5 Moderate deviations in spaces of functions

5a Asymptotically quadratic generating functions

Let $p, q, \mu, S_n, \Lambda_n, \Lambda_\infty$ and A_n be as in Sect. 4b, $\int x^2\,\mu(\mathrm{d}x) = 1$ (that is, $\Lambda''_\mu(0) = 1$), and $p \le 2 \le q$ (see 4b4).

5a1 Proposition. For every $g \in L_q$,

$$\frac{1}{n\varepsilon^2}\Lambda_n(\varepsilon g) \to \frac{1}{2}\|g\|_2^2 \quad \text{as } \varepsilon \to 0, n \to \infty\,.$$

This is a two-dimensional limit; that is,

$$\forall \delta > 0\ \exists \varepsilon_0 > 0\ \exists n_0\ \forall \varepsilon \le \varepsilon_0\ \forall n \ge n_0 \quad \Big| \frac{1}{n\varepsilon^2}\Lambda_n(\varepsilon g) - \frac{1}{2}\|g\|_2^2 \Big| \le \delta\,.$$

Not the same as $\lim_\varepsilon \lim_n$ or $\lim_n \lim_\varepsilon$.

First, we improve 4b1, 4b2 for small arguments.

5a2 Lemma. $\Lambda'_\mu(t) \le \mathrm{const} \cdot \max\big(|t|, |t|^{q-1}\big)$ for all $t \in \mathbb{R}$.

Proof. For large t we have $\Lambda'_\mu(t) = \mathcal{O}(|t|^{q-1})$ by 4b1; for small t, $\Lambda'_\mu(t) = \mathcal{O}(|t|)$. □

5a3 Lemma. There exists C such that for all $g_1, g_2 \in L_q$,

$$\|\Lambda_\infty(g_1) - \Lambda_\infty(g_2)\| \le C\|g_1 - g_2\|_q \big(\|g_1\|_q + \|g_1\|_q^{q-1} + \|g_2\|_q + \|g_2\|_q^{q-1} \big)\,.$$

Proof. Using 5a2, we take C such that

$$\forall t_1, t_2\ \ |\Lambda_\mu(t_1) - \Lambda_\mu(t_2)| \le C|t_1 - t_2| \max(|t_1|, |t_1|^{q-1}, |t_2|, |t_2|^{q-1})\,;$$

then

$$\Big| \int_0^1 \Lambda_\mu\big(g_1(x)\big)\,\mathrm{d}x - \int_0^1 \Lambda_\mu\big(g_2(x)\big)\,\mathrm{d}x \Big| \le \int_0^1 |\Lambda_\mu\big(g_1(x)\big) - \Lambda_\mu\big(g_2(x)\big)|\,\mathrm{d}x \le$$

$$
\begin{aligned}
&\le C\langle |g_1-g_2|, \max(|g_1|,|g_1|^{q-1},|g_2|,|g_2|^{q-1})\rangle \le \\
&\qquad \le C\|g_1-g_2\|_q \|\max(|g_1|,|g_1|^{q-1},|g_2|,|g_2|^{q-1})\|_p \,,
\end{aligned}
$$

and

$$
\begin{aligned}
&\|\max(\dots)\|_p = \|\max(|g_1|^{p/q},|g_1|,|g_2|^{p/q},|g_2|)\|_q^{q-1} \le \\
&\le \| \, |g_1|^{p/q}+|g_1|+|g_2|^{p/q}+|g_2| \, \|_q^{q-1} \le \big(\| |g_1|^{p/q}\|_q+\|g_1\|_q+\| |g_2|^{p/q}\|_q+\|g_2\|_q\big)^{q-1} = \\
&= \big(\|g_1\|_p^{p/q}+\|g_1\|_q+\|g_2\|_p^{p/q}+\|g_2\|_q\big)^{q-1} \le \\
&\le \big(4\max(\|g_1\|_p^{p/q},\|g_1\|_q,\|g_2\|_p^{p/q},\|g_2\|_q)\big)^{q-1} = \\
&= 4^{q-1}\max\big(\|g_1\|_p,\|g_1\|_q^{q-1},\|g_2\|_p,\|g_2\|_q^{q-1}\big) \le \\
&\qquad \le 4^{q-1}\max\big(\|g_1\|_q,\|g_1\|_q^{q-1},\|g_2\|_q,\|g_2\|_q^{q-1}\,.\big)
\end{aligned}
$$

□

5a4 Lemma. For every $g \in L_q$,

$$
\frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g) \to \frac12 \|g\|_2^2 \quad \text{as } \varepsilon \to 0\,.
$$

Proof. First, the bounded case: $g \in L_\infty$; we have then

$$
\frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g) = \int_0^1 \frac{1}{\varepsilon^2}\Lambda_\mu\big(\varepsilon g(x)\big)\,\mathrm{d}x \to \int_0^1 \frac12 g^2(x)\,\mathrm{d}x\,,
$$

since $\frac{1}{\varepsilon^2}\Lambda_\mu(\varepsilon g(\cdot)) \to \frac12 g^2(\cdot)$ uniformly.

