Exam of 10.09.2001 — Solutions

1

1a

We have $Y = \varphi(X)$ where φ is defined by

$$\varphi(x) = 10 \int_x^{x+0.1} h(t)\, dt\,.$$

Also,

$$\mathbb{E}\varphi(X) = \int_0^1 \varphi(x)\, dx\,,$$

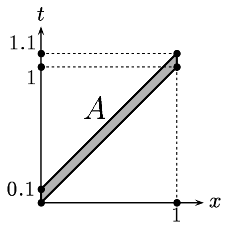
since X is distributed uniformly on $(0, 1)$. Fubini theorem gives

$$\mathbb{E}Y = \int_0^1 \left(10 \int_x^{x+0.1} h(t)\, dt \right) dx = 10 \iint_A h(t)\, dt\, dx\,,$$

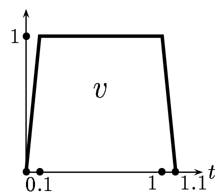
where $A = \{(x, t) : 0 \le x \le 1, x \le t \le x + 0.1\}$. Further,

$$10 \iint_A h(t)\, dt\, dx = 10 \int \left(\int h(t) \mathbf{1}_A(x, t)\, dx \right) dt = \int h(t) \underbrace{\left(10 \int \mathbf{1}_A(x, t)\, dx \right)}_{v(t)} dt\,.$$

1b



$$v(t) = \begin{cases} 10t & \text{for } 0 \le t \le 0.1, \\ 1 & \text{for } 0.1 \le t \le 1, \\ 11 - 10t & \text{for } 1 \le t \le 1.1, \\ 0 & \text{otherwise.} \end{cases}$$



1c

The function v satisfies $v(t) \ge 0$ for all t, and $\int v(t)\, dt = 1$. Therefore it is the density of some random variable Z. We have

$$\mathbb{E}h(Z) = \int h(t) v(t)\, dt = \mathbb{E}Y\,.$$

2

2a

The conditional distribution of X given $N = n$ is the distribution of $\frac{1}{2}(Z_1^2 + \dots + Z_{2n}^2)$. However, $\frac{1}{2}Z_1^2 \sim \text{Gamma}(0.5)$; thus $\frac{1}{2}(Z_1^2 + \dots + Z_{2n}^2) \sim \text{Gamma}(2n \cdot \frac{1}{2}) = \text{Gamma}(n)$, and so,

$$f_{X|N=n}(x) = \frac{1}{(n-1)!} x^{n-1} e^{-x} \,.$$

The marginal density is

$$f_X(x) = \mathbb{E} f_{X|N}(x) = \sum_n f_{X|N=n}(x) \mathbb{P}\left(N = n \right) = \sum_{n=1}^{\infty} \frac{1}{(n-1)!} x^{n-1} e^{-x} \cdot pq^{n-1} =$$

$$= e^{-x} p \sum_{k=0}^{\infty} \frac{1}{k!} (qx)^k = e^{-x} p e^{qx} = p e^{-(1-q)x} = p e^{-px} \,,$$

thus X has an exponential distribution, $X \sim \text{Exp}(p)$.

2b

$$p_{N|X=x}(n) = \frac{f_{X|N=n}(x) \mathbb{P}\left(N = n \right)}{f_X(x)} = \frac{\frac{1}{(n-1)!} x^{n-1} e^{-x} \cdot pq^{n-1}}{pe^{-px}} = \frac{1}{(n-1)!} (qx)^{n-1} e^{-qx} \,;$$

thus, $\mathbb{P}\left(N - 1 = k \mid X = x \right) = p_{N|X=x}(k+1) = \frac{1}{k!}(qx)^k e^{-qx}$, which means that the conditional distribution of $N - 1$ (given $X = x$) is a Poisson distribution, $P(qx)$.

2c

We have $\mathbb{E}\left(N - 1 \mid X = x \right) = qx$ (according to the Poisson distribution), that is, $\mathbb{E}\left(N \mid X = x \right) = qx + 1$, or $\mathbb{E}\left(N \mid X \right) = qX + 1$. However, $\mathbb{E}X = 1/p$ (according to the exponential distribution), and so,

$$\mathbb{E}\left(\mathbb{E}\left(N \mid X \right) \right) = \mathbb{E}(qX + 1) = q\mathbb{E}X + 1 = q \cdot \frac{1}{p} + 1 = \frac{1}{p} \,.$$

Also $\mathbb{E}N = \frac{1}{p}$ (according to the geometric distribution). So, $\mathbb{E}\left(\mathbb{E}\left(N \mid X \right) \right) = \mathbb{E}N$ indeed.

3

3a

By Borel-Cantelli lemma(s), finiteness of the set $\{n : X_n^2 + Y_n^2 < 100\}$ depends on convergence of the series

$$\sum_n \mathbb{P}\left(X_n^2 + Y_n^2 < 100 \right).$$

However,

$$\mathbb{P}\left(X_n^2 + Y_n^2 < 100 \right) = \frac{\pi \cdot 100}{\pi r_n^2} = \frac{100}{r_n^2}$$

for n large enough.

If $r_n = n$ then the series converges, and the set is finite (almost surely).

If $r_n = \sqrt{n}$ then the series diverges, and the set is infinite (almost surely).

3b ...

