6 Dissipation

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In a slightly non-quasistatic process, the entropy increases.

6a Introduction

All quasistatic processes are reversible. In contrast, everyday life is full of irreversible processes. For quasistatic processes, 'adiabatic' means 'isentropic' (constant entropy), while an irreversible process can be adiabatic (perfect thermal insulation) but not isentropic.

Reversible processes are investigated by equilibrium statistical mechanics; irreversible processes — by nonequilibrium statistical mechanics.

In order to avoid the limiting procedure $n \to \infty$ we define the equilibrium state at a given inverse temperature β as the probability measure $\nu = \frac{\mathrm{e}^{-\beta h} \cdot \mu}{\int \mathrm{e}^{-\beta h} \, \mathrm{d}\mu}$ (the canonical ensemble), where (Ω, μ) is the given measure space (the phase space), not necessarily a power of another (one-particle) space; and h is the Hamiltonian.

The temperature is well-defined only for equilibrium states. Then, what is the meaning of the famous heat equation ($\frac{\partial}{\partial t} u = \frac{\partial^2}{\partial x^2} u$, etc)? It is about the so-called local equilibrium, — approximate equilibrium within 'cells' that are macroscopically small (in space and time) but microscopically large. (When needed, matter and radiation are treated via separate cells.) The local relaxation time is much shorter than global.

Thus, in most cases it is enough to understand near-equilibrium states.

We'll consider an adiabatic process driven by a Hamiltonian of the form

$$h_t = h + \varepsilon \varphi(t) g \,, \quad t \in [0, 1] \,, \tag{6a1}$$

where $h, g : \Omega \to \mathbb{R}$, $\varphi : [0,1] \to \mathbb{R}$, $\varphi(0) = 0 = \varphi(1)$. Denote by u_0 the initial energy and u_1 the final energy. It appears that

$$u_1 - u_0 = \varepsilon^2 \beta Q(\varphi) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0\,, \tag{6a2}$$

where Q is a *nonnegative* quadratic form (generally non-zero). Thus, the energy only increases, provided that $\beta > 0$. The differential entropy is constant, since relaxation is not stipulated in this theory. In practice, the state returns to equilibrium by relaxation, and the increase of energy leads to an increase of entropy.

In the case $\beta < 0$ (negative temperature) the energy decreases. Especially, in a laser, thermal energy turns into mechanical energy. However, it still leads to increase of entropy.

In order to get (6a2) we need substantially richer structure on the phase space than the measure space structure used before.

6b Symplectic manifold: dimension two

A Hamiltonian $h : \Omega \to \mathbb{R}$ generates dynamics via a vector field. The vector $\dot{x}$ at a point $x \in \Omega$ results from the differential $\mathrm{d}h(x)$ via a linear operator. In contrast to the gradient vector, $\dot{x}$ is tangent to a level surface of h, since the dynamics preserves energy.

An observable $f : \Omega \to \mathbb{R}$ changes dynamically: $\dot{f}(x) = \nabla_{\dot{x}} f(x)$; this is a bilinear function of f and h denoted by $\{f, h\}$ and called the Poisson bracket[1] of f and h. So,

$$\dot{f}(x) = \nabla_{\dot{x}} f(x) = \{f, h\}(x)\,; \quad \dot{f} = \{f, h\}\,. \tag{6b1}$$

Clearly, $\{\text{const}, h\} = 0$. The energy conservation gives $\{h, h\} = 0$.

6b2 Example. A one-dimensional particle of mass m: $\Omega = \mathbb{R}^2$; $h(q,p) = \frac{1}{2m} p^2 + U(q)$ (kinetic and potential energy). The usual motion equations

$$\dot{q} = \frac{1}{m} p\,, \quad \dot{p} = -U'(q)$$

result from Poisson brackets

$$\{q, p\} = 1\,.$$

More formally, $\{Q, P\} = 1$ where $Q(q,p) = q$ and $P(q,p) = p$. Indeed, $h = \frac{1}{2m} P^2 + U(Q)$. Near a point (q,p) in the linear approximation $U \approx$

[1] Denoted also $[f,h]$ or (f,h), sometimes also $[h,f]$ or (h,f).