Second, the general case; given $\delta > 0$, we take $g_\delta \in L_\infty$ such that $\|g_\delta - g\|_q \le \delta$; by 5a3, $|\Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta)| \le \text{const} \cdot \varepsilon^2 \delta$ with a constant that depends on g but does not depend on ε, δ (as long as $|\varepsilon| \le 1$, $\delta \le 1$). We get

$$
\begin{aligned}
&\Big|\frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g) - \frac12\|g\|_2^2\Big| \le \\
&\quad \le \underbrace{\frac{1}{\varepsilon^2}\Big|\Lambda_\infty(\varepsilon g) - \Lambda_\infty(\varepsilon g_\delta)\Big|}_{\le \text{const}\cdot\delta} + \underbrace{\Big|\frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g_\delta) - \frac12\|g\|_2^2\Big|}_{\to 0 \text{ as } \varepsilon\to 0} + \underbrace{\Big|\frac12\|g_\delta\|_2^2 - \frac12\|g\|_2^2\Big|}_{\le\text{const}\cdot\delta}\,,
\end{aligned}
$$

thus, $\limsup_{\varepsilon\to 0}\big|\frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g) - \frac12\|g\|_2^2\big| \le \text{const}\cdot\delta$ for all δ. □

***Proof** of Prop. 5a1.* $\frac1n\Lambda_n(\varepsilon g) = \Lambda_\infty(A_n \varepsilon g)$; by 5a3, $|\Lambda_\infty(\varepsilon A_n g) - \Lambda_\infty(\varepsilon g)| \le \text{const}\cdot\varepsilon^2\|A_n g - g\|_q$ with a constant that depends on g but does not depend on ε, n (as long as $|\varepsilon| \le 1$). Thus, $\big|\frac{1}{n\varepsilon^2}\Lambda_n(\varepsilon g) - \frac{1}{\varepsilon^2}\Lambda_\infty(\varepsilon g)\big| \to 0$ as $n \to \infty$, uniformly on $|\varepsilon| \le 1$. It remains to use 5a4. □

For the (one-dimensional) distribution ν_n of $\langle S_n, g\rangle$, similarly to (4b7), we get

$$\frac{1}{n\varepsilon^2}\Lambda_{\nu_n}(\varepsilon t) \to \frac{1}{2}\|g\|_2^2 t^2 \quad \text{as } \varepsilon \to 0, n \to \infty\,, \tag{5a5}$$

since $\Lambda_{\nu_n}(\varepsilon t) = \ln \mathbb{E} \exp\big(\varepsilon t\langle S_n, g\rangle\big) = \Lambda_n(\varepsilon t g)$.

5b Gärtner-Ellis, again

Dimension 1

Let probability measures $\nu_1, \nu_2, \dots$ on $\mathbb{R}$ be such that

$$\frac{1}{n\varepsilon^2}\Lambda_{\nu_n}(\varepsilon t) \to \frac{1}{2}t^2 \quad \text{as } \varepsilon \to 0, n \to \infty \tag{5b1}$$

for all $t \in \mathbb{R}$. (In particular, $\nu_n = \nu^{*n}$ satisfy it, provided that $\int x\,\nu(\mathrm{d}x) = 0$ and $\int x^2\,\nu(\mathrm{d}x) = 1$, since $\frac{1}{n\varepsilon^2}\Lambda_{\nu_n}(\varepsilon t) = \frac{1}{\varepsilon^2}\Lambda_\nu(\varepsilon t) \to \frac{1}{2}t^2$.)

5b2 Example. It may seem that (4c1) with $\Lambda(t) \sim \frac{1}{2}t^2$ (for $t \to 0$) implies (5b1). But this is an illusion. Here is a counterexample.

Let $\frac{1}{\sqrt{n}} \ll a_n \ll 1$ (that is: $a_n \to 0$ and $\sqrt{n}a_n \to \infty$), and

$$\nu_n = \frac{1}{2}\mu^{*n} + \frac{1}{4}(\delta_{-na_n} + \delta_{na_n})\,;$$

here $\mu = N(0,1)$ is the standard normal distribution (thus, $\mu^{*n} = N(0,n)$), and δ_x is the unit atom at x. Then

$$\Lambda_{\nu_n}(t) = \ln\Big(\frac{1}{2}\exp\frac{nt^2}{2} + \frac{1}{2}\cosh na_n t\Big)\,.$$

On one hand,

$$\frac{1}{n}\Lambda_{\nu_n}(t) \to \frac{1}{2}t^2 \quad \text{as } n \to \infty\,,$$

since for $t = 0$ this holds trivially, otherwise $na_n t = o(nt^2)$ for large n.

