for each $x \in [a,b]$ the joint distribution of $f(x)$, $f'(x)$, $f''(x)$ has a density p_x;

(4a1)
$$\int \varphi(y,y',y'')p_x(y,y',y'')\,\mathrm{d}y\mathrm{d}y'\mathrm{d}y'' = \int_{C^2[a,b]} \varphi(f(x),f'(x),f''(x))\,\mu(\mathrm{d}f)$$

for every bounded Borel function $\varphi : \mathbb{R}^3 \to \mathbb{R}$. (Once again, the function $(x,y,y',y'') \mapsto p_x(y,y',y'')$ on $[a,b]\times\mathbb{R}^3$ may be chosen to be measurable.)

The second assumption:

(4a2)
$$\iint_{[a,b]\times C^2[a,b]} |f''(x)|\,\mathrm{d}x\,\mu(\mathrm{d}f) < \infty\,.$$

Once again, μ can be an arbitrary nondegenerate Gaussian measure on the (finite-dimensional linear) space of trigonometric (or algebraic) polynomials of degree n (provided that its dimension is at least 3; the toy model (3a1) does not fit, but see 4a7 and notes after it).

4a3 Exercise. For every bounded Borel functions $\varphi,\psi : \mathbb{R}\to\mathbb{R}$ and every $f \in C^2[a,b]$,

$$\int \mathrm{d}y'\psi(y') \sum_{x:f'(x)=y'} \varphi(f(x))\operatorname{sgn} f''(x) = \int_a^b \mathrm{d}x\,\psi(f'(x))\varphi(f(x))f''(x)\,.$$

Prove it.

Hint: (3b7) for f' and $\psi(f'(\cdot))\varphi(f(\cdot))$ instead of f and g.

4a4 Exercise. For all bounded Borel functions $\varphi, \psi : \mathbb{R} \to \mathbb{R}$,

$$\int \mathrm{d}y' \psi(y') \, \mathbb{E} \sum_{x:f'(x)=y'} \varphi(f(x)) \operatorname{sgn} f''(x) =$$

$$= \int_a^b \mathrm{d}x \iiint \mathrm{d}y \mathrm{d}y' \mathrm{d}y'' p_x(y,y',y'') \psi(y') \varphi(y) y'' \, .$$

Prove it.

Hint: 4a3 and Fubini (and do not forget integrability).

4a5 Exercise. For all bounded Borel functions $\varphi : \mathbb{R} \to \mathbb{R}$,

$$\mathbb{E} \sum_{x:f'(x)=y'} \varphi(f(x)) \operatorname{sgn} f''(x) = \int_a^b \mathrm{d}x \int \mathrm{d}y \, \varphi(y) \int \mathrm{d}y'' \, p_x(y,y',y'') y''$$

for almost all $y' \in \mathbb{R}$.

Prove it.

Similarly, one may get (if needed)

$$\mathbb{E} \sum_{x:f'(x)=y'} \varphi(f(x)) = \int_a^b \mathrm{d}x \int \mathrm{d}y \, \varphi(y) \int \mathrm{d}y'' \, p_x(y,y',y'') |y''| \, .$$

In terms of marginal and conditional densities

$$p_x(y,y') = \int \mathrm{d}y'' \, p_x(y,y',y'') \, , \quad p_x(y''|y,y') = \frac{p_x(y,y',y'')}{p_x(y,y')} \, ,$$

$$p_x(y') = \int \mathrm{d}y \, p_x(y,y') \, , \quad p_x(y|y') = \frac{p_x(y,y')}{p_x(y')}$$

and the conditional expectation

$$\mathbb{E} \big(f''(x) \, \big| \, f(x) = y, f'(x) = y' \big) = \int \mathrm{d}y'' \, p_x(y''|y,y') y''$$

we have

$$\int \mathrm{d}y'' \, p_x(y,y',y'') y'' = p_x(y,y') \mathbb{E} \big(f''(x) \, \big| \, f(x) = y, f'(x) = y' \big) \, ;$$

4a5 becomes

$$\text{(4a6)} \quad \mathbb{E} \sum_{x:f'(x)=y'} \varphi(f(x)) \operatorname{sgn} f''(x) =$$

$$= \int_a^b \mathrm{d}x \, p_x(y') \int_{\mathbb{R}} \mathrm{d}y \, p_x(y|y') \varphi(y) \mathbb{E} \big(f''(x) \, \big| \, f(x) = y, f'(x) = y' \big)$$

for almost all y'.

