Tel Aviv University

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Floquet theory

for elliptic equations

Thesis submitted for the degree "Doctor of Philosophy"

by

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Acknowledgments

I wish to express my deep gratitude to my thesis advisor Professor V.Matsaev for his valuable guidance, essential support and permanent attention.

It is also a pleasure to thank to Professor N.Kopachevsky who introduced me to the wonderful world of Mathematics and support me during many years. I am also indebted to Professor I.Gohberg for many suggestions and encouragement.

I thank the staff of Tel Aviv University for the permanent help in technical problems.

Last but not least I want to thank my wife Natalie, my parents and my children for patience and warm moral support.

Abstract

The Floquet approach is a main tool of the theory of linear ordinary differential equations (o.d.e.) with periodic coefficients. Such equations arise in many physical and technical applications. At the same time, a lot of theoretical and applied problems that appear in quantum mechanics, hydrodynamics, solid state physics, theory of wave conductors, parametric resonance theory, etc. lead to periodic partial differential equations. The case of ordinary periodical differential equations is in a sharp contrast with a partial one, because a space of solutions of partial differential equation is, generally, an infinite-dimensional space. Methods of the classical Floquet theory (i.e. method of monodromy operator and method of substitution) in general are not applicable and new tools and technique are necessary.

The main idea of this thesis is to use the old technique of monodromy operator but from a new point of view. This approach turned out quite effective in the case of elliptic equations periodic with respect to one of variables and allowed to obtain in this case the same results as in the classical o.d.e. one (so, basis property of the set of Floquet solutions and distribution of Floquet multipliers).

In Chapter 2 we define the monodromy operator for general elliptic selfadjoint periodic problem as a shift operator on the space of solutions of this problem, similarly to the ordinary case. It is very difficult to deal with this operator since its definition is connected with solving the Cauchy problem for elliptic equations but we overcome this difficulty by considering appropriate elliptic boundary value problem instead of the Cauchy problem. In this way we reduce the spectral problem for the monodromy operator to the spectral problem for the quadratic operator pencil. Then we study this pencil by indefinite inner product techniques. The most of results obtained are related to the symmetric case, that is the case when the domain of the problem has a plane of symmetry and all coefficients of the problem are even w.r.t. this plane.

The main goal of Chapter 3 is the detailed studying the monodromy operator in the general case (without symmetry). First, we prove that the monodromy operator is closed and has a trivial kernel and a dense domain . Then we establish the quasi-isometric property for the monodromy operator and obtain some useful corollaries related to Floquet multipliers and Floquet solutions of the problem. Next, we study properties of Floquet multipliers (that is, spectral properties of the monodromy operator) in a more delicate way. For this purpose we classify the multipliers. Such a classification is standard in the indefinite scalar product theory. This approach has been applied by M.G.Krein for periodic systems of linear differential equations (i.e., in a finite-dimensional case).

In the next (and last) Chapter 4 we deal with elliptic selfadjoint periodic problems with a parameter. We are interested in a motion of the multipliers of such problems. Behavior of multipliers under small perturbation is very important in various theoretical and applied problems, generally speaking it is important in all problems that lead to periodic partial differential equations. To describe the motion of the multipliers we use the results obtained in the previous part of our work.

Contents

Chapter 1

Introduction

1. In this thesis we consider second order elliptic problems periodic with respect to one of variables and the Floquet theory for such problems.

The Floquet approach is a main tool of the theory of linear ordinary differential equations (o.d.e.) with periodic coefficients. Such equations arise in many physical and technical applications (see e.g. [17]). The following result, known as the *Floquet-Lyapunov theorem*, is central in this theory.

Consider the system

$$\frac{dy}{dt} = A(t)\, y, \tag{1.1}$$

where $A(t)$ is a 1-periodic $n \times n$ matrix-function, i.e. $A(t+1) = A(t)$ for all t. Then in the space of solutions of system (1.1) there exists a basis that consists of functions of the form

$$y(t) = e^{\mu t} \sum_{k=0}^{r} u_k(t) t^k, \qquad u_k(t+1) = u_k(t) \ \text{ for all } \ t, k. \tag{1.2}$$