On the other hand, taking ε_n such that $\frac{1}{\sqrt{n}} \ll \varepsilon_n \ll a_n$ we get

$$\frac{1}{n\varepsilon_n^2}\Lambda_{\nu_n}(\varepsilon_n t) \ge \frac{1}{n\varepsilon_n^2}\ln\Big(\frac{1}{4}\exp na_n\varepsilon_n t\Big) = \frac{a_n}{\varepsilon_n}t + \mathcal{O}\Big(\frac{1}{n\varepsilon_n^2}\Big) \to \infty \quad \text{as } n \to \infty\,.$$

By the way, these ν_n violate 5b3 below.

The Legendre transform of $\Lambda(t) = \frac{1}{2}t^2$ is $\Lambda^*(x) = \frac{1}{2}x^2$ (recall 2c6).

5b3 Exercise.

$$\nu_n[n\varepsilon x,\infty) \le \exp\big(-\tfrac12 x^2 n\varepsilon^2 + o(n\varepsilon^2)\big) \quad \text{for } x \ge 0\,;$$
$$\nu_n(-\infty,n\varepsilon x] \le \exp\big(-\tfrac12 x^2 n\varepsilon^2 + o(n\varepsilon^2)\big) \quad \text{for } x \le 0\,.$$

Of course, these $o(\dots)$ are meant for $\varepsilon \to 0, n \to \infty$.

Prove it.[1]

It follows that $\nu_n(n\varepsilon a, n\varepsilon b) \to 1$ as $\varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty$, whenever $a < 0 < b$.

For tilted measures $\nu_{n,\varepsilon t}$ we have $\Lambda_{\nu_{n,\varepsilon t}}(\varepsilon s) = \Lambda_{\nu_n}(\varepsilon t + \varepsilon s) - \Lambda_{\nu_n}(\varepsilon t)$, thus $\frac{1}{n\varepsilon^2}\Lambda_{\nu_{n,\varepsilon t}}(\varepsilon s) \to \frac12 (t+s)^2 - \frac12 t^2 = ts + \frac12 s^2$; the corresponding Legendre transform is $\Lambda_t^*(x) = \frac12 (x-t)^2$ (since generally $\Lambda_t^*(x) = \Lambda^*(x) - tx + \Lambda(t)$, as noted after 4c2). Similarly to (4c3),

(5b4)
$$\nu_{n,\varepsilon t}(n\varepsilon a, n\varepsilon b) \to 1 \quad \text{as } \varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty, \text{whenever } a < t < b\,.$$

Taking into account that

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}\nu_{n,\varepsilon t}}(\varepsilon x) = \exp\big(-\varepsilon t \varepsilon x + \Lambda_{\nu_n}(\varepsilon t)\big) \ge \exp\big(-n\varepsilon^2 \max(ta,tb) + \Lambda_{\nu_n}(\varepsilon t)\big)$$

for $x \in (na, nb)$, we get, similarly to (4e4),

$$\nu_n(n\varepsilon a, n\varepsilon b) \ge \exp\big(-n\varepsilon^2 \max(ta,tb) + n\varepsilon^2 \cdot \tfrac12 t^2 + o(n\varepsilon^2)\big) \tag{5b5}$$

whenever $a < t < b$.

Similarly to 4c5 (but simpler), if $x \ge 0$ and $\delta > 0$ then

$$\nu_n\big(n\varepsilon x, n\varepsilon(x+\delta)\big) \ge \exp\big(-\tfrac12 x^2 n\varepsilon^2 + o(n\varepsilon^2)\big)\,,$$

and similarly to 4c6,

$$\nu_n\big(n\varepsilon x, n\varepsilon(x+\delta)\big) = \exp\big(-\tfrac12 x^2 n\varepsilon^2 + o(n\varepsilon^2)\big)\,.$$

Dimension d

All limits, as well as symbols $o(\dots)$, $\mathcal{O}(\dots)$ are taken for $\varepsilon \to 0, n \to \infty, n\varepsilon^2 \to \infty$ (unless stated otherwise).

Let probability measures $\nu_1, \nu_2, \dots$ on $\mathbb{R}^d$ be such that

$$\frac{1}{n\varepsilon^2}\Lambda_{\nu_n}(\varepsilon t) \to \frac12 |t|^2 \quad \text{for all } t \in \mathbb{R}^d\,. \tag{5b6}$$

[1] Hint: similar to 4c2.