4a7 Exercise. Prove (4a6) assuming less than (4a1), namely, existence of the joint density $p_x(y,y')$ of $f(x)$, $f'(x)$ and the *regression function* $(y,y') \mapsto \mathbb{E}\big(f''(x)\,\big|\,f(x)=y, f'(x)=y'\big)$ (for each x) such that

$$\mathbb{E}\,\varphi(f(x))\psi(f'(x))f''(x) = \\ = \iint \mathrm{d}y\mathrm{d}y'\, p_x(y,y')\varphi(y)\psi(y')\mathbb{E}\big(f''(x)\,\big|\,f(x)=y, f'(x)=y'\big)$$

for all bounded Borel functions $\varphi,\psi:\mathbb{R}\to\mathbb{R}$ and all $x\in[a,b]$.

Now we may apply 4a6 to the toy model (3a1). Here $p_x(y,y',y'')$ does not exist, since $y''=-y$ always. However, $\mathbb{E}\big(f''(x)\,\big|\,f(x)=y, f'(x)=y'\big) = -y$; also, both $p_x(y')$ and $p_x(y|y')$ is just the standard normal density; we get

$$\mathbb{E}\sum_{x:f'(x)=y'} \varphi(f(x))\,\mathrm{sgn}\, f''(x) = \\ = \frac{1}{2\pi}\int_0^{2\pi} \mathrm{d}x\, \mathrm{e}^{-y'^2/2} \int \mathrm{d}y\, \mathrm{e}^{-y^2/2}\varphi(y)\cdot(-y) = -\mathrm{e}^{-y'^2/2}\int y\mathrm{e}^{-y^2/2}\varphi(y)\,\mathrm{d}y\,;$$

for $y'=0$ it means

$$\mathbb{E}\sum_{x:f'(x)=0} \varphi(f(x))\,\mathrm{sgn}\, f''(x) = -\int y\mathrm{e}^{-y^2/2}\varphi(y)\,\mathrm{d}y\,.$$

In fact, $f'(\cdot)$ vanishes at two points, the minimum and the maximum. Here $f(x)=\pm M$ and $f''(x)=-f(x)$, thus $\sum_{x:f'(x)=0}\varphi(f(x))\,\mathrm{sgn}\,f''(x) = \varphi(-M)-\varphi(M)$, and the expectation is $\int_0^\infty \big(\varphi(-u)-\varphi(u)\big)f_M(u)\,\mathrm{d}u$; recall (3a5).

4b Gaussian case

Let γ be a (centered) Gaussian measure on $C^2[a,b]$ such that for every $x\in[a,b]$

(4b1) $$\int_{C^2[a,b]} |f(x)|^2\,\gamma(\mathrm{d}f) = 1\,,$$

(4b2) $$\int_{C^2[a,b]} |f'(x)|^2\,\gamma(\mathrm{d}f) = \sigma^2(x) > 0$$

for some $\sigma:[a,b]\to(0,\infty)$. We know (recall 3d3) that the function $\sigma(\cdot)$ is continuous. Similarly, the function $x\mapsto\int|f''(x)|^2\,\gamma(\mathrm{d}f)$ is continuous, therefore bounded, which ensures (4a2). Also (recall (3d5),

(4b3) $$p_x(y,y') = \frac{1}{2\pi\sigma(x)}\exp\bigg(-\frac{y^2}{2}-\frac{y'^2}{2\sigma^2(x)}\bigg)$$

is the joint density of $f(x)$ and $f'(x)$.

The joint distribution of $f(x), f'(x), f''(x)$ is a Gaussian measure on $\mathbb{R}^3$ (maybe, degenerate). The normal correlation theorem (recall 1c) gives us a *linear* regression function (for each x)

(4b4) $$(y, y') \mapsto \mathbb{E}\big(f''(x) \,\big|\, f(x) = y, f'(x) = y'\big) = A(x)y + B(x)y' \,.$$

By 4a7 we may use (4a6):

(4b5) $$\mathbb{E} \sum_{x:f'(x)=y'} \varphi(f(x)) \operatorname{sgn} f''(x) = \\ = \frac{1}{2\pi} \int_a^b \mathrm{d}x \, \frac{1}{\sigma(x)} \exp\Big(-\frac{y'^2}{2\sigma^2(x)} \Big) \int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{-y^2/2} \varphi(y)(A(x)y + B(x)y')$$

for almost all y'; here $\varphi : \mathbb{R} \to \mathbb{R}$ is an arbitrary bounded Borel function.

