5 Random functions of two variables

5a Introductory remarks

The sphere $S^2 = \{x \in \mathbb{R}^3 : |x| = 1\}$ is an example of a 2-dimensional manifold. We will consider the space $C^2(S^2)$ of twice continuously differentiable functions $S^2 \to \mathbb{R}$ and Gaussian measures on this space.

For example, linear functions $x \mapsto \langle x, \xi \rangle$ on $\mathbb{R}^3$, restricted to S^2, are a 3-dimensional subspace H_{lin} of $C^2(S^2)$, parameterized by $\xi \in \mathbb{R}^3$. If ξ is random, distributed γ^3, we get a rotation-invariant 3-dimensional Gaussian measure on $C^2(S^2)$, denote it γ_{lin}.

5a1 Exercise. (a) Calculate $\mathbb{E}\, f(x)f(y)$, that is,

$$\int_{C^2(S^2)} f(x)f(y)\, \gamma_{\text{lin}}(\mathrm{d}f)\,,$$

for $x, y \in S^2$.

(b) Prove that the norm $|\cdot|_{\gamma_{\text{lin}}}$ (recall page 7) is the restriction to H_{lin} of the norm of $L_2(S^2, 3\mu)$, where μ is the uniform distribution (in other words, normalized area measure) on S^2.

Quadratic forms on $\mathbb{R}^3$, restricted to S^2, are a 6-dimensional subspace of $C^2(S^2)$, containing $1 = x^2 + y^2 + z^2$. Forms orthogonal to 1 in $L_2(S^2, \mu)$ are a 5-dimensional subspace H_{quad}. The norm of $C^2(S^2)$ does not turn it into a Euclidean space, but the norm of $L_2(S^2, 5\mu)$ does. This norm is $|\cdot|_\gamma$ for some (unique) rotation-invariant 5-dimensional Gaussian measure on $C^2(S^2)$, denote it γ_{quad}.

5a2 Exercise. (a) Prove that the restriction of f to any great circle of S^2 is distributed like the random trigonometric polynomial

$$\frac{1}{2}\zeta_0 + \frac{\sqrt{3}}{2}(\zeta_2 \cos 2t + \eta_2 \sin 2t) \,,$$

where ζ_0, ζ_2, η_2 are independent $\mathrm{N}(0,1)$ random variables.

(b) Calculate $\mathbb{E}\, f(x) f(y)$ w.r.t. γ_{quad} for $x, y \in S^2$.

Hint. First, $\int_{S^2} f(x)\,\mu(\mathrm{d}x\mathrm{d}y\mathrm{d}z) = \frac{1}{2}\int_{-1}^{1} f(x)\,\mathrm{d}x$ for any f. Second, $\int_{S^2} x^2 y^2\,\mu(\mathrm{d}x\mathrm{d}y\mathrm{d}z) = \frac{1}{15}$, since $(x^2+y^2+z^2)^2 = 1$ on S^2. Third, here is a convenient orthonormal basis of $H_{\mathrm{quad}} \subset L_2(S^2, 5\mu)$:[1]

$$\frac{3z^2-1}{2}; \quad \sqrt{3}zx,\, \sqrt{3}zy; \quad \sqrt{3}xy,\, \sqrt{3}\frac{x^2-y^2}{2}\,.$$

We may consider, say, $f = g + \frac{1}{2}h$ where $g \sim \gamma_{\mathrm{lin}}$ and $h \sim \gamma_{\mathrm{quad}}$ are independent; such f is still a rotation-invariant Gaussian random field on S^2. And, of course, we may use higher (cubic, ...) forms.[2]

The similar one-dimensional construction (over S^1) leads to random trigonometric polynomials, in the spirit of 1c12.

The torus $\mathbb{T}^2 = S^1 \times S^1$ is another example of a 2-dimensional manifold. The space $C^2(\mathbb{T}^2)$ may be identified with the space of double-periodic smooth functions $\mathbb{R}^2 \to \mathbb{R}$.

For example, trigonometric polynomials, spanned by $\cos kx \cos ly$, $\cos kx \sin ly$, $\sin kx \cos ly$, $\sin kx \sin ly$ with $k + l \le n$ (for a given n) are a finite-dimensional subspace of $C^2(\mathbb{T}^2)$. The norm of $L^2(S^2)$ turns it into a Euclidean space and leads to a stationary (that is, shift-invariant) Gaussian measure on $C^2(\mathbb{T}^2)$.

We may also treat $\mathbb{T}^2$ as $\{(x,y,u,v) : x^2+y^2 = 1,\, u^2+v^2 = 1\} \subset \mathbb{R}^4$. Algebraic polynomials on $\mathbb{R}^4$ turn into trigonometric polynomials on $\mathbb{T}^2$.

Other 2-dimensional manifolds could be used, but we restrict ourselves to S^2 and $\mathbb{T}^2$.

See [1, p. 9] for a figure showing the cosmic microwave background radiation treated as a Gaussian random field on S^2 (the sky). See also [1, p. 10] for tomographic brain images.

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[1] Accordingly, H_{quad} decomposes into a line and two planes; rotating S^2 by α around the z axis, we rotate these planes by α and 2α respectively. In terms of spin 1 these correspond to eigenvalues of the z projection of the spin: 0 (the line), ± 1 (the first plane), ± 2 (the second plane).

[2] Harmonic homogeneous forms of degree n correspond to an eigenspace of the spherical Laplacian, spherical functions, and spin n.

5b Excursions: some topology

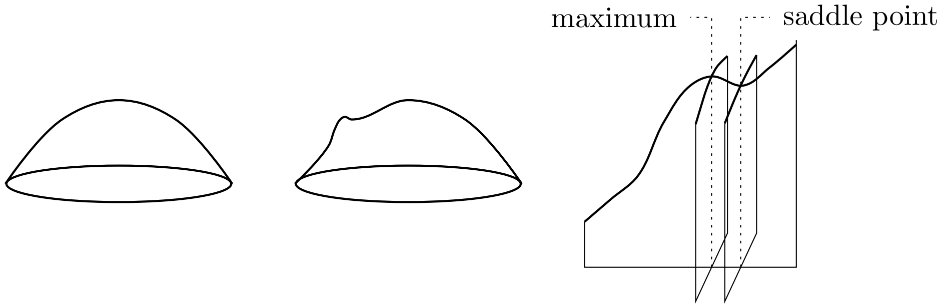
We return for a short while to dimension one. Let $f \in C^2(S^1)$, and $y \in \mathbb{R}$ be such that $f^{-1}(y)$ is a finite nonempty set not containing critical points of f. The set $\{x \in S^1 : f(x) \ge y\}$ is the union of a finite set of disjoint intervals. These intervals may be called *excursions* of f (above y). How to calculate the expected number of excursions for a random f?

Clearly, the number of excursions is equal to $\frac12 \# f^{-1}(y)$, and we may use Rice's formula. Unfortunately, this approach does not work in dimension two, since $f^{-1}(y)$ fails to be a discrete set.