5b7 Theorem. (a) For every nonempty closed set $F \subset \mathbb{R}^d$,

$$\limsup \frac{1}{n\varepsilon^2} \ln \nu_n(n\varepsilon F) \le - \min_{x \in F} \frac{1}{2}|x|^2 \, .$$

(b) For every open set $U \subset \mathbb{R}^d$,

$$\liminf \frac{1}{n\varepsilon^2} \ln \nu_n(n\varepsilon U) \ge - \inf_{x \in U} \frac{1}{2}|x|^2 \, .$$

5b8 Exercise (upper bound for a half-space)**.**

$$\nu_n\bigl(\{n\varepsilon x : \langle t, x \rangle - \tfrac{1}{2}|t|^2 \ge c\}\bigr) \le \exp\bigl(-cn\varepsilon^2 + o(n\varepsilon^2)\bigr)$$

for all $t \in \mathbb{R}^d$ and $c \ge 0$.

Prove it.

5b9 Exercise (half-space not containing the expectation)**.** If $c > 0$, then

$$\exists \delta > 0 \;\; \nu_n(\{n\varepsilon x : \langle t, x \rangle \ge c\}) = \mathcal{O}(\mathrm{e}^{-\delta n \varepsilon^2}) \, .$$

Prove it.

5b10 Exercise (lower bound)**.** If $U \subset \mathbb{R}^d$ is open, then

$$\ln \nu_n(n\varepsilon U) \ge -n\varepsilon^2 \inf_{x \in U} \frac{1}{2}|x|^2 + o(n\varepsilon^2) \, .$$

Prove it.

5b11 Exercise. Prove Theorem 5b7.[1]

The simple rate function $\frac{1}{2}|\cdot|^2$ leads to a simple formula for half-spaces. Every closed half-space $H \subset \mathbb{R}^d$ not containing 0 is

$$H = \{x : \langle x, x_H \rangle \ge |x_H|^2\}$$

where x_H is the point of H closest to 0. Now, 5b8 with $t = x_H$ and $c = \frac{1}{2}|x_H|^2$ gives

$$\nu_n(n\varepsilon H) \le \exp\bigl(-\tfrac{1}{2}|x_H|^2 n\varepsilon^2 + o(n\varepsilon^2)\bigr) \, ; \tag{5b12}$$

we see very clearly that every $x \ne 0$ belongs to (a) a closed half-space that satisfies the upper bound with rate $\frac{1}{2}|x|^2$, and (b) an open half-space that satisfies the upper bound with rate arbitrarily close to $\frac{1}{2}|x|^2$.

[1] Hint: recall the proof of 4c10(a).

5c Exponential tightness

What about a weakly compact set $K \subset L_p$ such that $\mathbb{P}\big(S_n \notin n\varepsilon K\big) \le \exp\big(-Cn\varepsilon^2 + o(n\varepsilon^2)\big)$ (for a given C)? No, this cannot happen. Indeed, on one hand, K must be bounded, that is, $K \subset \{f : \|f\|_p \le R\}$ for some R; on the other hand, $\|S_n\|_1 = |X_1| + \dots + |X_n|$; $\mathbb{E}\,\|S_n\|_1 = n\mathbb{E}\,|X_1|$; $\mathbb{P}\big(S_n \in n\varepsilon K\big) \le \mathbb{P}\big(\|S_n\|_p \le n\varepsilon R\big) \le \mathbb{P}\big(\|S_n\|_1 \le n\varepsilon R\big)$ is close to 0 (rather than 1) when $n\varepsilon R \ll \mathbb{E}\,\|S_n\|_1$, that is, $\varepsilon \ll \mathbb{E}\,|X_1|/R$.

The joint compactification introduced in Sect. 4b and used successfully for large deviations, fails for moderate deviations. We need another joint compactification. The L_p-norm feels only absolute values of $X_1, \dots, X_n$. But we have $\mathbb{E}\,X_1 = 0$, and cancellation of positive and negative summands should not be ignored.

We sacrifice invariance under permutations of the random variables $X_1, \dots, X_n$ (thus, by the way, complicating generalization to, say, two-dimensional arrays of random variables) and take indefinite integrals of the functions S_n (and others). We move to the space $C_0[0,1]$ of all continuous functions on $[0,1]$ vanishing at 0, and redefine the random function S_n as such piecewise-linear function of $C_0[0,1]$:

$$S_n(x) = \int_0^x \big(nX_1 1\!\!1_{(0,\frac1n)} + \dots + nX_n 1\!\!1_{(\frac{n-1}{n},1)} \big)\,.$$

Note that indefinite integrals of functions of L_p (or L_1) are absolutely continuous; they are dense in the space $C_0[0,1]$, but far not the whole space. In this sense, we really move to a larger space.