The right-hand side of (4b5) is continuous in y'. Similarly to 3d12, in order to prove (4b5) for all y' we will prove (assuming continuity of φ) that the left-hand side is also continuous in y'. Similarly to 3d11, it is sufficient to check continuity of the function[1]

$$y' \mapsto \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathrm{d}u \, \mathrm{e}^{-u^2/2} \sum_{x:f'_u(x)=y'} \varphi(f_u(x)) \operatorname{sgn} f''_u(x) \,,$$

where $f_u(\cdot) = g(\cdot) + uh(\cdot)$; $g, h \in C^2[a,b]$ and $h'(x) \ne 0$ for all $x \in [a,b]$. To this end we transform the integral in u into an integral in x:

(4b6)
$$\int_{\mathbb{R}} \big(\mathrm{d}\Phi(u)\big) \sum_{x:f'_u(x)=y'} \varphi(f_u(x)) \operatorname{sgn} f''_u(x) = \pm \int_a^b \varphi\big(f_{U(x)}(x)\big) \, \mathrm{d}\Phi(U(x)) \,;$$

here Φ is the cumulative distribution function of $\mathrm{N}(0,1)$; $U(x) = (y' - g'(x))/h'(x)$; and the sign is '$-$' if $h'(\cdot) > 0$ on $[a,b]$, but '$+$' if $h'(\cdot) < 0$ on $[a,b]$. Clearly, the latter integral is continuous in y' (assuming continuity of φ). The equality (4b6) follows from (3b7) applied to $U(x)$ instead of $f(x)$ and $\varphi(f_{U(x)}(x))\Phi'(U(x))$ instead of $g(x)$:

$$\int_{\mathbb{R}} \mathrm{d}u \sum_{x \in U^{-1}(u)} \varphi(f_{U(x)}(x))\Phi'(U(x)) \operatorname{sgn} U'(x) = \int_a^b \mathrm{d}x \, U'(x) \varphi(f_{U(x)}(x))\Phi'(U(x)) \,;$$

taking into account that $x \in U^{-1}(u) \iff f'_u(x) = y'$ we get

$$\int_{\mathbb{R}} \mathrm{d}u \, \Phi'(u) \sum_{x:f'_u(x)=y'} \varphi(f_u(x)) \operatorname{sgn} U'(x) = \int_a^b \varphi(f_{U(x)}(x))\Phi'(U(x))U'(x) \, \mathrm{d}x \,.$$

[1] And in addition, integrability of its supremum in y' (running over a bounded interval).

It remains to note that $f''_{U(x)}(x) = -h'(x)U'(x)$, which follows from the equality $f'_{U(x)}(x) = y'$ by differentiation (in x).

Thus, (4b5) holds for all y', especially, for $y' = 0$:

(4b7)
$$\mathbb{E} \sum_{x:f'(x)=0} \varphi(f(x)) \operatorname{sgn} f''(x) = \frac{1}{2\pi} \biggl(\int_a^b \frac{\mathrm{d}x}{\sigma(x)} A(x) \biggr) \biggl(\int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{-y^2/2} \varphi(y) y \biggr) ;$$

here $A(x)$ is defined by the Gaussian regression, $\mathbb{E}\bigl(f''(x) \bigm| f(x) = y, f'(x) = 0\bigr) = A(x)y$. Being proved for bounded continuous φ, (4b7) holds for all bounded Borel functions φ, since it is in fact an equality between (finite) measures,

$$\text{(4b8)} \quad \mathbb{E} \sum_{x:f'(x)=0} \bigl(\operatorname{sgn} f''(x)\bigr) \delta_{f(x)} = \frac{1}{2\pi} \biggl(\int_a^b \frac{\mathrm{d}x}{\sigma(x)} A(x) \biggr) \biggl(\int_{\mathbb{R}} \mathrm{d}y \, \mathrm{e}^{-y^2/2} y \delta_y \biggr) ;$$

you see, $\mathbb{E} \, \#\{x : f'(x) = 0\} = \frac{1}{\sqrt{2\pi}} \mathbb{E} \int_a^b |f''(x)| \, \mathrm{d}x < \infty$ by (3d13) and (4a2).

Especially, the case $\varphi = \mathbf{1}_{(y,\infty)}$ gives

$$\text{(4b9)} \qquad \mathbb{E} \sum_{x:f'(x)=0, f(x)>y} \operatorname{sgn} f''(x) = \frac{1}{2\pi} \mathrm{e}^{-y^2/2} \int_a^b \frac{\mathrm{d}x}{\sigma(x)} A(x)$$

for all $y \in \mathbb{R}$.

4c Natural parameter

The general case of 4b may be reduced to the special case $\sigma(\cdot) = 1$, that is,

$$\text{(4c1)} \qquad \int_{C^2[a,b]} |f'(x)|^2 \, \gamma(\mathrm{d}f) = 1 \quad \text{for all } x,$$

by a change of variable, $x_{\text{new}} = \int_0^x \sigma(x_1) \, \mathrm{d}x_1$. Clearly, the left-hand side of (4b9) is invariant under such change of variable. Now we assume (4c1).

4c2 Exercise. $\mathbb{E}\bigl(f(x)f''(x)\bigr) = -1$, that is,

$$\int_{C^2[a,b]} f(x)f''(x) \, \gamma(\mathrm{d}f) = -1 \quad \text{for all } x.$$

Prove it.