Each excursion contains a critical point, namely, a maximum of f. It may contain k maxima, but only in combination with $k-1$ minima. Thus,

$$\begin{pmatrix} \text{number of} \\ \text{excursions} \\ \text{above } y \end{pmatrix} = \begin{pmatrix} \text{number of} \\ \text{maxima} \\ \text{above } y \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{minima} \\ \text{above } y \end{pmatrix}.$$

This approach could work in dimension two, since critical points are still a discrete set. However, the combination (#maxima) − (#minima) fails to be 1 (or another constant) within a single two-dimensional excursion.



A local perturbation of f (near a non-critical point) can create a new maximum in combination with a new saddle point. Similarly it can create a new minimum in combination with a new saddle point. Here is the only expression that has a chance to be insensitive to perturbations:

$$\begin{pmatrix} \text{number of} \\ \text{maxima} \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{saddle points} \end{pmatrix} + \begin{pmatrix} \text{number of} \\ \text{minima} \end{pmatrix}.$$

Theorem 5b1 below states that it really is.

Let $f \in C^2(S^2)$ (however, $\mathbb{T}^2$ may be used equally well);

* a point $x \in S^2$ is called a *critical point* (of f), if the first derivatives of f at x vanish; equivalently, if $\frac{f(x_1)-f(x)}{\operatorname{dist}(x_1,x)} \to 0$ as $x_1 \to x$;
* a number $y \in \mathbb{R}$ is called a *critical value* (of f), if $y = f(x)$ for some critical point x;

* a critical point x is *nondegenerate* if the matrix of the second derivatives of f at x is nondegenerate, that is, its determinant does not vanish;[1]
* the *index* $i_f(x)$ of a nondegenerate critical point x of f is, by definition, the sign (± 1) of the determinant mentioned above;
* a nondegenerate critical point of index $+1$ is an extremum (maximum or minimum); a nondegenerate critical point of index -1 is a saddle point;
* f is called a *Morse function* if all its critical points are nondegenerate.

See [5, Sect. 1.2–1.4].[2] Below, '$\nabla f(x) = 0$' means that x is a critical point of f.

5b1 Theorem. Let $f, g \in C^2(S^2)$ be Morse functions such that

$$\forall x \in S^2 \quad \operatorname{sgn} f(x) = \operatorname{sgn} g(x)$$

and 0 is not a critical value of f, nor of g. Then

$$\sum_{x: f(x)<0, \nabla f(x)=0} i_f(x) = \sum_{x: g(x)<0, \nabla g(x)=0} i_g(x) \, ,$$

$$\sum_{x: f(x)>0, \nabla f(x)=0} i_f(x) = \sum_{x: g(x)>0, \nabla g(x)=0} i_g(x) \, .$$

The same holds for $\mathbb{T}^2$.

I give no proof. Theorem 5b1 is a simple consequence of Morse theory, see Poincaré-Hopf theorem in [6, Sect. 6]. A rather elementary proof of the two-dimensional case is given in [4, Sect. 3.4 and 11.2] for the planar case, that is, when the domains $\{x : f(x) < 0\}$, $\{x : f(x) > 0\}$ can be embedded into $\mathbb{R}^2$. This is enough for S^2 but not $\mathbb{T}^2$; see also [4, Sect. 4.7]. Beyond the planar case we cannot define the rotation of the vector field ∇f on the curve $f^{-1}(0)$, but still, we can define the difference between two such rotations (of ∇f and ∇g).

The topological invariant disclosed by Theorem 5b1 is called the *Euler* (-Poincare) *characteristic* and denoted by χ;[3]

$$\text{(5b2)} \qquad \sum_{x: f(x)>0, \nabla f(x)=0} i_f(x) = \chi\big(\{x : f(x) \ge 0\}\big) \, .$$

[1] The determinant depends on a local coordinate system, but its nondegeneracy and sign do not.

[2] The same notion of index is used in [6, Sect. 6] and [4, Sect. 11.2]. A different (but related) notion of index is used in [1] and [5], see also [3, Sect. 6.1]. In terms of their index $\operatorname{Ind}_f(x) \in \{0, 1, 2\}$, our index is $i_f(x) = (-1)^{\operatorname{Ind}_f(x)}$.

[3] Another definition is, (number of triangles) − (number of edges) + (number of vertices) in any triangulation, see [5, Th. 4.11], where also a third, homological definition can be found. In addition, $\chi(A \cup B) = \chi(A) + \chi(B) - \chi(A \cap B)$, see [5, Prop. 4.13].

Clearly, if $\{x : f(x) \ge 0\}$ is not connected then its Euler characteristic decomposes into the sum (over connected components). A connected region $D \subset S^2$ is S^2 with k holes (for some $k \in \{0, 1, 2, \dots\}$), and $\chi(D) = 2 - k$ [3, Th. 9.3.7 for $p = 0$]; especially, $\chi(S^2) = 2$, $\chi(\text{disk}) = 1$, $\chi(\text{annulus}) = 0$ (and note that negative values are also possible). The torus $\mathbb{T}^2$ with k holes is a non-planary connected region D; in this case $\chi(D) = -k$ [3, Th. 9.3.7 for $p = 1$]. Especially, $\chi(\mathbb{T}^2) = 0$.

A trivial generalization of (5b2),

$$\sum_{x: f(x) > y, \nabla f(x) = 0} i_f(x) = \chi\big(\{x : f(x) \ge y\}\big) \tag{5b3}$$

(assuming that f is a Morse function, and y is not a critical value of f), will be applied to a random function f. Calculating the expectation of the sum we will get the *expected Euler characteristic,* $\mathbb{E}\,\chi\big(\{x : f(x) \ge y\}\big)$. True, this is not the expected number of excursions above y (that is, connected components of the excursion set $\{x : f(x) \ge y\}$). However, for a high level y we have usually no excursion at all, and sometimes a single, small excursion (roughly, ellipse) with no holes inside; its Euler characteristic equals 1. Other cases are relatively rare. This is why the expected Euler characteristic can give a valuable information about (the tail of) the distribution of the maximum of a smooth random field.

Dealing with a small excursion D with $1 - \chi(D)$ holes inside we may call these holes 'antiexcursions'. In this sense,

$$\chi\big(\{x : f(x) \ge y\}\big) = \begin{pmatrix} \text{number of} \\ \text{excursions} \end{pmatrix} - \begin{pmatrix} \text{number of} \\ \text{antiexcursions} \end{pmatrix}.$$

5c Nonrandom function

Taking into account that a small piece of S^2 or $\mathbb{T}^2$ is a planar domain, for now we consider functions on a bounded open set $D \subset \mathbb{R}^2$ or its closure $\overline{D}$.

Recall that every map $f \in C^1(\overline{D}, \mathbb{R}^2)$ has the Jacobian $J_f \in C(\overline{D}, \mathbb{R})$;

$$J_f(x) = \begin{vmatrix} \frac{\partial y_1}{\partial x_1} & \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_1}{\partial x_2} & \frac{\partial y_2}{\partial x_2} \end{vmatrix}, \tag{5c1}$$

where $(y_1, y_2) = y = f(x) = f(x_1, x_2)$. If f is one-to-one on D then

$$\operatorname{mes}_2 f(D) = \int_D |J_f(x)| \, \mathrm{d}x \tag{5c2}$$

('mes_2' is the two-dimensional Lebesgue measure) and moreover,

$$\int_{f(D)} \varphi(y) \, \mathrm{d}y = \int_D \varphi(f(x)) |J_f(x)| \, \mathrm{d}x \tag{5c3}$$

for every bounded Borel function $\varphi : f(D) \to \mathbb{R}$.