In another words, there is a substitution $x = F(t)y$ with an invertible 1-periodic $n \times n$ matrix $F(t)$ that converts (1.1) into a system with constant coefficients and $Y(t) = e^{tK} F(t)$ (here $Y(t)$ is a fundamental system of solutions for (1.1) and K is a $n \times n$ constant matrix). Non-zero complex numbers $\lambda = e^{\mu}$ are said to be *Floquet multipliers* of equation (1.1). It is not difficult to show that the Floquet multipliers are eigenvalues of *monodromy operator* acting on the space of solutions of problem (1.1) by

$$U(y(t)) = y(t+1). \tag{1.3}$$

Moreover, the set of *Floquet solutions* (i.e. solutions of the form (1.2)) of problem (1.1) corresponds to the Jordan representation of the monodromy operator. As is well known [3, 17, 24] one can deduce numerous properties of equation (1.1) (stability, solvability of nonhomogeneous equations, exponential dichotomy, spectral theory, etc.) from the distribution of Floquet multipliers.

2. At the same time, a lot of theoretical and applied problems that appear in quantum mechanics, hydrodynamics, solid state physics, theory of wave conductors, parametric resonance theory, etc. lead to periodic partial differential equations. The first result of this type has been proved by Bloch [5]. It pertains to Bloch waves which are plane waves multiplied by periodic functions, and they have formed the basis of the theory of electrons in crystals, i.e. of the theory of solids. Further numerous extensions and generalizations of Bloch's idea gave a good tool for investigation the Schrödinger equation and related topics, see e.g. [15, 26, 29]. However, the case of ordinary periodical differential equations is in a sharp contrast with a partial one, because a space of solutions of partial differential equation is, generally, infinite-dimensional space. Methods of the classical Floquet theory (i.e. method of monodromy operator and method of substitution) in general are not applicable and new tools and technique are necessary.

Recently this problem attracts a big deal of attention, see e.g. monograph [21] and references there in. Series of deep results analogous to those from the classical Floquet theory (e.g. completeness of Floquet solutions in special classes of solutions of periodic partial differential equations) are received for elliptic and hypoelliptic equations and systems, parabolic problems, etc. However, the results obtained are not complete.

The main idea of our work is to use the old technique of monodromy operator but from a new point of view. This approach turned out quite effective in the case of elliptic equations periodic with respect to one of variables and allowed to obtain in this case numerous results concerning the set of Floquet solutions and the set of Floquet multipliers and to get the complete "general picture" as in the classical o.d.e. case.

3. Basis property of the set of Floquet solutions and distribution of Floquet multipliers are done in Chapter 2 of the thesis. In Sections 2.1 and 2.2 general elliptic selfadjoint problem of second order periodic with respect to one of variables is studying by monodromy operator technique. Section 2.1 is devoted to a particular case, namely to the case of Dirichlet problem for Schrödinger type equation $-\Delta u + qu = 0$. It is the simplest case but transition to general one doesn't demand any additional idea. However to generalize our considerations some technical complication is necessary. It is done in Section 2.2. The main points of these Sections are follows. An analogue of the monodromy operator is constructed on the space of solution of elliptic problem, namely , there is an operator U that acts similarly to (1.3). One can get the answer to the questions raised above (so, basis and distribution properties) by means of studying the spectrum and the set of eigenfunctions and associated functions of the operator U. However, it is very difficult to deal with this operator since its definition is connected with solving the Cauchy problem for elliptic equation. To overcome this difficulty we consider appropriate elliptic boundary value problem instead of the Cauchy problem. Solving this problem we come to quadratic operator pencil with a spectral problem equivalent to one for the monodromy operator in certain sense. The pencil is

$$\mathcal{L}(\lambda) = \lambda^2 A - \lambda B + A^*, \tag{1.4}$$

where A is a compact Hilbert-Schmidt operator with trivial kernel, B is a selfadjoint bounded from below operator with compact resolvent.