$U'(q)Q + \text{const}$ and $\frac{1}{2m}P^2 \approx \frac{p}{m}P + \text{const}$. Thus,

$$\dot{Q}(q,p) = \Big\{ Q, \frac{1}{2m}P^2 \Big\}(q,p) = \Big\{ Q, \frac{p}{m}P \Big\}(q,p) = \frac{p}{m}\{Q,P\}(q,p) = \frac{1}{m}p\,;$$
$$\dot{P}(q,p) = \{P, U(Q)\}(q,p) = \{P, U'(q)Q\}(q,p) = U'(q)\{P,Q\}(q,p) = -U'(q)\,.$$

A symplectic manifold is usually defined via differential forms, but can be defined equivalently via Poisson brackets.

6b3 Definition. A two-dimensional symplectic manifold consists of a two-dimensional smooth manifold Ω and a bilinear map $\{\cdot,\cdot\} : C^\infty(\Omega)\times C^\infty(\Omega) \to C^\infty(\Omega)$ such that

(a) for every $x \in \Omega$, $\{g,h\}(x)$ is a bilinear form of $\mathrm{d}g(x)$ and $\mathrm{d}h(x)$;

(b) $\{h,h\} = 0$ for all h;

(c) for every $x \in \Omega$ there exist $q,p \in C^\infty(\Omega)$ such that $\{q,p\} = 1$ in a neighborhood of x.[1]

Till the end of Sect. 6b, Ω is assumed to be a two-dimensional symplectic manifold.

6b4 Exercise. Show that $\{g,f\} = -\{f,g\}$.

We note that $C^\infty(\Omega)$ is not only a linear space but also a commutative algebra.

6b5 Exercise. $\{fg,h\} = f\{g,h\} + g\{f,h\}$; prove it.

6b6 Exercise. $\{\varphi(f),h\} = \varphi'(f)\{f,h\}$ for any smooth $\varphi : \mathbb{R} \to \mathbb{R}$. Prove it.

6b7 Example. A one-dimensional particle: $\Omega = \mathbb{R}^2$,

$$\{g,h\}(q,p) = \frac{\partial g(q,p)}{\partial q}\frac{\partial h(q,p)}{\partial p} - \frac{\partial g(q,p)}{\partial p}\frac{\partial h(q,p)}{\partial q}\,,$$

which leads to the well-known equations of motions,

$$\dot{q} = \frac{\partial h}{\partial p}\,, \quad \dot{p} = -\frac{\partial h}{\partial q}\,.$$

6b8 Example. Classical spin:[2] $\Omega = S^2 \subset \mathbb{R}^3$;

$$\{g,h\}(x) = \det\big(x, \operatorname{grad} g(x), \operatorname{grad} h(x)\big)\,;$$

here $x \in S^2$ is treated as a vector normal to the sphere, while $\operatorname{grad} g(x)$ and $\operatorname{grad} h(x)$ are treated as vectors tangent to the sphere.

[1] In every point of the neighborhood, not just at x.

[2] Not the same as the discrete classical spin used before.

6b9 Exercise. For the classical spin check the condition $\{q,p\} = 1$ for q,p defined by

$$q\big(\sqrt{1-z^2}\cos\varphi, \sqrt{1-z^2}\sin\varphi, z\big) = \varphi\,,$$
$$p\big(\sqrt{1-z^2}\cos\varphi, \sqrt{1-z^2}\sin\varphi, z\big) = z$$

whenever $-1 < z < 1$ and $-\pi < \varphi < \pi$.

In this case, if the Hamiltonian is z then the dynamics is rotation.