We also need Hölder spaces $C_{0,\alpha}$ and Hölder norms $\|\cdot\|_\alpha$ for $\alpha \in (0,1)$,

$$\|f\|_\alpha = \sup_{0<x<y<1} \frac{|f(y)-f(x)|}{(y-x)^\alpha} \in [0,\infty] \quad \text{for } f \in C_0[0,1]\,,$$
$$C_{0,\alpha} = \{ f \in C_0[0,1] : \|f\|_\alpha < \infty \}\,.$$

For $0 < \alpha \le \beta < 1$ we have $\|\cdot\|_\alpha \le \|\cdot\|_\beta$ and $C_{0,\alpha} \supset C_{0,\beta}$.

The unit ball $B_\alpha = \{f : \|f\|_\alpha \le 1\}$ is separable, but not compact (in $C_{0,\alpha}$).[1] However, B_α is compact in $C_0[0,1]$.[2] Note that Hölder functions need not be absolutely continuous.

We also redefine operators A_n; now $A_n f$ is the function linear on $[0,\frac1n]$, $[\frac1n,\frac2n]$, …, $[\frac{n-1}{n},1]$ and equal to f at $0, \frac1n, \frac2n, \dots, \frac{n-1}{n}, 1$.

[1] Try $f_n(x) = \min(x^\alpha, 1/n)$.

[2] Hint: in this situation, convergence on a dense countable set implies uniform convergence. In fact, moreover, B_β is compact in $C_{0,\alpha}$ whenever $0<\alpha<\beta<1$; hint: if $f,g \in B_\beta$ satisfy $|f(x)-g(x)| \le \frac1n$ for $x = \frac1n, \frac2n, \dots, \frac{n-1}{n}$, then $\|f-g\|_\alpha \le 4/n^{\beta-\alpha}$.

For a piecewise-linear function $f = A_n f$ we have

$$\|f\|_\alpha = \max_{0\le k<l\le n} \frac{1}{(\frac{l}{n} - \frac{k}{n})^\alpha} \Big| f\Big(\frac{l}{n}\Big) - f\Big(\frac{k}{n}\Big) \Big| ;$$

indeed, $\frac{|f(y)-f(x)|}{(y-x)^\alpha}$ cannot be maximal between the nodes $\frac{0}{n}, \frac{1}{n}, \dots, \frac{n}{n}$ due to concavity of the function $x \mapsto x^\alpha$. For such f,

$$\|f\|_\alpha = \max_{0\le k<l\le n} |\langle f', g_{k,l} \rangle| \quad \text{where } g_{k,l} = \frac{n^\alpha}{(l-k)^\alpha} 1\!\!1_{(\frac{k}{n}, \frac{l}{n})} \, .$$

We note that $\|g_{k,l}\|_q = \big(\frac{l-k}{n} \big)^{\frac{1}{q} - \alpha} \le 1$ for $\alpha \le 1/q$. We use 5b3,

$$\mathbb{P}\big(\|S_n\|_\alpha \ge n\varepsilon x \big) \le$$
$$\le \sum_{k,l} \mathbb{P}\big(|\langle S'_n, g_{k,l} \rangle| \ge n\varepsilon x \big) \le 2 \binom{n+1}{2} \exp\Big(-\frac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2) \Big) \, ,$$

and get

$$\mathbb{P}\big(\|S_n\|_\alpha \ge n\varepsilon x \big) \le \exp\big(-\tfrac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2) + \mathcal{O}(\ln n) \big)$$

for $\alpha \le 1/q$.

From now on, all limits, as well as symbols $o(\dots)$, $\mathcal{O}(\dots)$ are taken for $\varepsilon \to 0, n \to \infty, \frac{n\varepsilon^2}{\ln n} \to \infty$ (unless stated otherwise). Note the logarithmic gap between moderate deviations and central limit theorem.

Now, for $\alpha \le 1/q$ we have

(5c1) $$\mathbb{P}\big(\|S_n\|_\alpha \ge n\varepsilon x \big) \le \exp\big(-\tfrac{1}{2} x^2 n\varepsilon^2 + o(n\varepsilon^2) \big) \, ,$$

which is exponential tightness; K_C is the ball xB_α (with x such that $x^2/2 = C$) endowed with the compact topology from $C_0[0,1]$.

