# Exam of 12.10.2007 — Solutions

## 1

### 1a

The absorbing states 0 and $N/8$ are reachable from every other state, therefore all other states are transient. A transient state is visited only a finite number of times (almost surely); the same holds for the finite set of transient states.

### 1b

$p(x, x-1) = x/N$, thus

$$\mathbb{P}\big(B\big) = p(i, i-1)p(i-1, i-2)\dots p(1,0) = \frac{i}{N}\frac{i-1}{N}\dots\frac{1}{N} = \frac{i!}{N^i}\,.$$

### 1c

We have to check that $h(x) \ge p(x, x-1)h(x-1) + p(x, x+1)h(x+1)$ for $0 < x < N/8$; that is,

$$x!(N-x)!\Big(1+\frac{4x}{N}\Big) \ge \frac{x}{N}(x-1)!(N-x+1)!\Big(1+\frac{4(x-1)}{N}\Big)+\frac{N-x}{N}(x+1)!(N-x-1)!\Big(1+\frac{4(x+1)}{N}\Big)\,.$$

We cancel $x!(N-x)!$ and simplify:

$$1+\frac{4x}{N} \ge \frac{N-x+1}{N}\Big(1+\frac{4x-4}{N}\Big) + \frac{x+1}{N}\Big(1+\frac{4x+4}{N}\Big)\,;$$

$$1+\frac{4x}{N} \ge \frac{N+2}{N}\Big(1+\frac{4x}{N}\Big) - \frac{N-2x}{N}\cdot\frac{4}{N}\,;$$

$$\frac{N-2x}{N}\cdot\frac{4}{N} \ge \frac{2}{N}\Big(1+\frac{4x}{N}\Big)\,;$$

$$\frac{N-2x}{N}\cdot 2 \ge 1+\frac{4x}{N}\,; \quad 2N-4x \ge N+4x\,; \quad N \ge 8x\,,$$

which is true.

### 1d

By the stopping theorem for bounded supermartingales,

$$\mathbb{E}\,M_0 \ge \mathbb{E}\,M_T\,,$$

where $T$ is the first time $X_n$ visits 0 or $N/8$. We have

$$\mathbb{E}\,M_0 = h(i)\,,$$

$$\mathbb{E}\,M_T = h(0)\mathbb{P}\big(A\big) + h(N/8)(1-\mathbb{P}\big(A\big)) \ge h(0)\mathbb{P}\big(A\big)\,.$$

Therefore $h(i) \ge h(0)\mathbb{P}\big(A\big)$;

$$\mathbb{P}\big(A\big) \le \frac{h(i)}{h(0)} = \frac{i!(N-i)!(1+\frac{4i}{N})}{N!} = \frac{1+\frac{4i}{N}}{\binom{N}{i}} .$$

## 1e

Yes, $\mathbb{P}\big(B\big|A\big)$ converges to 1 as $N \to \infty$, since

$$\mathbb{P}\big(B\big|A\big) = \frac{\mathbb{P}\big(B\big)}{\mathbb{P}\big(A\big)} \ge \frac{i!}{N^i} \cdot \frac{\binom{N}{i}}{1+\frac{4i}{N}} = \frac{N(N-1)\dots(N-i+1)}{N^i} \cdot \frac{1}{1+\frac{4i}{N}} \to 1 .$$

# 2

## 2a

We have jump rates $q(1,2) = \lambda_1$, $q(2,1) = \lambda_2$. The stationary distribution $\pi(1), \pi(2)$ satisfies $\pi_1 q(1,2) = \pi_2 q(2,1)$, that is, $\pi_1\lambda_1 = \pi_2\lambda_2$. Thus,

$$\frac{\pi_1}{1/\lambda_1} = \frac{\pi_2}{1/\lambda_2} \quad \text{and} \quad \pi_1 + \pi_2 = 1 ;$$

$$\pi_1 = \frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2}} = \frac{\lambda_2}{\lambda_1\lambda_2} .$$

## 2b

Similarly,

$$q(1,2) = \lambda_1 , \quad q(2,3) = \lambda_2 , \quad q(3,1) = \lambda_3 ;$$

$$\pi_1\lambda_1 = \pi_2\lambda_2 = \pi_3\lambda_3 ; \quad \pi_k = \frac{\frac{1}{\lambda_k}}{\frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{1}{\lambda_3}} ;$$

$$\pi_1 = \frac{\lambda_2\lambda_3}{\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2} , \quad \pi_2 = \frac{\lambda_3\lambda_1}{\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2} , \quad \pi_3 = \frac{\lambda_1\lambda_2}{\lambda_2\lambda_3 + \lambda_3\lambda_1 + \lambda_1\lambda_2} .$$

## 2c

Now $q(1,2) = q(1,3) = \lambda_1/2$, $q(2,3) = q(2,1) = \lambda_2/2$, $q(3,1) = q(3,2) = \lambda_3/2$. Stationarity: $\pi_1\lambda_1 = (\pi_2\lambda_2 + \pi_3\lambda_3)/2$, $\pi_2\lambda_2 = (\pi_3\lambda_3 + \pi_1\lambda_1)/2$, $\pi_3\lambda_3 = (\pi_1\lambda_1 + \pi_2\lambda_2)/2$. It follows that $\pi_1\lambda_1 = \pi_2\lambda_2 = \pi_3\lambda_3$, which can be shown algebraically or just by noting that the greatest among the three numbers $\pi_k\lambda_k$ is strictly larger than the mean of the other two numbers, unless they all are equal. Thus, $\pi_1, \pi_2, \pi_3$ are the same as in the previous case.

## 2d

This time $q(1,2) = \lambda_1$, $q(2,3) = \lambda_2$ and $q(3,1) = q(3,2) = \lambda_3/2$. Stationarity: $\pi_1\lambda_1 = \pi_3\lambda_3/2$, $\pi_2\lambda_2 = \pi_1\lambda_1 + 0.5\pi_3\lambda_3$, $\pi_3\lambda_3 = \pi_2\lambda_2$. Thus, $0.5\pi_2\lambda_2 = \pi_1\lambda_1$;

$$\frac{\pi_1}{1/\lambda_1} = \frac{\pi_2}{2/\lambda_2} = \frac{\pi_3}{2/\lambda_3} \quad \text{and} \quad \pi_1 + \pi_2 + \pi_3 = 1\,;$$

$$\pi_1 = \frac{\frac{1}{\lambda_1}}{\frac{1}{\lambda_1} + \frac{2}{\lambda_2} + \frac{2}{\lambda_3}} = \frac{\lambda_2\lambda_3}{\lambda_2\lambda_3 + 2\lambda_3\lambda_1 + 2\lambda_1\lambda_2}\,,$$

$$\pi_2 = \frac{\frac{2}{\lambda_2}}{\frac{1}{\lambda_1} + \frac{2}{\lambda_2} + \frac{2}{\lambda_3}} = \frac{2\lambda_1\lambda_3}{\lambda_2\lambda_3 + 2\lambda_3\lambda_1 + 2\lambda_1\lambda_2}\,,$$

$$\pi_3 = \frac{\frac{2}{\lambda_3}}{\frac{1}{\lambda_1} + \frac{2}{\lambda_2} + \frac{2}{\lambda_3}} = \frac{2\lambda_1\lambda_2}{\lambda_2\lambda_3 + 2\lambda_3\lambda_1 + 2\lambda_1\lambda_2}\,.$$

# 3

## 3a

Yes, it can happen. For example, let $X_n$ be a Markov chain with three values $-1, 0, +1$ and transition probabilities $p(-1,0) = p(0,1) = p(1,-1) = 1$. Then the process $Y_n = X_n^2$ is not Markov. Indeed, $\mathbb{P}\big(Y_2 = 0 \,\big|\, Y_1 = 1, Y_0 = 1\big) = 1$ but $\mathbb{P}\big(Y_2 = 0 \,\big|\, Y_1 = 1, Y_0 = 0\big) = 0$.

## 3b

Yes, it can happen. For example, let $X_n$ be a non-Markov process with two values $-1, +1$ (such processes exist, see the previous item). Then the process $Y_n = X_n^2$, taking only one value $+1$, is Markov.

## 3c

No, it cannot happen. Markovianity of the process $X_n$ is equivalent to Markovianity of the process $Y_n = X_n^3$, since the function $y = x^3$ is one-to-one.