5c4 Exercise. For every bounded Borel function $\varphi : D \times \mathbb{R}^2 \to \mathbb{R}$,

$$\int_{\mathbb{R}^2} \mathrm{d}y \sum_{x \in f^{-1}(y)} \varphi(x, y) = \int_D \varphi(x, f(x)) |J_f(x)| \, \mathrm{d}x \, .$$

Prove it.

Hint: partition the graph of f into a (finite or infinite) sequence of small parts and apply (5c3) on each part; consider positive and negative values of φ separately; and use (two-dimensional) Sard's theorem, similarly to the proof of 3b1.

A function $f \in C^2(\overline{D})$ has the gradient $\nabla f \in C^1(\overline{D}, \mathbb{R}^2)$,

$$\nabla f(x) = \Big(\frac{\partial y}{\partial x_1}, \frac{\partial y}{\partial x_2} \Big)$$

where $y = f(x) = f(x_1, x_2)$. A critical point of f is x such that $\nabla f(x) = 0$. The critical point x is nondegenerate if $J_{\nabla f}(x) \ne 0$. The index of a nondegenerate critical point x is

$$i_f(x) = \operatorname{sgn} J_{\nabla f}(x) \, .$$

5c5 Exercise. For all bounded Borel functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$ and all $f \in C^2(\overline{D})$,

$$\int_{\mathbb{R}^2} \mathrm{d}y' \psi(y') \sum_{x \in D : \nabla f(x) = y'} \varphi(f(x)) \operatorname{sgn} J_{\nabla f}(x) = \int_D \mathrm{d}x \, \psi(\nabla f(x)) \varphi(f(x)) J_{\nabla f}(x) \, .$$

Prove it.

Hint: similar to 4a3.

5d Random function

Similarly to 4a we consider a probability measure μ on $C^2(\overline{D})$ such that for each $x \in D$ some density $(y, y') \mapsto p_x(y, y')$ and some regression function $(y, y') \mapsto \mathbb{E} \big(J_{\nabla f}(x) \, \big| \, f(x) = y, \nabla f(x) = y' \big)$ satisfy the equalities

$$\text{(5d1)} \qquad \mathbb{E} \, \varphi(f(x)) \psi(\nabla f(x)) = \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}^2} \mathrm{d}y' \, p_x(y, y') \varphi(y) \psi(y')$$

and

$$\text{(5d2)}\quad \mathbb{E}\,\varphi(f(x))\psi(\nabla f(x))J_{\nabla f}(x) = \\ = \int_{\mathbb{R}} \mathrm{d}y \int_{\mathbb{R}^2} \mathrm{d}y'\, p_x(y,y')\varphi(y)\psi(y')\mathbb{E}\big(J_{\nabla f}(x)\,\big|\,f(x)=y, \nabla f(x)=y'\big)$$

for all bounded Borel functions $\varphi : \mathbb{R} \to \mathbb{R}$, $\psi : \mathbb{R}^2 \to \mathbb{R}$ (similarly to 4a7). We also assume that

$$\text{(5d3)}\qquad \mathbb{E} \int_D |J_{\nabla f}(x)|\,\mathrm{d}x < \infty\,.$$

Similarly to (4a6) we get

$$\text{(5d4)}\quad \mathbb{E} \sum_{x\in D:\nabla f(x)=y'} \varphi(f(x))\operatorname{sgn} J_{\nabla f}(x) = \\ = \int_D \mathrm{d}x\, p_x(y') \int_{\mathbb{R}} \mathrm{d}y\, p_x(y|y')\varphi(y)\mathbb{E}\big(J_{\nabla f}(x)\,\big|\,f(x)=y, \nabla f(x)=y'\big)$$

for almost all $y' \in \mathbb{R}^2$; as before,

$$p_x(y,y') = p_x(y')p_x(y|y')\,.$$

5e Gaussian case

Similarly to 4b we consider a (centered) Gaussian measure γ on $C^2(\overline{D})$ such that for every $x \in D$ (assuming that f is distributed γ),

(5e1) the joint distribution of $f(x)$ and $\nabla f(x)$ is a *nondegenerate* Gaussian measure on $\mathbb{R}^3$.

In other words,

- ∗ the variance of $f(x)$ does not vanish,
- ∗ conditionally, given $f(x)$, the variance of the directional derivative of f at x does not vanish

(at each point, in each direction). Compare it with (4b1), (4b2). For now the variance of $f(x)$ need not be equal to 1 (yet).

The conditions of 5d are thus ensured, and therefore (5d4) holds for almost all y'. In fact it holds for all y', as is shown below (similarly to 4b). Especially, for $y' = 0$, taking into account that $\operatorname{sgn} J_{\nabla f}(x) = i_f(x)$ (and letting $i_f(x) = 0$ if x is degenerate) we get

$$\text{(5e2)}\quad \mathbb{E} \sum_{x\in D:\nabla f(x)=0} \varphi(f(x))i_f(x) = \\ = \int_D \mathrm{d}x\, p_x(0) \int_{\mathbb{R}} \mathrm{d}y\, p_x(y|y'=0)\varphi(y)\mathbb{E}\big(J_{\nabla f}(x)\,\big|\,f(x)=y, \nabla f(x)=0\big)\,.$$

The right-hand side of (5d4) is continuous in y' (check it); we have to prove that the left-hand side

$$\mathbb{E} \sum_{x\in D:\nabla f(x)=y'} \varphi(f(x)) \operatorname{sgn} J_{\nabla f}(x)$$

is also continuous in y'. Similarly to 3d and 4b, it is sufficient to check continuity of the function[1]

$$y' \mapsto \int_{\mathbb{R}^2} \gamma^2(\mathrm{d}u) \sum_{x\in D:\nabla f_u(x)=y'} \varphi(f_u(x)) \operatorname{sgn} J_{\nabla f_u}(x) \, ,$$

where $f_u(\cdot) = f_{u_1,u_2}(\cdot) = g(\cdot) + u_1 h_1(\cdot) + u_2 h_2(\cdot)$; $g, h_1, h_2 \in C^2(\overline{D})$ and the two vectors $\nabla h_1(x), \nabla h_2(x)$ are linearly independent for all $x \in \overline{D}$. To this end we transform the integral in u into an integral in x (compare it with (4b6)):

(5e3)

$$\int_{\mathbb{R}^2} \gamma^2(\mathrm{d}u) \sum_{x\in D:\nabla f_u(x)=y'} \varphi(f_u(x)) \operatorname{sgn} J_{\nabla f_u}(x) = \pm \int_D \varphi\big(f_{U(x)}(x)\big) \gamma^2(\mathrm{d}U(x)) \, ;$$

here the sign is '+' if $J_h(\cdot) = \det(\nabla h_1, \nabla h_2) > 0$ on $\overline{D}$, or '−' if $J_h(\cdot) < 0$ on $\overline{D}$; and $U(x) = \big(U_1(x), U_2(x)\big)$ is the (unique) solution of the linear equation

(5e4) $$\nabla g(x) + U_1(x)\nabla h_1(x) + U_2(x)\nabla h_2(x) = y_1 \, .$$

(Note that $U \in C^1(\overline{D}, \mathbb{R}^2)$.) Clearly, the right-hand side of (5e3) is continuous in y' (assuming continuity of φ without loss of generality, recall (4b8)).

In order to get (5e3) we start with a two-dimensional counterpart of (3b7):

$$\int_{\mathbb{R}^2} \mathrm{d}y \sum_{x\in f^{-1}(y)} g(x) \operatorname{sgn} J_f(x) = \int_D g(x) J_f(x) \, \mathrm{d}x$$

for $f \in C^1(\overline{D}, \mathbb{R}^2)$ (which follows from 5c4). Replacing f with U and $g(x)$ with $\varphi(f_{U(x)}(x)) \frac{1}{2\pi} \exp\big(-\frac12 U_1^2(x) - \frac12 U_2^2(x)\big)$ we get

$$\int_{\mathbb{R}^2} \gamma^2(\mathrm{d}u) \sum_{x:U(x)=u} \varphi(f_u(x)) \operatorname{sgn} J_U(x) =$$
$$= \int_D \varphi(f_{U(x)}(x)) \frac{1}{2\pi} \exp\big(-\tfrac12 U_1^2(x) - \tfrac12 U_2^2(x)\big) J_U(x) \, \mathrm{d}x \, .$$

[1] And, in addition, integrability of its supremum over a bounded set.

However, $U(x) = u \iff \nabla f_u(x) = y'$, and $J_{\nabla f}(\cdot) = J_U(\cdot) J_h(\cdot)$. In order to get the latter equality we differentiate (5e4) in x getting $f_{,kl} = -h_{1,k}U_{1,l} - h_{2,k}U_{2,l}$ where

$$f_{,kl} = \Big(\frac{\partial^2}{\partial x_k \partial x_l} f_u(x_1, x_2) \Big) \Big|_{u=U(x_1,x_2)} .$$

Finally,

$$J_{\nabla f} = f_{,11} f_{,22} - f_{,12} f_{,21} = (h_{1,1}h_{2,2} - h_{1,2}h_{2,1})(U_{1,1}U_{2,2} - U_{1,2}U_{2,1}) = J_h J_U \,,$$

which completes the proof of (5e2).

5e5 Exercise. With probability 1, f is a Morse function.

Prove it. Do you need the whole (5e1), or something weaker?

Hint: $f_u = g + u_1 h_1 + u_2 h_2$ and $U(\cdot)$ as before; if x is a degenerate critical point of f_u, then u is a critical value of U, since $u = U(x)$ and

$$\nabla g(x + \Delta x) + U_1(x) \nabla h_1(x + \Delta x) + U_2(x) \nabla h_2(x + \Delta x) = o(|\Delta x|) \,,$$

therefore

$$(U_1(x+\Delta x) - U_1(x)) \nabla h_1(x+\Delta x) + (U_2(x+\Delta x) - U_2(x)) \nabla h_2(x+\Delta x) = o(|\Delta x|) \,.$$

5f Curvature appears

No kind of curvature can be detected by a bug on a curve. But if the bug moves to a surface, it can detect Gaussian curvature.

F. Morgan [7, Sect. 3.6 (p. 24)].

Upgrading 3e and 4d we consider a surface (rather than curve) on $S^{n-1} = \{z \in \mathbb{R}^n : |z| = 1\}$ parameterized by S^2;

$$Z \in C^2\big(S^2, \mathbb{R}^n\big) \,, \quad Z\big(S^2\big) \subset S^{n-1} \,.$$

We assume that Z is an immersion, that is [3, Sect. 1.3],

$$\nabla_v Z(x) \ne 0 \tag{5f1}$$

for every $x \in S^2$ and every vector $v \ne 0$ tangent to S^2 at x (that is, $\langle v, x \rangle = 0$); of course,

$$\nabla_v Z(x) = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon} \bigg(Z\Big(\frac{x + \varepsilon v}{|x + \varepsilon v|} \Big) - Z(x) \bigg) \,. \tag{5f2}$$

It leads to a Gaussian random vector in $C^2(S^2)$,

$$f(x) = \langle Z(x), \xi \rangle \,,$$

where ξ is distributed γ^n.

5f3 Exercise. (a) Some choice of n and Z makes f distributed γ_{lin} (recall 5a).

(b) The same holds for γ_{quad}.

(c) The same holds for the convolution of γ_{quad} and γ_{lin}, that is, the distribution of $f = g + h$ for independent $g \sim \gamma_{\text{quad}}$ and $h \sim \gamma_{\text{lin}}$.

Prove it. What about $ag + bh$?

Hint: (b) use 5a2; (c) take the orthogonal sum of the spaces used in (a) and (b).

A crucial distinction between curves and surfaces is that all curves are mutually isometric but surfaces are not. I mean, on every curve there exists a coordinate u such that

$$\operatorname{dist}(A, B) = |u(A) - u(B)| \cdot (1 + o(1))$$

as $\operatorname{dist}(A, B) \to 0$. In contrast, a surface generally does not admit coordinates u_1, u_2 such that

$$\operatorname{dist}(A, B) = \sqrt{|u_1(A) - u_1(B)|^2 + |u_2(A) - u_2(B)|^2} \cdot (1 + o(1))$$

as $\operatorname{dist}(A, B) \to 0$.

The natural parameter helped us a lot in 4c, but cannot help now. And do not blame the domain, S^2. Blame the range, $Z(S^2) \subset S^{n-1}$. The same difficulty appears for Gaussian random fields on planar domains.

Critical points of the function $x \mapsto f(x) = \langle Z(x), \xi \rangle$ on S^2 correspond to critical points of the function $z \mapsto \langle z, \xi \rangle$ on the two-dimensional surface $Z(S^2) \subset S^{n-1}$. True, Z need not be one-to-one (it is an immersion, not necessarily embedding [3, Sect. 1.3]), but this is not an obstacle. We may count critical points on small pieces of S^2; on such piece Z is one-to-one, and its inverse Z^{-1} is also smooth (C^2) on the corresponding piece of $Z(S^2)$. Thus, we may forget for a while about S^2 and Z and consider the random function $z \mapsto \langle z, \xi \rangle$ on a small piece $T \subset S^{n-1}$ of a two-dimensional surface. (Afterwards we will translate the result into the language of Z and S^2.)

Given a point of T, we want to choose coordinates that are as convenient as possible around this point. To this end we rotate the coordinate system of $\mathbb{R}^n$ so that the given point becomes $e_3 = (0, 0, 1, 0, \dots, 0)$ and the tangent plane to T at e_3 becomes $\mathbb{R}e_1 + \mathbb{R}e_2 + e_3$. We get

$$T = \{z_1 e_1 + z_2 e_2 + e_3 + h(z_1, z_2) : (z_1, z_2) \in D\}$$

where $D \subset \mathbb{R}^2$ is an open neighborhood of the origin (of $\mathbb{R}^2$), and $h \in$

$C^2(\overline{D}, \mathbb{R}^n)$ satisfies

$$\frac{h(z_1,z_2)}{\sqrt{z_1^2+z_2^2}} \to 0 \quad \text{as } z_1 \to 0,\ z_2 \to 0\,;$$
$$\langle h(z_1,z_2), e_1 \rangle = \langle h(z_1,z_2), e_2 \rangle = 0 \quad \text{for all } (z_1,z_2) \in D\,.$$

Introducing

$$a_{i,j} = \frac{\partial^2}{\partial z_i \partial z_j}\bigg|_{z_1=z_2=0} h(z_1,z_2) \in \mathbb{R}^n \quad \text{for } i,j \in \{1,2\}$$

we get $\langle a_{i,j}, e_1 \rangle = \langle a_{i,j}, e_2 \rangle = 0$ and

$$h(z_1,z_2) = \tfrac12 a_{1,1} z_1^2 + a_{1,2} z_1 z_2 + \tfrac12 a_{2,2} z_2^2 + o(z_1^2 + z_2^2)\,.$$

The *Gauss curvature* of the surface T at e_3 is, by definition,

$$K = \langle a_{1,1}, a_{2,2} \rangle - |a_{1,2}|^2\,; \tag{5f4}$$

see [7, p. 30] (there it is denoted by G instead of the traditional K).[1]

Instead of the random function $z \mapsto \langle z, \xi \rangle$ on T we consider the corresponding random function $f \in C^2(\overline{D})$,

$$f(z_1,z_2) = \langle z_1 e_1 + z_2 e_2 + e_3 + h(z_1,z_2), \xi \rangle\,;$$

as before, ξ is distributed γ^n.

5f5 Exercise. $\mathbb{E}\, J_{\nabla f}(0) = K$.

Prove it.

Hint: first, $\frac{\partial^2}{\partial z_i \partial z_j}\big|_{z_1=z_2=0} f(z_1,z_2) = \langle a_{i,j}, \xi \rangle$; second, $\mathbb{E}\big(\langle a, \xi \rangle \langle b, \xi \rangle\big) = \langle a, b \rangle$ for all $a, b \in \mathbb{R}^n$.

5f6 Exercise. $\langle a_{1,1}, e_3 \rangle = \langle a_{2,2}, e_3 \rangle = -1$ and $\langle a_{1,2}, e_3 \rangle = 0$.

Prove it.

Hint: $|z_1 e_1 + z_2 e_2 + e_3 + h(z_1,z_2)|^2 = 1$, therefore $\langle h(z_1,z_2), e_3 \rangle = -\frac12 \big(z_1^2 + z_2^2 + o(z_1^2 + z_2^2)\big)$.

5f7 Exercise. The following two three-dimensional random vectors are independent:

$$\big(f(0), f_{,1}(0), f_{,2}(0)\big)\,,$$
$$\big(f_{,11}(0) + f(0), f_{,22}(0) + f(0), f_{,12}(0)\big)\,;$$

[1] See also [7, Chap. 3] for surfaces in $\mathbb{R}^3$, the famous Gauss's *Theorema Egregium* ('remarkable' or 'excellent' theorem): Gaussian curvature is intrinsic [7, 3.6 and 4.3] and the area of a disc of intrinsic radius r: area $= \pi r^2 - G \frac{\pi}{12} r^4 + \dots$ [7, (3.8)].

here $f_{,i}(0) = \frac{\partial}{\partial z_i}\big|_{z_1=z_2=0} f(z_1,z_2)$ and $f_{,ij}(0) = \frac{\partial^2}{\partial z_i \partial z_j}\big|_{z_1=z_2=0} f(z_1,z_2)$.

Prove it.

Hint: each one of the three vectors $a_{1,1}+e_3$, $a_{2,2}+e_3$, $a_{1,2}$ is orthogonal to e_1, e_2, e_3.

5f8 Exercise. The regression function $\mathbb{E}\big(J_{\nabla f}(0)\,\big|\,f(0)=y, \nabla f(0)=y'\big) = y^2-1+K$ and the density $p_0(y,y') = (2\pi)^{-3/2}\exp\big(-\frac12 y^2 - \frac12|y'|^2\big)$ for $y\in\mathbb{R}$, $y'\in\mathbb{R}^2$ satisfy (5d1) and (5d2) (for $x=0$).

Prove it.

Hint: first, a formal calculation: $\mathbb{E}\big(f_{,11}(0)f_{,22}(0)\,\big|\,\dots\big) = \mathbb{E}\big(((f_{,11}(0)+f(0)-f(0))(f_{,22}(0)+f(0)-f(0))\,\big|\,\dots\big) = \mathbb{E}\,(f_{,11}(0)+f(0))(f_{,22}(0)+f(0)) - \dots$ etc; second, prove (5d1) and (5d2).

Now we are in position to evaluate the integrand of (the external integral of) (5e2) at the point 0 of D; it is equal to

(5f9) $$\frac{1}{2\pi}\int_{\mathbb{R}}(y^2-1+K)\varphi(y)\,\gamma^1(\mathrm{d}y)\,.$$

Still, we cannot evaluate the integral, since every point needs its own coordinate system!

5g Curvature disappears

Recall the notion 'surface area'; it may be calculated as

(5g1)
$$\sigma(T) = \int_D \sqrt{|a_1(z_1,z_2)|^2|a_2(z_1,z_2)|^2 - (\langle a_1(z_1,z_2), a_2(z_1,z_2)\rangle)^2}\,\mathrm{d}z_1\mathrm{d}z_2\,,$$
$$\text{where}\quad a_i(z_1,z_2) = e_i + \frac{\partial}{\partial z_i}h(z_1,z_2)\quad\text{for } i=1,2\,,$$

but it is 'geometric' in the sense that it does not depend on the choice of a coordinate system. Moreover, σ is a measure on T (still, 'geometric').

Another 'geometric' measure μ on T is defined by

(5g2) $$\mu(A) = \mathbb{E}\sum_{z\in A:\nabla f(z)=0}\varphi(f(z))i_f(z)\quad\text{for } A\subset T\,;$$

the formula (5e2) (in combination with (5g1)) shows that μ has a density $\mathrm{d}\mu/\mathrm{d}\sigma$, and in fact, the density is continuous. Clearly, $\mathrm{d}\mu/\mathrm{d}\sigma$ is also 'geometric'. It means that, when calculating $\mathrm{d}\mu/\mathrm{d}\sigma$, we may choose a convenient coordinate system for each point separately.

Taking into account that the integrand of (5g1) at 0 is equal to 1, we conclude from (5f9) that

$$\frac{\mathrm{d}\mu}{\mathrm{d}\sigma}(z) = \frac{1}{2\pi}\int_{\mathbb{R}} (y^2 - 1 + K(z))\varphi(y)\,\gamma^1(\mathrm{d}y) \tag{5g3}$$

for $z \in T$; here $K(z)$ is the Gauss curvature of T at z. Thus,

$$\mathbb{E} \sum_{z\in T:\nabla f(z)=0} \varphi(f(z)) i_f(z) = \frac{1}{2\pi}\sigma(T)\int_{\mathbb{R}} (y^2-1)\varphi(y)\,\gamma^1(\mathrm{d}y) + \\ + \frac{1}{2\pi}\Big(\int_T K(z)\,\sigma(\mathrm{d}z)\Big)\Big(\int_{\mathbb{R}}\varphi(y)\,\gamma^1(\mathrm{d}y)\Big).$$