Dynamics generated by a Hamiltonian h is a one-parameter group $(T_t)_{t\in\mathbb{R}}$ of diffeomorphisms $T_t : \Omega \to \Omega$;

$$T_{s+t} = T_s T_t \quad \text{for all } s,t \in \mathbb{R}\,; \quad \forall x \; T_0 x = x\,.$$

For every $x \in \Omega$ the path $(x_t)_{t\in\mathbb{R}}$, $x_t = T_t x$, satisfies the differential equation

$$\frac{\mathrm{d}}{\mathrm{d}t} x_t = \dot{x}_t\,;$$

it is well-known that a solution exists, is unique, and

(6b10) $$(x,t) \mapsto T_t x \quad \text{is a } C^\infty\text{-map } \Omega\times\mathbb{R} \to \Omega\,.$$

By (6b1),

(6b11) $$\frac{\mathrm{d}}{\mathrm{d}t} f(x_t) = \nabla_{\dot{x}_t} f(x_t) = \{f,h\}(x_t)\,.$$

Note that

(6b12) $$\forall t,x \quad h(T_t x) = h(x)\,.$$

6b13 Lemma. Let $x \in \Omega$, $h \in C^\infty(\Omega)$, $\mathrm{d}h(x) \ne 0$. Then there exists $g \in C^\infty(\Omega)$ such that $\{g,h\} = 1$ in some neighborhood of x.

Proof (sketch). We use local coordinates q,p as in 6b3(c), assume $\frac{\partial h}{\partial p}(x) \ne 0$ (otherwise use $p, -q$ instead of q,p), and $q(x) = 0$, $p(x) = 0$. We define g by

$$g\big(T_t(0,p)\big) = t$$

for t and p small enough. Then $\{g,h\}\big(T_t(0,p)\big) = \frac{\mathrm{d}}{\mathrm{d}t} g\big(T_t(0,p)\big) = 1$. $\square$

Every h leads to a linear operator $f \mapsto \{f,h\}$ on $C^\infty(\Omega)$; by 6b5, this operator is a differentiation. The product $f \mapsto \{\{f,g\},h\}$ of two differentiations is not a differentiation, but the commutator

$$f \mapsto \{\{f,g\},h\} - \{\{f,h\},g\}$$

of two differentiations is a differentiation.[1] Surprisingly, it is $f \mapsto \{f, \{g, h\}\}$, which is well-known as the Jacobi identity,

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0\,. \tag{6b14}$$

6b15 Exercise. Prove the Jacobi identity.

6b16 Lemma. $\{f, g\}(T_t(\cdot)) = \{f(T_t(\cdot)), g(T_t(\cdot))\}$.

Proof (sketch). Using again local coordinates q, p as in 6b13(c) we see that the equality holds on a neighborhood of x for all t small enough, and for all f, g. If it holds for s and t then it holds for $s + t$. □

6b17 Exercise. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} f(T_t(\cdot)) = \{f(T_t(\cdot)), h\}\,.$$

Let q_1, p_1 be local coordinates around x, satisfying $\{q_1, p_1\} = 1$, and the same for q_2, p_2. Then the Jacobian

$$\begin{vmatrix} \frac{\partial q_2}{\partial q_1} & \frac{\partial q_2}{\partial p_1} \\ \frac{\partial p_2}{\partial q_1} & \frac{\partial p_2}{\partial p_1} \end{vmatrix} = 1$$

by 6b7, and so, integration in $\mathrm{d}q_1 \mathrm{d}p_1$ gives the same result as integration in $\mathrm{d}q_2 \mathrm{d}p_2$. We see that Ω carries a special measure, the so-called Liouville measure.

6b18 Lemma. Diffeomorphisms T_t preserve the Liouville measure.

Proof (sketch). Similar to the proof of 6b16. □

6c Symplectic manifold: dimension $2n$

A $2n$-dimensional symplectic manifold is defined similarly to 6b3, but (c) stipulates $q_1, \dots, q_n$ and $p_1, \dots, p_n$ such that $\{q_k, p_k\} = 1$ for all k, and $\{q_k, p_l\} = 0$, $\{q_k, q_l\} = 0$, $\{p_k, p_l\} = 0$ whenever $k \ne l$.