Hint: $(f(x)f'(x))' = f'(x)f'(x) + f(x)f''(x)$; recall (3d4).

By (3d4) applied to f and also to f',

(4c3) $$\mathbb{E}\big(f(x)f'(x)\big) = 0 \quad \text{and} \quad \mathbb{E}\big(f'(x)f''(x)\big) = 0 \,.$$

We see that the three random variables

(4c4) $$f(x),\ f'(x),\ f(x) + f''(x) \text{ are orthogonal.}$$

Therefore $\mathbb{E}\big(f(x) + f''(x) \,\big|\, f(x) = y, f'(x) = y'\big) = 0$, and

(4c5) $$\mathbb{E}\big(f''(x) \,\big|\, f(x) = y, f'(x) = y'\big) = -y \,;$$

in terms of (4b4) it means that $A(x) = -1$, $B(x) = 0$. Now (4b9) becomes

(4c6) $$\mathbb{E} \sum_{x:f'(x)=0, f(x)>y} \operatorname{sgn} f''(x) = -\frac{b-a}{2\pi} \mathrm{e}^{-y^2/2} \,.$$

On the other hand, Rice's formula 3d6 gives

$$\mathbb{E}\left(\#f^{-1}(y)\right) = \frac{b-a}{\pi} \mathrm{e}^{-y^2/2} \,,$$

and we see that

(4c7) $$\mathbb{E}\left(\#f^{-1}(y)\right) = -2\, \mathbb{E} \sum_{x:f'(x)=0, f(x)>y} \operatorname{sgn} f''(x) \,.$$

Here is a simple explanation of (4c7). First (irrespective of any randomness), for every $f \in C^2[a,b]$,[1]

$$\#f^{-1}(y) + 2 \sum_{x:f'(x)=0, f(x)>y} \operatorname{sgn} f''(x) = \\ = \mathbf{1}_{(y,\infty)}(f(b)) \operatorname{sgn} f'(b) - \mathbf{1}_{(y,\infty)}(f(a)) \operatorname{sgn} f'(a)$$

(think, why), provided that the following degenerate cases are excluded:

$$f'(a) = 0 \,; \\ f'(b) = 0 \,; \\ f'(x) = f''(x) = 0 \quad \text{for some } x \in [a,b] \,.$$

Second, the expectation of the right-hand side vanishes, since $f'(a)$ is independent of $f(a)$ (and the same holds for b).

[1]The right-hand side disappears on the circle, that is, for 2π-periodic functions restricted to $[0, 2\pi]$.

4c8 Exercise. The degenerate cases are excluded for γ-almost all f.

Prove it.

Hint: consider again $f_u(\cdot) = g(\cdot) + uh(\cdot)$ for $g, h \in C^2[a,b]$ and $h'(x) \ne 0$ for all $x \in [a,b]$; if $f'_u(x) = f''_u(x) = 0$ for some x then u is a critical value of $-g'(\cdot)/h'(\cdot)$; use Sard's theorem.

Note that (4b1) is essential for (4c7).

We see that (4c6) follows easily from Rice's formula. However, the approach of Sect. 4 is important in dimension two (and higher).

4d Some integral geometry

Similarly to 3e we consider a curve on $S^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}$ parameterized by some $[a,b]$;

$$Z \in C^2\big([a,b], \mathbb{R}^n\big)\,, \quad Z\big([a,b]\big) \subset S^{n-1}\,, \quad Z'(\cdot) \ne 0\,.$$

It leads to a Gaussian random vector in $C^2[a,b]$,

$$f(x) = \langle Z(x), \xi \rangle\,,$$

where ξ is distributed γ^n.

Extrema of $f(\cdot)$ are extrema of the distance between a point of the curve and the random hyperplane $\{z \in \mathbb{R}^n : \langle z, \xi \rangle = 0\}$. The (unsigned) distance is maximal when $f'(x) = 0$ and $\operatorname{sgn} f(x) \operatorname{sgn} f''(x) < 0$; it is minimal when $f'(x) = 0$ and $\operatorname{sgn} f(x) \operatorname{sgn} f''(x) > 0$. (Degenerate cases, $f'(x) = f(x) f''(x) = 0$, are excluded almost surely, recall 4c8.) Using the natural parameter we have

$$\mathbb{E} \sum_{x:f'(x)=0} \big(-\operatorname{sgn} f(x) \operatorname{sgn} f''(x) \big) = -2\, \mathbb{E} \sum_{x:f'(x)=0, f(x)>0} \operatorname{sgn} f''(x) = \frac{b-a}{\pi}$$

by (4c6); and $b-a$ is the length of the curve. Thus,

$$\text{(4d1)} \qquad \frac{\text{the mean number of maxima} - \text{the mean number of minima}}{\text{the length of the curve}} = \frac{1}{\pi}\,.$$

Think, what happens for such a curve: