
A fair coin is tossed repeatedly with results $Y_0, Y_1, Y_2, \dots$ that are 0 or 1 with probability 1/2 each. For $n \ge 1$ let $X_n = Y_n + Y_{n-1}$ be the number of 1's in the $(n-1)$th and nth tosses. Is X_n a Markov chain?

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No, X_n is not a Markov chain. For example, $\mathbb{P}\left(X_3 = 2 \,\middle|\, X_2 = 1, X_1 = 2 \right) = \mathbb{P}\left(Y_3 = 1, Y_2 = 1 \,\middle|\, Y_2 = 0, Y_1 = 1, Y_0 = 1 \right) = 0$, but $\mathbb{P}\left(X_3 = 2 \,\middle|\, X_2 = 1, X_1 = 0 \right) = \mathbb{P}\left(Y_3 = 1, Y_2 = 1 \,\middle|\, Y_2 = 1, Y_1 = 0, Y_0 = 0 \right) = 0.5$. Thus, $\mathbb{P}\left(X_3 = x_3 \,\middle|\, X_2 = x_2, X_1 = x_1 \right)$ depends on x_1.

Five white balls and five black balls are distributed in two urns in such a way that each urn contains five balls. At each step we draw one ball from each urn and exchange them. Let X_n be the number of white balls in the left urn at time n. Compute the transition probability for X_n.

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The drawn balls are both white or both black with (conditional) probability $\frac{X_n}{5} \cdot \frac{5-X_n}{5} + \frac{5-X_n}{5} \cdot \frac{X_n}{5}$; thus, $\mathbb{P}\left(X_{n+1} = X_n \,\middle|\, X_n \right) = \frac{2}{25} X_n (5 - X_n)$. The combination "white and black" appears with probability $\frac{X_n}{5} \cdot \frac{X_n}{5}$, thus, $\mathbb{P}\left(X_{n+1} = X_n - 1 \,\middle|\, X_n \right) = \frac{1}{25} X_n^2$. The combination "black and white" appears with probability $\frac{5-X_n}{5} \cdot \frac{5-X_n}{5}$, thus, $\mathbb{P}\left(X_{n+1} = X_n + 1 \,\middle|\, X_n \right) = \frac{1}{25}(5 - X_n)^2$. We get

$$p = \frac{1}{25} \cdot \begin{pmatrix} 2 \cdot 0 \cdot 5 & 5^2 & 0 & 0 & 0 & 0 \\ 1^2 & 2 \cdot 1 \cdot 4 & 4^2 & 0 & 0 & 0 \\ 0 & 2^2 & 2 \cdot 2 \cdot 3 & 3^2 & 0 & 0 \\ 0 & 0 & 3^2 & 2 \cdot 3 \cdot 2 & 2^2 & 0 \\ 0 & 0 & 0 & 4^2 & 2 \cdot 4 \cdot 1 & 1^2 \\ 0 & 0 & 0 & 0 & 5^2 & 2 \cdot 5 \cdot 0 \end{pmatrix} =$$

$$= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0.04 & 0.32 & 0.64 & 0 & 0 & 0 \\ 0 & 0.16 & 0.48 & 0.36 & 0 & 0 \\ 0 & 0 & 0.36 & 0.48 & 0.16 & 0 \\ 0 & 0 & 0 & 0.64 & 0.32 & 0.04 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$