Formulas 6b4, 6b5, 6b6 and (6b14) still hold.

Lemma 6b13 generalizes as follows: $q_1, \dots, q_n$ and $p_1, \dots, p_n$ as above can be chosen such that $p_1 = h$.

Lemma 6b16 still holds.

The Liouville measure is $\mathrm{d}q_1 \mathrm{d}p_1 \dots \mathrm{d}q_n \mathrm{d}p_n$.

Lemma 6b18 still holds.

---

[1] Let $A(fg) = fAg + gAf$ and the same for B, then $(AB - BA)(fg) = A(fBg + gBf) - B(fAg + gAf) = \dots = f(AB - BA)g + g(AB - BA)f$.

6d The theorem formulated

In order to avoid some technicalities we restrict ourselves to *compact* symplectic manifolds. (For example, S^2 fits, as well as $S^2 \times S^2$; also a torus.)

Given $h, g \in C^\infty(\Omega)$ and a C^∞-smooth $\varphi : [0,1] \to \mathbb{R}$, $\varphi(0) = 0 = \varphi(1)$, we consider the time-dependent Hamiltonian (6a1) and the corresponding nonstationary dynamics: $S_t : \Omega \to \Omega$ for $t \in [0,1]$ such that for every $x \in \Omega$

$$\text{(6d1)} \qquad \frac{\mathrm{d}}{\mathrm{d}t} f(S_t x) = \{f, h_t\}(S_t x) = \{f, h\}(S_t x) + \varepsilon\varphi(t)\{f, g\}(S_t x)$$

for all t and all $f \in C^\infty(\Omega)$; and $S_0 x = x$. Once again, it is well-known that such S_t exist, are unique, and $(x, t) \mapsto S_t x$ is a C^∞-map $\Omega \times \mathbb{R} \to \Omega$ (recall (6b10)).

Given $\beta \in \mathbb{R}$ we consider the probability measure ν on Ω whose density w.r.t. the Liouville measure is $\mathrm{const} \cdot \mathrm{e}^{-\beta h}$.

6d2 Theorem.

$$\int_\Omega h\bigl(S_1(\cdot)\bigr)\,\mathrm{d}\nu - \int_\Omega h\,\mathrm{d}\nu = \varepsilon^2 \beta Q(\varphi) + o(\varepsilon^2) \quad \text{as } \varepsilon \to 0$$

where Q is a nonnegative quadratic form.[1]

6e Some integral relations

Still, Ω is a *compact* symplectic manifold. Also, μ is the Liouville measure.

6e1 Exercise. For all $f, g \in C^\infty(\Omega)$,

$$\int \{f, g\}\,\mathrm{d}\mu = 0\,.$$

Prove it.

6e2 Exercise. For all $f, g, h \in C^\infty(\Omega)$,

$$\int \{f, g\} h\,\mathrm{d}\mu = \int \{g, h\} f\,\mathrm{d}\mu = \int \{h, f\} g\,\mathrm{d}\mu\,.$$

Prove it.

6e3 Exercise. For all $f, \rho, h, h_1, h_2 \in C^\infty(\Omega)$,

$$\int \{f, h\}\rho\,\mathrm{d}\mu = \int f\{h, \rho\}\,\mathrm{d}\mu\,,$$

$$\int \{\{f, h_2\}, h_1\}\rho\,\mathrm{d}\mu = \int f\{h_2, \{h_1, \rho\}\}\,\mathrm{d}\mu\,.$$

Prove it.

[1] Q depends on β.

6f Some perturbation theory

The maps S_t of Sect. 6d are close to T_t (for small ε). Thus it helps to introduce

$$R_t = T_t^{-1} S_t \,, \quad R_t : \Omega \to \Omega \,;$$
$$S_t = T_t R_t \,, \quad \text{that is,} \quad \forall x \;\; S_t x = T_t(R_t x) \,;$$

note that T_t are a group, but S_t and R_t are not.

6f1 Exercise. For all $f \in C^\infty(\Omega)$ and $x \in \Omega$,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(S_t x) = \{f, h\}(S_t x) + \frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} f(T_t R_s x) \,.$$

Prove it.