First conclusion on distribution of the Floquet multipliers follows from this equivalence immediately:

Corollary 2.1.6 (and Theorem 2.2.2) *The set of Floquet multipliers of second order elliptic selfadjoint problem periodic with respect to one of the variables (see problem (2.13)) is or* $\mathbf{C}$ *(the whole complex plane) either a countable subset of* $\mathbf{C}$ *with accumulation points* 0 *and* ∞*. This set is symmetric with respect to the unit circle, that is if* λ *is the Floquet multiplier of problem (2.13) then* $\overline{\lambda}^{-1}$ *is too, and the multiplicity of* λ *is the same as the multiplicity of* $\overline{\lambda}^{-1}$*.*

In Section 2.3 operator pencil (1.4) is studied by indefinite inner product techniques. Under one essential restriction, namely, some kind of symmetry for domain and coefficients of our problem, the main results of Chapter 1 are obtained. These results are formulated in the following two theorems:

Theorem 2.3.2

1) Under the symmetry assumption (see assumption (A) in Section 2.3) the set of Floquet multipliers of second order elliptic selfadjoint problem periodic with respect to one of the variables (see problem (2.13)) is a discrete set with two accumulation points 0 *and* ∞*, this set has double symmetry with respect to the real axis and with respect to the unit circle;*

2) the set of non-real multipliers is finite, if κ *is the number (with multiplicity) of non-positive eigenvalues of operator* F_{22} *(see Section 2.1), then exist at most* 4κ *multipliers belonging to neither the real axis nor the unit circle;*

3) there are at most 2κ *multipliers different from* ± 1 *with Floquet solution of positive order and the order is not grater than* 2κ *(if* $\lambda = \pm 1$ *is a multiplier then its order can reach* $4\kappa + 1$*).*

Theorem 2.3.3 *Under assumption (A) an arbitrary solution of problem (2.13) can be expanded in a series over Floquet solutions, the series converges in the sense of* $W^s_{2,loc}(\Omega)$ *topology for all* $s \geq 0$ *and in the sense of* $C^\infty_{loc}(\Omega)$ *topology.*

4. In Chapter 3 we study the properties of the monodromy operator and distribution of the multipliers in more detail. First, we establish that the monodromy operator is closed and its domain and range are dense sets. It is done in Section 3.1 (see **Theorem 3.1.1**). In Section 3.2 we obtain a quasi-unitary property of the monodromy operator (see **Theorem 3.2.2**). It is the well-known property, in a finite-dimensional case it means that a transformation preserve symplectic structure defined by a skew-scalar product. In Section 3.2 we mention some useful results on spectral properties of the monodromy operator. They are direct consequences of its quasi-unitarity.

Section 3.3 is devoted to classification of the multipliers. Such a classification is standard in the indefinite scalar product theory. Namely, we introduce the following definition (see Definitions 3.3.1 and 3.3.2):

A function f *is called* **positive** *(***negative** *or* **neutral***) if* $i[f,f] > 0$ *(* $i[f,f] < 0$ *or* $[f,f] = 0$*, where the skew-scalar product* $[f,f]$ *is defined in Section 3.2 (see (3.9)).*

A Floquet multiplier is called multiplier of the **first kind** *(***second kind** *or* **neu-**

tral) *if all corresponding eigenfunctions are positive (negative or neutral).*

This classification was successfully applied for periodic differential equations by M.G.Krein. In work [20] he studied by this approach periodic systems of linear differential equations. We use here the idea but not the methods of Krein's paper because a finite-dimensional case of system of linear o.d.e is in a sharp contrast with an infinite-dimensional case of partial differential problems that we are interested on. For example, we can not represent the monodromy operator in the form of matriciant and such representation is essential for Krein's work. Therefore, we use the results obtained in the previous sections of this thesis to establish some important properties of the Floquet multipliers and the eigenfunctions of the monodromy operator and these properties are done in the terms of the classification defined above, see **Propositions 3.3.3, 3.3.4** and **3.3.6, Theorems 3.3.7** and **3.3.11, Corollaries 3.3.8 - 3.3.10**.

5. The main goal of the last Chapter 4 is to study the motion of multipliers of a periodic elliptic problem with a parameter. Here we refer, once again, to Krein's work [20]. Periodic differential problems with a parameter were studied thoroughly in this paper for systems of linear differential equations, so in a finite-dimensional case. This chapter is a generalization of [20] on an infinite-dimensional case of partial differential problems. Behavior of multipliers under small perturbation is very important in various theoretical and applied problems, generally speaking it is important in all problems that lead to periodic partial differential equations.

The results obtained in Section 3.3 and standard general reasons say us that the general properties of nonunimodular multipliers do not depend on small perturbation, therefore the only case of great interests is the case of multipliers belonging to the unit circle.