Summing up small pieces T of the surface $Z(S^2)$ and returning to the random function $x \mapsto f(x) = \langle Z(x), \xi\rangle$ on S^2 we get

$$\mathbb{E} \sum_{x\in S^2:\nabla f(x)=0} \varphi(f(x)) i_f(x) = \frac{1}{2\pi}\sigma(Z(S^2))\int_{\mathbb{R}} (y^2-1)\varphi(y)\,\gamma^1(\mathrm{d}y) + \\ + \frac{1}{2\pi}\Big(\int_{Z(S^2)} K(z)\,\sigma(\mathrm{d}z)\Big)\Big(\int_{\mathbb{R}}\varphi\,\mathrm{d}\gamma^1\Big).$$

This is correct if Z is an embedding, but for an immersion we should write $\mathrm{Area}_Z(S^2)$ rather than $\sigma(Z(S^2))$, and $\int_{S^2} K_Z(x)\,\mathrm{Area}_Z(\mathrm{d}x)$ rather than $\int_{Z(S^2)} K(z)\,\sigma(\mathrm{d}z)$. Here S^2 is equipped with a new Riemannian metric RiM_Z induced by the immersion Z (see below); Area_Z is the area measure corresponding to the new Riemannian metric; and $K_Z(x)$ is the Gauss curvature at x, corresponding to the new Riemannian metric.

For any smooth curve $(x_t)_{t\in[0,1]}$ on S^2, its length is $\int_0^1 |v_t|\,\mathrm{d}t$ where $v_t = \frac{\mathrm{d}}{\mathrm{d}t}x_t$. The length of the corresponding curve $\big(Z(x_t)\big)_{t\in[0,1]}$ on S^{n-1} is

$$\int_0^1 \Big|\frac{\mathrm{d}}{\mathrm{d}t}Z(x_t)\Big|\,\mathrm{d}t = \int_0^1 |\nabla_{v_t} Z(x_t)|\,\mathrm{d}t = \int_0^1 \sqrt{\mathrm{RiM}_Z(x_t)(v_t)}\,\mathrm{d}t$$

(recall (5f2)), where $\mathrm{RiM}_Z(x)$ is the quadratic form[1] $v \mapsto \mathrm{RiM}_Z(x)(v) = |\nabla_v Z(x)|^2$ on the tangent plane to S^2 at x. The family $\big(\mathrm{RiM}_Z(x)\big)_{x\in S^2}$ of these quadratic forms is, by definition, the Riemannian metric RiM_Z. (In general, a Riemannian metric is a smooth family of strictly positive quadratic forms on tangent spaces.)

Both Area_Z and K_Z are uniquely determined by RiM_Z (which is the meaning of the term 'intrinsic').

[1] It is quadratic, since $\nabla_v Z(x)$ is linear in v.

5g4 Exercise. (a) For γ_{lin} (recall 5f3(a)) the new Riemannian metric RiM_Z is equal to the old (usual) Riemannian metric RiM of S^2.

(b) For γ_{quad} (recall 5f3(b)), $\mathrm{RiM}_Z = \sqrt{3}\,\mathrm{RiM}$.

Prove it.

Hint: (b) $|Z(x) - Z(y)|^2 = 2 - 2\langle Z(x), Z(y)\rangle = 2\big(1 - \mathbb{E}\, f(x)f(y)\big)$; use 5a2.

Here comes a surprise: we do not need to calculate the curvature for every given Z, since the integral of the curvature is a topological invariant, namely,

$$\int_{S^2} K_Z(x)\,\mathrm{Area}_Z(\mathrm{d}x) = 4\pi\,; \tag{5g5}$$

this is a special case of the famous Gauss-Bonnet theorem [7, Sect. 8.2]. Thus,

$$\mathbb{E} \sum_{x\in S^2:\nabla f(x)=0} \varphi(f(x)) i_f(x) = \frac{1}{2\pi}\,\mathrm{Area}_Z(S^2) \int_{\mathbb{R}} (y^2-1)\varphi(y)\,\gamma^1(\mathrm{d}y) + \\ + 2\int_{\mathbb{R}} \varphi\,\mathrm{d}\gamma^1\,.$$

Finally, using (5b3), 5e5 and taking into account that $(y^2-1)\mathrm{e}^{-y^2/2} = -(y\mathrm{e}^{-y^2/2})'$ we get

$$\mathbb{E}\,\chi\big(\{x\in S^2 : f(x)\ge y\}\big) = (2\pi)^{-3/2} y \mathrm{e}^{-y^2/2}\,\mathrm{Area}_Z(S^2) + 2\gamma^1\big([y,\infty)\big) \tag{5g6}$$

for every $y\in\mathbb{R}$ and every random process of the type introduced in (the beginning of) 5f. In particular, for $y=0$,

$$\mathbb{E}\,\chi\big(\{x\in S^2 : f(x)\ge 0\}\big) = 1$$

irrespective of $\mathrm{Area}_Z(S^2)$, but this fact is rather a simple consequence of the symmetry of γ under $f\mapsto(-f)$.[1] Note also the limiting cases $y\to-\infty$ and $y\to+\infty$. And one more limiting case: $\mathrm{Area}_Z(S^2)\to 0$, the constant process.

All said above holds also for a surface on S^{n-1} parameterized by the torus $\mathbb{T}^2$ (rather than the sphere S^2), except for (5g5); this time,[2]

$$\int_{\mathbb{T}^2} K_Z(x)\,\mathrm{Area}_Z(\mathrm{d}x) = 0\,, \tag{5g7}$$

and the last term of (5g6) disappears:

$$\mathbb{E}\,\chi\big(\{x\in \mathbb{T}^2 : f(x)\ge y\}\big) = (2\pi)^{-3/2} y \mathrm{e}^{-y^2/2}\,\mathrm{Area}_Z(\mathbb{T}^2)\,. \tag{5g8}$$

[1] Indeed, $\chi\big(\{x\in S^2 : f(x)\ge 0\}\big) + \chi\big(\{x\in S^2 : f(x)\le 0\}\big) = \chi(S^2) + \chi\big(\{x\in S^2 : f(x) = 0\}\big) = 2+0 = 2$.

[2] Since $\chi(\mathbb{T}^2)=0$. Generally, $\int_M K_Z(x)\,\mathrm{Area}_Z(\mathrm{d}x) = 2\pi\chi(M)$.

5h A generalization

Let γ be a (centered) Gaussian measure on $C^2(S^2)$ such that for every $x \in S^2$ (assuming that f is distributed γ),

$$\begin{gathered}\text{the distribution of } f(x) \text{ is } \mathrm{N}(0,1)\,,\\ \text{the distribution of } \nabla f(x) \text{ is a 2-dimensional Gaussian measure}\,.\end{gathered} \tag{5h1}$$

In other words, the variance of the directional derivative of f does not vanish (at each point, in each direction).

We consider the Hilbert space $H = L_2^{\text{lin}}(\gamma)$ of γ-measurable linear functionals (or rather, their equivalence classes), its unit sphere $S(H) = \{z \in H : \|z\| = 1\}$ and the map

$$\begin{gathered}Z \in C^2(S^2, H)\,, \quad Z(S^2) \subset S(H)\,,\\ Z(x)(f) = f(x) \quad \text{for } x \in S^2,\ f \in C^2(S^2)\,.\end{gathered}$$

5h2 Exercise. Prove that the map $Z : S^2 \to H$ is indeed twice continuously differentiable.

Hint: recall (the hint to) 3d3.

Do not think that the necessary condition $Z \in C^2(S^2, H)$ is also sufficient. Some γ on $C^1(S^2)$ satisfy this condition but do not fit into $C^2(S^2)$.[1]

5h3 Exercise. $f(x)$ and $\nabla f(x)$ are independent (for each $x \in S^2$ separately).

Prove it.

Hint: similar to (3d4).

Thus, γ satisfies (5e1) (in any smooth coordinate system on a small piece of S^2), which ensures (5e2). Now, all arguments of 5f and 5g work, giving (5g6). The only point that needs some attention is this: the extrinsic definition (5f4) of the Gauss curvature is equivalent to its intrinsic definition, based on the Riemannian metric RiM_Z. The proof is quite similar to the finite-dimensional case. The Gauss-Bonnet theorem works as before, since it is applied to (S^2, RiM_Z) (rather than $Z(S^2) \subset S(H)$).

Similarly, (5g8) holds for every Gaussian random function on the torus, satisfying (5h1).

[1] It is sufficient (but not necessary) that the second derivatives of Z satisfy some Hölder condition (in particular, $Z \in C^3(S^2, H)$ is far enough).

5i Final remarks

5i1 Exercise. Let $f \in C^2(S^2)$ be distributed γ_{lin} (recall 5a), and $M = \sup_{S^2} f$. Then

$$\chi\big(\{x \in S^2 : f(x) \ge y\}\big) = \begin{cases} 2 & \text{for } y \in (-\infty, -M), \\ 1 & \text{for } y \in (-M, M), \\ 0 & \text{for } y \in (M, \infty), \end{cases}$$

thus

$$\mathbb{E}\,\chi\big(\{x \in S^2 : f(x) \ge y\}\big) = \begin{cases} 2 - \mathbb{P}\big(M > -y\big) & \text{for } y < 0, \\ \mathbb{P}\big(M > y\big) & \text{for } y \ge 0, \end{cases}$$

and (5g6) gives

$$\mathbb{P}\big(M > y\big) = \frac{2}{\sqrt{2\pi}} y \mathrm{e}^{-y^2/2} + 2\gamma^1\big([y,\infty)\big) = \frac{2}{\sqrt{2\pi}} \int_y^\infty u^2 \mathrm{e}^{-u^2/2} \, \mathrm{d}u$$

for $y \ge 0$.

Check it. Give an elementary explanation, why the density of M is $f_M(u) = \frac{2}{\sqrt{2\pi}} u^2 \mathrm{e}^{-u^2/2}$ on $(0, \infty)$.

5i2 Exercise. Let $f \in C^2(S^2)$ be distributed γ_{quad} (recall 5a), and $M_1 > M_2 > M_3$ be the eigenvalues of the quadratic form f, in other words, the critical values of f on S^2 (two symmetric maxima, two symmetric saddle points and two symmetric minima). Then

$$\chi\big(\{x \in S^2 : f(x) \ge y\}\big) = \begin{cases} 2 & \text{for } y \in (-\infty, M_3), \\ 0 & \text{for } y \in (M_3, M_2), \\ 2 & \text{for } y \in (M_2, M_1), \\ 0 & \text{for } y \in (M_1, \infty), \end{cases}$$

thus

$$\mathbb{E}\,\chi\big(\{x \in S^2 : f(x) \ge y\}\big) = 2\mathbb{P}\big(M_1 > y\big) - 2\mathbb{P}\big(M_2 > y\big) + 2\mathbb{P}\big(M_3 > y\big) \, ,$$

and (5g6) gives

$$\mathbb{P}\big(M_1 > y\big) - \mathbb{P}\big(M_2 > y\big) + \mathbb{P}\big(M_3 > y\big) = \frac{3}{\sqrt{2\pi}} y \mathrm{e}^{-y^2/2} + \gamma^1\big([y,\infty)\big) = \\ = \frac{1}{\sqrt{2\pi}} \int_y^\infty (3u^2 - 2) \mathrm{e}^{-u^2/2} \, \mathrm{d}u$$

for $y \in \mathbb{R}$. In terms of densities,

$$f_{M_1}(u) - f_{M_2}(u) + f_{M_3}(u) = \frac{1}{\sqrt{2\pi}}(3u^2 - 2)\mathrm{e}^{-u^2/2} \,.$$

Check it.

Here we have no exact formula for $\mathbb{P}\big(M > y \big)$, where $M = \sup_{S^2} f = M_1$. However, we have an inequality,

$$\mathbb{P}\big(M > y \big) \ge \frac{3}{\sqrt{2\pi}} y \mathrm{e}^{-y^2/2} + \gamma^1\big([y,\infty) \big) \,,$$

and in fact, the right-hand side is a very good approximation of $\mathbb{P}\big(M > y \big)$ for large y.

5i3 Exercise. Consider a linear combination $f = a\zeta + bg + ch$ of independent ζ, g, h distributed $\mathrm{N}(0,1)$, γ_{lin} and γ_{quad} respectively (ζ is treated as a random constant function), assuming that $a^2 + b^2 + c^2 = 1$.

(a) f satisfies (5h1).

(b) $\mathrm{RiM}_f = \sqrt{b^2 + 3c^2}\, \mathrm{RiM}$.

Check it.

We see that different rotation-invariant Gaussian measures on $C^2(S^2)$ may lead to the same Riemannian metric. For example, g and $\sqrt{2/3}\,\zeta + \sqrt{1/3}\,h$. In fact, the same (rotation-invariant) Riemannian metric results also from many Gaussian measures that are not rotation-invariant.

From now on we assume that f and g are random functions of $C^2(S^2)$ whose distributions satisfy the conditions of 5h.

Given a smooth curve $C \subset S^2$, we denote by $\mathrm{Len}_f(C)$ its length according to the Riemannian metric RiM_f.

5i4 Exercise. For any smooth closed curve $C \subset S^2$,

$$\mathbb{E}\, \#\big(C \cap f^{-1}(0) \big) = \mathrm{Len}_f(C) \,.$$

Prove it.

Hint: parameterize C by S^1 and apply 3d6.

5i5 Exercise. The degenerate case

$$f(x) = 0 \quad \text{and} \quad \nabla f(x) = 0$$

is excluded for almost all f.

Prove it.

Hint: in the spirit of 5e, introduce $f_u(\cdot) = g(\cdot)+u_1h_1(\cdot)+u_2h_2(\cdot)+u_3h_3(\cdot)$ where

$$\begin{vmatrix} h_1(\cdot) & h_2(\cdot) & h_3(\cdot) \\ h_{1,1}(\cdot) & h_{2,1}(\cdot) & h_{3,1}(\cdot) \\ h_{1,2}(\cdot) & h_{2,2}(\cdot) & h_{3,2}(\cdot) \end{vmatrix} \ne 0 \,,$$

and define $U \in C^1(\overline{D}, \mathbb{R}^3)$ by solving (in u) the system $f_u(x) = 0$, $\nabla f_u(x) = 0$.

It follows that the set $f^{-1}(0)$, known as the nodal line, consists of a finite number of disjoint simple (that is, non-self-intersecting) smooth closed curves on S^2.

5i6 Exercise.

$$\mathbb{E}\,\mathrm{Len}_f\, g^{-1}(0) = \pi \mathbb{E}\,\#\big(f^{-1}(0) \cap g^{-1}(0)\big) = \mathbb{E}\,\mathrm{Len}_g\, f^{-1}(0)\,.$$

Prove it.

Hint: 5i4, and Fubini.

Choosing $g \sim \gamma_{\mathrm{lin}}$ we observe that $g^{-1}(0)$ is a random great circle, and $\mathrm{Len}_g = \mathrm{Len}$ is the usual length. Thus,

$$\mathbb{E}\,\mathrm{Len}\, f^{-1}(0) = \mathbb{E}\,\mathrm{Len}_f(\text{great circle})\,; \tag{5i7}$$

the expected length of the nodal line is equal to the averaged Riemannian length of a great circle.

The condition

$$\mathrm{RiM}_f = C_f \cdot \mathrm{RiM} \quad \text{for some } C_f \in (0,\infty) \tag{5i8}$$

is weaker than rotation-invariance; it means that the directional derivative is distributed $\mathrm{N}(0, C_f^2)$ at each point, in each direction. In this case (5i7) becomes

$$\mathbb{E}\,\mathrm{Len}\, f^{-1}(0) = 2\pi C_f\,. \tag{5i9}$$

On the other hand, $\mathrm{Area}_f(S^2) = C_f^2\,\mathrm{Area}(S^2) = 4\pi C_f^2$, thus,

$$\begin{aligned} \mathbb{E}\,\chi\big(\{x \in S^2 : f(x) \ge y\}\big) &= 2(2\pi)^{-1/2} y \mathrm{e}^{-y^2/2} C_f^2 + 2\gamma^1([y,\infty)) = \\ &= 2(2\pi)^{-5/2} y \mathrm{e}^{-y^2/2} \big(\mathbb{E}\,\mathrm{Len}\, f^{-1}(0)\big)^2 + 2\gamma^1([y,\infty))\,, \end{aligned} \tag{5i10}$$

a nontrivial relation between the mean length of a nodal line and the mean Euler characteristic.

Waiving (5i8) we have no direct relation between $\operatorname{Area}_f(S^2)$ and $\mathbb{E}\operatorname{Len}_f(\text{great circle})$, but still, we may get a nontrivial relation as follows.

Let f, g be identically distributed (and independent, as before). It appears that

$$\mathbb{E}\,\#\big(f^{-1}(0)\cap g^{-1}(0)\big) = \frac{1}{2\pi}\operatorname{Area}_f(S^2)\,. \tag{5i11}$$

This is a special case of a two-dimensional counterpart of Rice's formula; I do not prove it. (Think, what happens if $f \sim \gamma_{\text{lin}}$.) Using (5i11) we get

$$\begin{aligned}\mathbb{E}\,\chi\big(\{x\in S^2 : f(x)\ge y\}\big) = (2\pi)^{-1/2}y\mathrm{e}^{-y^2/2}\mathbb{E}\,\#\big(f^{-1}(0)\cap g^{-1}(0)\big)+\\ +2\gamma^1([y,\infty))\,,\end{aligned} \tag{5i12}$$

a very general nontrivial relation between the mean Euler characteristic and the mean number of intersections between two independent nodal lines. Assuming also (5i8) we get $\mathbb{E}\,\#\big(f^{-1}(0)\cap g^{-1}(0)\big) = 2C_f^2$.

The same holds for the torus, except for the last term;

$$\mathbb{E}\,\chi\big(\{x\in \mathbb{T}^2 : f(x)\ge y\}\big) = (2\pi)^{-1/2}y\mathrm{e}^{-y^2/2}\mathbb{E}\,\#\big(f^{-1}(0)\cap g^{-1}(0)\big)\,. \tag{5i13}$$

A remarkable theorem of Taylor, Takemura and Adler [8, Th. 4.3] shows that the expected Euler characteristic is an excellent approximation for (the tail of) the distribution of $M_f = \max_{S^2} f$; namely, there exists $\varepsilon > 0$ (depending on the distribution of f) such that

$$\big|\mathbb{P}\big(M_f\ge y\big) - \mathbb{E}\,\chi\big(\{x\in S^2 : f(x)\ge y\}\big)\big| \le \exp\Big(-\frac{1+\varepsilon}{2}y^2\Big)$$

for all y large enough. It is assumed that the maximizer is unique a.s.[1] A sufficient condition: $\mathbb{E}\,|f(x_1)-f(x_2)|^2 > 0$ whenever $x_1\ne x_2$.

A much, much weaker statement,

$$\mathbb{P}\big(M_f\ge y\big) \sim (2\pi)^{-3/2}y\mathrm{e}^{-y^2/2}\operatorname{Area}_f(S^2) \sim \frac{\operatorname{Area}_f(S^2)}{2\pi}y^2\gamma^1([y,\infty))\,,$$

shows that

$$\frac{\mathbb{P}\big(M_f\ge y\big)}{\mathbb{P}\big(M_g\ge y\big)} \to \frac{\operatorname{Area}_f(S^2)}{\operatorname{Area}_g(S^2)} \quad \text{as } y\to\infty\,.$$

5i14 Corollary. If

$$\frac{\mathbb{E}\,|f(x_1)-f(x_2)|^2}{\mathbb{E}\,|g(x_1)-g(x_2)|^2} \to 1 \quad \text{as } \operatorname{dist}(x_1,x_2)\to 0$$

[1]Otherwise the approximation may fail; 5i2 gives a counterexample.

and $\mathbb{E}\,|f(x_1)-f(x_2)|^2>0$, $\mathbb{E}\,|g(x_1)-g(x_2)|^2>0$ whenever $x_1\ne x_2$, then

$$\frac{\mathbb{P}\big(\max f\ge y\big)}{\mathbb{P}\big(\max g\ge y\big)}\to 1\quad\text{as } y\to\infty\,.$$

The proof involves differential geometry, but the formulation does not!

5i15 Question. Is any differential structure essential for 5i14?

Let f,g be Gaussian random continuous functions on a compact metric space T, satisfying $f(t)\sim\mathrm{N}(0,1)$ and $g(t)\sim\mathrm{N}(0,1)$ for all $t\in T$. Does 5i14 hold in this generality?

The same question applies to a stronger claim: if

$$\frac{\mathbb{E}\,|f(x_1)-f(x_2)|^2}{\mathbb{E}\,|g(x_1)-g(x_2)|^2}\to C^2\quad\text{as } \operatorname{dist}(x_1,x_2)\to 0\,,$$

for some $C\in(0,\infty)$, and $\mathbb{E}\,|f(x_1)-f(x_2)|^2>0$, $\mathbb{E}\,|g(x_1)-g(x_2)|^2>0$ whenever $x_1\ne x_2$, then

$$\frac{\mathbb{P}\big(\max f\ge y\big)}{\mathbb{P}\big(\max g\ge y\big)}\to C^2\quad\text{as } y\to\infty\,.$$

The theory presented here is relatively easy, since we restrict ourselves to (some) smooth compact two-dimensional manifolds without boundary. The theory of Adler and Taylor [2] is much harder, since it covers piecewise smooth n-dimensional manifolds with boundary.

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Index