Comparing it with (6d1) we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} f(T_t R_s x) = \varepsilon\varphi(t)\{f, g\}(T_t R_t x) = \varepsilon\varphi(t)\{f(T_t(\cdot)), g(T_t(\cdot))\}(R_t x)$$

for all f; replacing $f(T_t(\cdot))$ with f we get

$$\frac{\mathrm{d}}{\mathrm{d}s}\Big|_{s=t} f(R_s x) = \varepsilon\varphi(t)\{f, g(T_t(\cdot))\}(R_t x) \,,$$

that is,

$$\frac{\mathrm{d}}{\mathrm{d}t} f(R_t x) = \varepsilon\varphi(t)\{f, g_t\}(R_t x) \,,$$

where

$$g_t = g(T_t(\cdot)) \,;$$

this is the differential equation for R_t, with a small parameter ε. It can be solved iteratively:

$$f(R_t x) = f(x) + \varepsilon \int_0^t \varphi(s)\{f, g_s\}(R_s x) \, \mathrm{d}s \,,$$

thus,[1]

$$\begin{aligned}
f(R_t x) &= f(x) + O(\varepsilon) \,; \\
\{f, g_t\}(R_t x) &= \{f, g_t\}(x) + O(\varepsilon) \,; \\
f(R_t x) &= f(x) + \varepsilon \int_0^t \varphi(s)\{f, g_s\}(x) \, \mathrm{d}s + O(\varepsilon^2) \,; \\
\{f, g_t\}(R_t x) &= \{f, g_t\}(x) + \varepsilon \int_0^t \varphi(s)\{\{f, g_t\}, g_s\}(x) \, \mathrm{d}s + O(\varepsilon^2) \,;
\end{aligned}$$

[1] These $O(\varepsilon^k)$ are uniform in $x \in \Omega$ and $t \in [0, 1]$.

iterations may be continued, but we exit here:

$$f(R_1x) = f(x)+\varepsilon \int\limits_{0<t<1} \varphi(t)\{f,g_t\}(x)\,\mathrm{d}t+\varepsilon^2 \iint\limits_{0<s<t<1} \varphi(s)\varphi(t)\{\{f,g_t\},g_s\}(x)\,\mathrm{d}s\mathrm{d}t+O(\varepsilon^3)\,.$$

In terms of $f_1 = f(T_1(\cdot))$ we get $f(S_1x) = f(T_1R_1x) = f_1(R_1x)$, thus

$$f(S_1x) = f_1(x)+\varepsilon \int\limits_{0<t<1} \varphi(t)\{f_1,g_t\}(x)\,\mathrm{d}t+\varepsilon^2 \iint\limits_{0<s<t<1} \varphi(s)\varphi(t)\{\{f_1,g_t\},g_s\}(x)\,\mathrm{d}s\mathrm{d}t+O(\varepsilon^3)\,.$$

Using 6e2, 6e3 we get for every $\rho \in C^\infty(\Omega)$,[1]

$$\int f(S_1(\cdot))\rho\,\mathrm{d}\mu - \int f_1\rho\,\mathrm{d}\mu =$$
$$= \varepsilon \int\limits_{0<t<1} \mathrm{d}t\varphi(t) \int f_1\{g_t,\rho\}\,\mathrm{d}\mu+\varepsilon^2 \iint\limits_{0<s<t<1} \mathrm{d}s\mathrm{d}t\varphi(s)\varphi(t) \int f_1\{g_t,\{g_s,\rho\}\}\,\mathrm{d}\mu+O(\varepsilon^3)\,.$$

6g Proving the theorem

We take $f = h$ (the Hamiltonian that generates T_t); then $f_1 = h$ by (6b12). We also take $\rho = \mathrm{const}\cdot \mathrm{e}^{-\beta h}$ such that $\nu = \rho\cdot\mu$.