5d Mogulskii's theorem, again

We interpret $\|f'\|_2$ as $+\infty$ if f is not the indefinite integral of a function of $L_2[0,1]$. As before, all limits, as well as symbols $o(\dots)$, $\mathcal{O}(\dots)$ are taken for $\varepsilon \to 0, n \to \infty, \frac{n\varepsilon^2}{\ln n} \to \infty$ (unless stated otherwise). Also, $1 < p \le 2 \le q < \infty$, $\frac{1}{p} + \frac{1}{q} = 1$, and $\alpha \le 1/q$.

5d1 Theorem. (a) For every nonempty closed set $F \subset C_0[0,1]$,

$$\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in F \Big) \le - \min_{f \in F} \frac{1}{2} \|f'\|_2^2 \, .$$

(b) For every open set $U \subset C_0[0,1]$,

$$\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\frac{1}{n\varepsilon} S_n \in U \Big) \ge - \inf_{f \in U} \frac{1}{2} \|f'\|_2^2 \, .$$

5d2 Remark. Weaker conditions on F and U are sufficient for the theorem (and the proof): for all $R > 0$,

$$F \cap RB_\alpha \text{ is closed,}$$
$$U \cap RB_\alpha \text{ is relatively open in } RB_\alpha\,;$$

here $RB_\alpha = \{Rf : f \in B_\alpha\} = \{f : \|f\|_\alpha \le R\}$.

We choose a dense sequence $x_1, x_2, \dots \in [0,1]$ and denote $g_k = 1\!\!1_{(0,x_k)}$. If $f \in C_0[0,1]$ is the indefinite integral of a function of $L_2[0,1]$,

$$f(x) = \int_0^x f'(u)\,\mathrm{d}u\,,$$

then clearly $f(x_k) = \langle f', g_k\rangle$. It is convenient to denote $\langle f', g_k\rangle = f(x_k)$ for arbitrary $f \in C_0[0,1]$ (even though f' is ill-defined). We note that

$$(f_n \to f \text{ in } C_0[0,1]) \quad \Longleftrightarrow \quad \forall k \;\; \langle f'_n, g_k\rangle \xrightarrow[n\to\infty]{} \langle f', g_k\rangle$$

for all $f, f_1, f_2, \dots \in B_\alpha$.

We fix d for a while, and enumerate $x_1, \dots, x_d$ in ascending order:

$$\{x_1, \dots, x_d\} = \{y_1, \dots, y_d\}\,, \quad 0 < y_1 < \dots < y_d < 1\,.$$

Here is an orthonormal basis in the d-dimensional space spanned by $g_1, \dots, g_d$:

$$h_1 = \frac{1}{\sqrt{y_1}} 1\!\!1_{(0,y_1)},\; h_2 = \frac{1}{\sqrt{y_2 - y_1}} 1\!\!1_{(y_1,y_2)},\; \dots,\; h_d = \frac{1}{\sqrt{y_d - y_{d-1}}} 1\!\!1_{(y_{d-1},y_d)}\,.$$

Naturally, we let $\langle f', h_i\rangle = \frac{1}{\sqrt{y_i - y_{i-1}}}\big(f(y_i) - f(y_{i-1})\big)$ (where $y_0 = 0$). We introduce linear operators $T_d : C_0[0,1] \to \mathbb{R}^d$ by

$$T_d f = \big(\langle f', h_1\rangle, \dots, \langle f', h_d\rangle\big)\,;$$

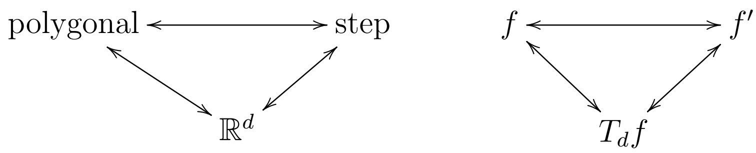
they are continuous.

Similarly to A_n, we define operator $\tilde{A}_d : C_0[0,1] \to C_0[0,1]$; $\tilde{A}_d f$ is the function linear on $[0,y_1]$, $[y_1,y_2]$, $\dots$, $[y_{d-1},y_d]$, equal to f at $0, y_1, \dots, y_d$, and constant on $[y_d,1]$. Thus, $(\tilde{A}_d f)' = \langle f', h_1\rangle h_1 + \dots + \langle f', h_d\rangle h_d$ and $\langle f', (\tilde{A}_d g)'\rangle = \langle (\tilde{A}_d f)', (\tilde{A}_d g)'\rangle = \langle (\tilde{A}_d f)', g'\rangle$ (like the orthogonal projection, but f', g' are ill-defined). Note that $\|(\tilde{A}_d f)'\|_2 = \|T_d f\|_2$ and $\langle (\tilde{A}_d f)', (\tilde{A}_d g)'\rangle = \langle T_d f, T_d g\rangle$.