We assume that all coefficients of the problem (3.1) depend on a complex parameter ϵ in such a way that the monodromy operator U is an analytic function of ϵ in a neighborhood of $\epsilon = 0$, the domain of definition of the operator $U(\epsilon)$ does not depend on ϵ and the conditions *(i) - (iv)* of (3.1) hold true for real values of ϵ. We assume, as well, that the energy of the problem (see (4.5)) is an increasing function of the parameter. Under these assumption we obtain the complete local description of motion of all multipliers which have not neutral eigenfunctions, namely:

All unimodular multipliers which have not neutral eigenfunctions remain on the unit circle under small perturbation, multipliers of the first kind move clockwise on the unit circle as ϵ increases and multipliers of the second kind move counterclockwise.

Thus a multiplier can jump off the unit circle only if it has a neutral eigenfunction.

The indefinite metric approach that we used in Chapter 3 and Chapter 4 was applied successfully to elliptic periodic problems by V.Derguzov, see [9]-[11]. But our case differs from Derguzov's one by three very essential points. First, Derguzov studied the problems periodic with respect to all variables and the problems periodic with respect to one of the variables, but such that the domains of the operator coefficients do not depend on this variable. It is impossible to reduce our problems to the problems of Derguzov. Second, quasi-unitary property does not imply automatically any property of the multipliers because the monodromy operator and its inverse are not bounded. To obtain the properties of the multipliers we need additional accurate reasonings, in Derguzov's works these reasonings and these properties are missing (actually, Derguzov does not prove even quasi-unitarity). Third, Derguzov studied the motion of multipliers for the case of linear dependence on the parameter only. Thus the results of Chapter 3 and Chapter 4 are new and can not be obtained from the papers of Derguzov.

The part of the results of this thesis were obtained jointly with V.Matsaev.

Chapter 2

Distribution of Floqet multipliers, property of the set of Floquet solutions

2.1 The case of Schrödinger type equation

1. We consider the following problem:

$$\begin{cases} -\Delta u + qu = 0 & (\Omega) \\ u = 0 & (\partial\Omega) \end{cases}, \tag{2.1}$$

where $\Omega \subset \mathbf{R}^3$ is an infinite and 1-periodic with respect to x_1 ($\mathbf{R}^3 \ni x = (x_1, x_2, x_3)$) domain with infinitely smooth boundary, such that any section of Ω by plane orthogonal to x_1 is bounded, $q(x)$ is a real infinitely smooth 1-periodic with respect to x_1 function, i.e. $q(x_1 + 1, x_2, x_3) = q(x_1, x_2, x_3)$ for all x.

Similarly to (1.2) the *Floquet solution* of problem (2.1) is said to be a solution of the form

$$u(x_1, x_2, x_3) = e^{\mu x_1} \sum_{k=0}^{r} x_1^k v_k(x_1, x_2, x_3),$$

$$v_k(x_1 + 1, x_2, x_3) = v_k(x_1, x_2, x_3) \quad \text{for all} \quad x, k.$$

As in the ordinary case the numbers e^{μ} are called the *Floquet multipliers* of the problem.

Properties of the Floquet solutions and the Floquet multipliers are strongly connected with spectral properties of the monodromy operator which we define in the following way.

Let us take the section of Ω by a smooth enough simple surface Γ_1 non-tangential to $\partial\Omega$. Let Γ_2 be the shift of Γ_1 by 1 along x_1. We denote the part of Ω between these sections by $\tilde{\Omega}$. Thus $\tilde{\Omega}$ is the *elementary cell* of the domain Ω or the fundamental domain of the group of shifts by $z \in \mathbf{Z}$ on Ω.

Let function $u(x)$ satisfy the equation and the boundary condition from (2.1) on the closure of the domain $\tilde{\Omega}$. We define the *monodromy operator* as

$$U \begin{pmatrix} u_1 \\ u_{n_1} \end{pmatrix} = \begin{pmatrix} u_2 \\ u_{n_2} \end{pmatrix}, \tag{2.2}$$

where u_j is the trace of $u(x)$ on Γ_j and u_{nj} is the normal derivative of $u(x)$ on Γ_j, $j = 1, 2$. We take here normals in the same directions, namely n_1 (on Γ_1) is an internal normal for the domain $\tilde{\Omega}$ and n_2 (on Γ_2) is an external one.