6g1 Exercise. For every $f \in C^\infty(\Omega)$,

$$\{f,\rho\} = -\beta\rho\{f,h\}\,.$$

Prove it.

In particular, $\{h,\rho\} = 0$.

Using 6e2, $\int f_1\{g_t,\rho\}\,\mathrm{d}\mu = \int \{g_t,\rho\}h\,\mathrm{d}\mu = \int \{\rho,h\}g_t\,\mathrm{d}\mu = 0$. Thus, the first-order term disappears; the energy change is

$$\int h(S_1(\cdot))\,\mathrm{d}\nu - \int h\,\mathrm{d}\nu = \int h(S_1(\cdot))\rho\,\mathrm{d}\mu - \int h\rho\,\mathrm{d}\mu =$$
$$= \varepsilon^2 \iint\limits_{0<s<t<1} \mathrm{d}s\mathrm{d}t\varphi(s)\varphi(t) \int h\{g_t,\{g_s,\rho\}\}\,\mathrm{d}\mu + O(\varepsilon^3)\,.$$

By 6e2,

$$\int h\{g_t,\{g_s,\rho\}\}\,\mathrm{d}\mu = \int \{g_s,\rho\}\{h,g_t\}\,\mathrm{d}\mu\,.$$

[1] No problem with the order of integration, since the integrand is continuous on a compactum.

By 6g1,
$$\{g_s, \rho\} = -\beta\rho\{g_s, h\} \,.$$
Thus,
$$\int h\{g_t, \{g_s, \rho\}\} \, \mathrm{d}\mu = \beta \int \{g_s, h\}\{g_t, h\}\rho \, \mathrm{d}\mu \,.$$
By 6b17,
$$g'_t = \frac{\mathrm{d}}{\mathrm{d}t} g_t = \{g_t, h\} \,.$$
Thus,
$$\int h\{g_t, \{g_s, \rho\}\} \, \mathrm{d}\mu = \beta \int g'_s g'_t \, \mathrm{d}\nu \,,$$
and the energy change is, up to $O(\varepsilon^3)$,[1]

$$\varepsilon^2 \iint\limits_{0<s<t<1} \mathrm{d}s\mathrm{d}t \varphi(s)\varphi(t)\beta \int g'_s g'_t \, \mathrm{d}\nu = \frac{\varepsilon^2}{2}\beta \int \mathrm{d}\nu \iint\limits_{0<s,t<1} \mathrm{d}s\mathrm{d}t \varphi(s)\varphi(t) g'_s g'_t =$$
$$= \frac{\varepsilon^2}{2}\beta \int \mathrm{d}\nu \biggl(\int_0^1 \varphi(s) g'_s \, \mathrm{d}s \biggr) \biggl(\int_0^1 \varphi(t) g'_t \, \mathrm{d}t \biggr) =$$
$$= \frac{\varepsilon^2}{2}\beta \int \mathrm{d}\nu \biggl(\int_0^1 \varphi(t) g'_t \, \mathrm{d}t \biggr)^2 = \varepsilon^2 \beta Q(\varphi) \,,$$

where
$$Q(\varphi) = \frac{1}{2} \int \mathrm{d}\nu \biggl(\int_0^1 \varphi(t) g'_t \, \mathrm{d}t \biggr)^2 \ge 0 \,.$$

6h Fluctuation-dissipation relation

The quadratic form Q was introduced in relation to the perturbed Hamiltonian (6a1), but appeared to be also related to the unperturbed Hamiltonian h (and its dynamics T_t):

$$Q(\varphi) = \frac{1}{2} \int \mathrm{d}\nu \biggl(\int_0^1 \varphi'(t) g(T_t(\cdot)) \, \mathrm{d}t \biggr)^2 = \frac{1}{2} \mathbb{E}\, G^2 \,,$$

where G is a random variable defined on the probability space (Ω, ν) by

$$G(x) = \int_0^1 \varphi'(t) g(T_t(x)) \, \mathrm{d}t \,.$$

[1] As before, no problem with the order of integration, since the integrand is continuous on a compactum.

That is, G is an integral linear combination of the values $g(T_t(x))$ of the observable g at various times $t \in [0,1]$, where the initial state x is chosen at random from the canonical ensemble $\nu = \text{const} \cdot \mathrm{e}^{-\beta h} \cdot \mu$.

6h1 Exercise. Prove that $\mathbb{E}\, G = 0$.

Thus,

$$Q(\varphi) = \frac{1}{2} \operatorname{Var} G \,.$$

Clearly, the quadratic form Q is not identically 0, unless g is constant on almost every trajectory.

The relation between the energy dissipation and the variance of an observable is well-known as fluctuation-dissipation relation (FDR) or fluctuation-dissipation theorem.

> Its importance comes first from its great generality: very few assumptions are necessary for its derivation. Moreover, it exhibits a beautiful link between equilibrium and nonequilibrium statistical mechanics. Finally, it provides us with simple expressions for microscopic quantities in terms of macroscopic observables.[1]

In many cases the values $g(T_t(x))$ at macroscopically different times t are nearly independent random variables,[2] thus $\operatorname{Var} G \approx \text{const} \cdot \int_0^1 \varphi'^2(t) \, \mathrm{d}t$ and

$$Q(\varphi) \approx \text{const} \cdot \int_0^1 \varphi'^2(t) \, \mathrm{d}t$$

provided that φ' is not too large. This is a very simple picture of dissipation: it is local in time; at time t the energy dissipates at the rate $\text{const} \cdot \varphi'^2(t)$.

Sometimes it happens that[3] $\operatorname{Var} G \approx \text{const} \cdot \int_0^1 \varphi''^2(t) \, \mathrm{d}t$, and then the dissipation rate is $\text{const} \cdot \varphi''^2(t)$.

The FDR generalizes easily to

$$h_t = h + \varepsilon\varphi_1(t) g_1 + \dots + \varepsilon\varphi_k(t) g_k$$

and further, to

$$h_t = h + \varepsilon\varphi(t) \,,$$

this time $\varphi(t) \in C^\infty(\Omega)$ for each t.

---

[1] R. Balescu, "Equilibrium and nonequilibrium statistical mechanics", 1975, Sect. 21.3 "The fluctuation-dissipation theorem".

[2] That is, $(g_t)_t$ is nearly proportional to the white noise.

[3] That is, $(g_t)_t$ is nearly proportional to the derivative of the white noise.

6i Hints to exercises

6b4: consider $\{f+g, f+g\} - \{f-g, f-g\}$.

6b5: $\mathrm{d}(fg) = \dots$

6b9: for $\varphi = 0$ we have $x = (\sqrt{1-z^2}, 0, z)$, $\operatorname{grad} q(x) = (0, 1/\sqrt{1-z^2}, 0)$ and $\operatorname{grad} p(x) = (-z\sqrt{1-z^2}, 0, 1-z^2)$.

6b15: calculating in local coordinates (q,p) such that $\{q,p\} = 1$ and $h(q,p) = p$ we have $\{g,h\} = g_1$ (it means, $\frac{\partial}{\partial q} g$), $\{f,g\} = f_1 g_2 - f_2 g_1$ by 6b7, $\{\{f,g\},h\} = f_{11} g_2 + f_1 g_{21} - f_{21} g_1 - f_2 g_{11}$, ...

6b17: (6b11), (6b12) and 6b16.

6e1: $\frac{\mathrm{d}}{\mathrm{d}t} \int (T_t f) \, \mathrm{d}\mu = 0$ by 6b18.

6e2: $\int \{fg, h\} \, \mathrm{d}\mu = 0$; use 6b5.

6f1: $\frac{\mathrm{d}}{\mathrm{d}t} f(T_t R_t x) = \frac{\partial}{\partial s}|_{s=t} f(T_s R_t x) + \frac{\partial}{\partial s}|_{s=t} f(T_t R_s x)$.

6g1: 6b6.

6h1: $\int g(T_t(\cdot)) \, \mathrm{d}\nu$ does not depend on t.

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