Now we have three "incarnations" of the d-dimensional Euclidean vector space:

- $*$ $\mathbb{R}^d$;
- $*$ subspace of $L_2[0,1]$ spanned by $g_1, \dots, g_d$ or, equivalently, by $h_1, \dots, h_d$, with the norm $\| \cdot \|_2$ (step functions);
- $*$ subspace $\{f : \tilde{A}_d f = f\}$ of $C_0[0,1]$, with the norm $f \mapsto \|f'\|_2$ (polygonal functions).

They are intertwined by a commutative diagram of linear isometries:



We turn to $d \to \infty$. Clearly,

$$f_n \to f \text{ in } C_0[0,1] \quad \Longleftrightarrow \quad \forall d \;\; T_d f_n \xrightarrow[n\to\infty]{} T_d f$$

for all $f, f_1, f_2, \dots \in B_\alpha$.

If $d_1 \le d_2$, then $\tilde{A}_{d_1} \tilde{A}_{d_2} = \tilde{A}_{d_1} = \tilde{A}_{d_2} \tilde{A}_{d_1}$, and $(\tilde{A}_{d_1} f)'$ is the orthogonal projection of $(\tilde{A}_{d_2} f)'$. Thus, $\|(\tilde{A}_d f)'\|_2$ is increasing (in d).

5d3 Lemma. $\|(\tilde{A}_d f)'\|_2 \uparrow \|f'\|_2$ (be it finite or infinite) as $d \to \infty$.

Proof. On one hand, if $f' \in L_2$, then $(\tilde{A}_d f)'$ is the orthogonal projection of f' to the subspace spanned by $g_1, \dots, g_d$; the union of these subspaces is dense in L_2, thus, $\|(\tilde{A}_d f)'\|_2 \uparrow \|f'\|_2$.

On the other hand, assume that $\lim_d \|(\tilde{A}_d f)'\|_2 < \infty$; we have to prove that $f' \in L_2$. The series

$$(\tilde{A}_1 f)' + (\tilde{A}_2 f - \tilde{A}_1 f)' + (\tilde{A}_3 f - \tilde{A}_2 f)' + \dots$$

consists of orthogonal summands, and its partial sums are bounded. It follows easily that these partial sums are a Cauchy sequence. Thus, the series converges:

$$(\tilde{A}_d f)' \to \varphi \quad \text{for some } \varphi \in L_2 \, .$$

We note that $\langle (\tilde{A}_k f)', g_d \rangle = \langle f', g_d \rangle$ when $k \ge d$; thus, it equals $\langle \varphi, g_d \rangle$; that is, $\int_0^{x_d} \varphi(u) \, \mathrm{d}u = f(x_d)$ for all d; this shows that $\varphi = f'$. $\square$

Denote by $\nu_{d,n}$ the distribution of $T_d S_n$. By 5a1,

$$\frac{1}{n\varepsilon^2} \Lambda_{\nu_{d,n}}(\varepsilon t_1, \dots, \varepsilon t_d) \to \frac{1}{2}(t_1^2 + \dots + t_d^2) \quad \text{as } n \to \infty$$

for all $(t_1, \dots, t_d) \in \mathbb{R}^d$, since $\Lambda_{\nu_{d,n}}(t_1, \dots, t_d) = \ln \mathbb{E} \exp\bigl(\varepsilon t_1 \langle S_n, h_1 \rangle + \dots + \varepsilon t_d \langle S_n, h_d \rangle\bigr) = \ln \mathbb{E} \exp \langle S_n, \varepsilon t_1 h_1 + \dots + \varepsilon t_d h_d \rangle = \Lambda_n(\varepsilon t_1 h_1 + \dots + \varepsilon t_d h_d)$.

Thus, Theorem 5b7 (as well as 5b8–(5b12)) applies to $\nu_{d,n}$ for given d. That theorem is formulated for $\mathbb{R}^d$, but may be transferred readily to the "step" or "polygonal" space. In all cases, the rate function is $\frac12 \|\cdot\|^2$.

5d4 Exercise. Let $g \in C_0[0,1]$ satisfy $g = \tilde A_d g$ (for a given d), and $H = \{f \in C_0[0,1] : \langle f', g' \rangle \ge \|g'\|_2^2\}$ (even though f' is ill-defined...). Then

(a) $H = \{f \in C_0[0,1] : \langle T_d f, T_d g \rangle \ge |T_d g|^2\}$;

(b) $\mathbb{P}\big(S_n \in n\varepsilon H\big) \le \exp\big(-\frac12 \|g'\|_2^2 n\varepsilon^2 + o(n\varepsilon^2)\big)$.

Prove it.