The result of (a) holds for the set $\{n : X_n^2 + Y_n^2 < r^2\}$ for every $r \in (0, \infty)$ (not just $r = 10$).

If $r_n = n$ then $\sqrt{X_n^2 + Y_n^2} \ge r$ for all n large enough; it means that $\sqrt{X_n^2 + Y_n^2} \to \infty$.

If $r_n = \sqrt{n}$ then the inequality $\sqrt{X_n^2 + Y_n^2} \ge 1$ is violated for infinitely many n,[1] therefore $\sqrt{X_n^2 + Y_n^2}$ does not tend to infinity.[2] ('Almost surely' is meant.)

3c ...

If $r_n = n$ then the set is not dense, since its intersection with the disk $x^2 + y^2 < 1$ is finite.[1]

If $r_n = \sqrt{n}$ then the set contains infinitely many points inside any disk $x^2 + y^2 < r^2$ (with the center at the origin). More generally, the same holds for any other disk, $(x-a)^2 + (y-b)^2 < r^2$ (with the center at (a, b)), except for a set of probability 0. We take an appropriate countable set of such disks (say, a, b, r run over rational numbers), exclude the union of corresponding sets of probability 0, and see that (X_n, Y_n) are dense (almost surely).

4

4a ...

The random variable $Y_1 = -\ln X_1$ has an exponential distribution, $Y_1 \sim \mathrm{Exp}(1)$; indeed,

$$\mathbb{P}\left(-\ln X_1 \le y \right) = \mathbb{P}\left(X_1 \ge e^{-y} \right) = 1 - e^{-y} \,.$$

Thus, $\mathbb{E} Y_1 = 1$ and $\mathrm{Var}(Y_1) = \sigma^2 \in (0, \infty)$. (In fact, $\sigma = 1$, but it does not matter.) By the central limit theorem, the distribution of Z_n (see the given hint) converges to the normal distribution $N(0, \sigma^2)$. We have

$$\mathbb{P}\left(X_1 \dots X_n \le e^{-n} \right) = \mathbb{P}\left(\ln X_1 + \dots + \ln X_n \le -n \right) = \\ = \mathbb{P}\left(Y_1 + \dots + Y_n \ge n \right) = \mathbb{P}\left(Z_n \ge 0 \right) \xrightarrow[n \to \infty]{} \Phi(0) = \frac{1}{2} \,,$$

so, $\lim_{n\to\infty} \mathbb{P}\left(X_1 \dots X_n \le e^{-n} \right) = 0.5$.

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[1] I use $r = 1$ here; any other number may be used equally well.

[2] Though, it is unbounded, of course.

4b ....

We have

$$\frac{1}{2} = \mathbb{P}\big(X_1 \dots X_n \le b_n \big) = \mathbb{P}\big(Y_1 + \dots + Y_n \ge -\ln b_n \big) =$$
$$= \mathbb{P}\Big(\frac{Y_1 + \dots + Y_n - n}{\sqrt{n}} \ge \frac{-\ln b_n - n}{\sqrt{n}} \Big) = \mathbb{P}\Big(Z_n \ge -\frac{n + \ln b_n}{\sqrt{n}} \Big) ,$$

which means that $-(n+\ln b_n)/\sqrt{n}$ is a median of Z_n. Therefore it converges (for $n \to \infty$) to the median of the normal distribution $N(0,\sigma^2)$,

$$-\frac{n + \ln b_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} \sigma \Phi^{-1}(0.5) = 0 \, .$$

Similarly, quartiles of Z_n converge to normal quartiles,

$$-\frac{n + \ln a_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} \sigma \Phi^{-1}(0.75) \, ,$$
$$-\frac{n + \ln c_n}{\sqrt{n}} \xrightarrow[n\to\infty]{} \sigma \Phi^{-1}(0.25) \, .$$

In other words,

$$\ln a_n = -n - \sqrt{n}\sigma\Phi^{-1}(0.75) + o(\sqrt{n}) \, ,$$
$$\ln b_n = -n - \sqrt{n}\sigma\Phi^{-1}(0.5) + o(\sqrt{n}) = -n + o(\sqrt{n}) \, ,$$
$$\ln c_n = -n - \sqrt{n}\sigma\Phi^{-1}(0.25) + o(\sqrt{n}) \, .$$

Therefore

$$\ln b_n - \ln a_n = \sqrt{n}\sigma\big(\Phi^{-1}(0.75) - \Phi^{-1}(0.5)\big) + o(\sqrt{n}) \xrightarrow[n\to\infty]{} +\infty \, ,$$

which means that $a_n/b_n \to 0$. Similarly $b_n/c_n \to 0$.

4c ....

By independence,

$$d_n = \mathbb{E}\,(X_1 \dots X_n) = (\mathbb{E}\,X_1) \dots (\mathbb{E}\,X_n) = \Big(\frac{1}{2}\Big)^n \, ,$$

that is,

$$\ln d_n = -n \ln 2 \, .$$

Therefore

$$\ln d_n - \ln c_n = -n\ln 2 + n + O(\sqrt{n}) = n \ln \frac{e}{2} + o(n) \xrightarrow[n\to\infty]{} +\infty \, ,$$

which means that $c_n/d_n \to 0$. So,

$$a_n < b_n < c_n < d_n$$

for all n large enough.