Our space $C_0[0,1]$ is not a finite-dimensional Euclidean space, nor a Hilbert space, and still, every $f \ne 0$ belongs to an open half-space that satisfies the upper bound with rate arbitrarily close to $\Lambda_\infty^*(f)$. Indeed, if $c < \Lambda_\infty^*(f)$ (being the latter finite or infinite), then $\frac12 \|(\tilde A_d f)'\|_2^2 > c$ for d large enough; we take such d, and introduce $g = (1-\delta)\tilde A_d f$ with $\delta > 0$ small enough, then $\frac12 \|g'\|_2^2 \ge c$ and $g = \tilde A_d g$; the half-space $H = \{f_1 \in C_0[0,1] : \langle f_1', g' \rangle > \|g'\|_2^2\}$ is open in $C_0[0,1]$ (think, why), $f \in H$ (think, why), and $\mathbb{P}\big(S_n \in n\varepsilon H\big) \le \exp\big(-cn\varepsilon^2 + o(n\varepsilon^2)\big)$ by 5d4(b).

5d5 Exercise. Prove Theorem 5d1(a).

5d6 Exercise. Let $U \subset C_0[0,1]$ be open, and $f_0 \in U \cap B_\alpha$. Then there exist d and $\delta > 0$ such that

$$\forall f \in B_\alpha \ \big(|T_d f - T_d f_0| \le \delta \implies f \in U \big) .$$

Prove it.[1]

5d7 Exercise. $\|f\|_{1/2} \le \|f'\|_2$ for all $f \in C_0[0,1]$ (be the norms finite or infinite). (Here $\|\cdot\|_{1/2}$ is the Hölder norm for $\alpha = 1/2$, while $\|\cdot\|_2$ is the L_2 norm.)

Prove it.

Also, $\alpha \le \frac1q$ and $p \le 2 \le q$, thus, $\|f\|_\alpha \le \|f\|_{1/2} \le \|f'\|_2$.

***Proof** of Theorem 5d1(b).* [2] Let $f_0 \in U$; we'll prove that $\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\big(S_n \in n\varepsilon U\big) \ge -\frac12 \|f_0'\|_2^2$, assuming $\|f_0'\|_2 < \infty$ (otherwise the claim is void). We take $R > \|f_0'\|_2$, then $f_0 \in RB_\alpha$ by 5d7, and $\limsup \frac{1}{n\varepsilon^2} \ln \mathbb{P}\big(\|S_n\|_\alpha \ge Rn\varepsilon\big) < -\frac12 \|f_0'\|_2^2$ by 5c1. Exercise 5d6 gives d and $\delta > 0$ such that $\forall f \in RB_\alpha \ \big(|T_d f - T_d f_0| \le \delta \implies f \in U\big)$. It is sufficient to prove that

$$\liminf \frac{1}{n\varepsilon^2} \ln \mathbb{P}\Big(\Big\| T_d \frac{S_n}{n\varepsilon} - T_d f_0 \Big\| < \delta \Big) \ge - \inf_{x : \|x - T_d f_0\| < \delta} \frac12 |x|^2 ,$$

[1] Hint: similar to 4e7.

[2] Quite similar to the proof of Theorem 4e1(b).

since $\inf_{x:|x-T_d f_0|<\delta} \frac12 |x|^2 \le \frac12 |T_d f_0|^2 = \frac12 \|(\tilde A_d f_0)'\|_2^2 \le \frac12 \|f_0'\|^2$. Theorem 5b7(b) gives the needed inequality, since $\nu_{d,n}\bigl(\{n\varepsilon x : |x - T_d f_0| < \delta\}\bigr) = \mathbb{P}\bigl(|T_d \frac{S_n}{n\varepsilon} - T_d f_0| < \delta\bigr)$. $\square$

5d8 Exercise. A fair coin is tossed n times, giving $(\beta_1, \dots, \beta_n) \in \{0,1\}^n$. Given a continuous $\varphi : [0,1] \to (0,\infty)$, consider

$$p_n = \mathbb{P}\Bigl(\forall k = 1, \dots, n \quad \frac{2(\beta_1 + \dots + \beta_k) - k}{n^{2/3}} \le \varphi\bigl(\tfrac{k}{n}\bigr)\Bigr)\,.$$

(a) Prove that

$$p_n = 1 - \exp\bigl(-a n^{1/3}(1 + o(1))\bigr) \quad \text{for } n \to \infty$$

for some $a > 0$;

(b) find a when $\varphi(x) = 1 + vx$ for a given $v > 0$;

(c) find a when $\varphi(x) = \max(1 + vx, y)$ for given $v > 0$ and $y > 1$;

(d) find a when $\varphi(x) = 1 + cx^2$ for a given $c > 0$;

(e) find a when $\varphi(x) = 1 + c\sqrt{x}$ for a given $c > 0$.

Index