This definition is correct due to uniqueness theorem for Cauchy problem for second order elliptic equations (this theorem states that if u vanishes of infinite order at some point then u vanishes identically, see e.g. [16, theorem 17.2.6]), thus U is the linear operator on an infinite-dimensional space. But U is an unbounded operator connected with Cauchy problem so, as mentioned in the Introduction, it is very difficult to study U by direct methods. However it is possible to overcome this difficulty. Namely, instead of Cauchy problem we consider the elliptic boundary value problem in the elementary cell $\tilde{\Omega}$:

$$\begin{cases} -\Delta u + qu = 0 & (\tilde{\Omega}) \\ u = 0 & (\partial\Omega) \\ u = u_1 & (\Gamma_1) \\ u = u_2 & (\Gamma_2) \end{cases} . \tag{2.3}$$

Remark. The domain $\tilde{\Omega}$ has non-smooth boundary. The nature of the solution u of the Dirichlet problem changes as the domain becomes less smooth and the existence of the solution requires additional study. The solution is best described in terms of a notion called harmonic measure (see [8] and [18]). The domain $\tilde{\Omega}$ is Lipschitz domain and it is proved in [8] that on Lipschitz domain harmonic measure and surface measure are mutually absolutely continuous. Using this fact one can solve the Dirichlet problem on Lipschitz domain for all $u \in L_2(\partial\tilde{\Omega})$. The details can be found in [8] and [18].

2. In order to not complicate our reasonings by minor technical details, let us assume that homogeneous problem (2.3) (i.e. $u_1 = u_2 = 0$) has only the trivial solution. To overcome this unessential restriction and pass to the general case one can use generalized Green's function, for details see Remark 1 at the end of this section.

Under this assumption problem (2.3) has unique (in the sense of $L_2(\tilde{\Omega})$) solution. Let us consider the operator

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{n_1} \\ u_{n_2} \end{pmatrix}, \quad \text{or in the matrix form} \quad \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix}\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{n_1} \\ u_{n_2} \end{pmatrix}.$$

Thus F acts as follows: we solve problem (2.3) and take the normal derivatives of the solution on Γ_1 and Γ_2, this couple is the image of the couple of boundary data (u_1, u_2) under the action of the operator F. Obviously, the operators U and F are strongly connected and we can study the spectral problem for the operator U with help of the operator F.

Our nearest goal is to study properties of the operators F_{jk}. We can identify Γ_1 and Γ_2, so let us consider the functions u_k and $F_{jk}(u_k)$ as functions on the surface $\Gamma_1 = \Gamma_2$ with the standard two-dimensional Lebesgue measure.

Let the operators F_{jk} act on the space of such functions $L_2(\Gamma_1) = L_2(\Gamma_2)$. Note that F_{11} acts as follows: we solve (2.3) with $u = 0$ on Γ_2 and then take the normal derivative of the solution on Γ_1, this function is the image of the boundary data on Γ_1 under the action of F_{11}. Similarly, to find the image of the boundary data on Γ_1 under the action of F_{21}, we solve the same boundary problem as for F_{11} but take the normal derivative of the solution on Γ_2. To find the image of the boundary data on Γ_2 under the action of F_{22}, we solve (2.3) with $u = 0$ on Γ_1 and then take the normal derivative of the solution on Γ_2, to find the image of the boundary data on Γ_2 under the action of F_{12}, we solve the same problem as for F_{22} but take the normal derivative of the solution on Γ_1. The operators F_{jk}, $j = 1, 2$, $k = 1, 2$ are closed operators.

Proposition 2.1.1 *The operators F_{12} and F_{21} are compact Hilbert-Schmidt operators (i.e. operators of the class $\mathcal{S}_2$) on the space $L_2(\Gamma_1)$ (or, the same, $L_2(\Gamma_2)$).*

Proof. Due to Green's function of problem (2.3) we have for any smooth enough solution of (2.3):

$$u(y) = -\int_{\Gamma_1} u_1 \frac{\partial G(x,y)}{\partial n_x}\, dx \;+\; \int_{\Gamma_2} u_2 \frac{\partial G(x,y)}{\partial n_x}\, dx,$$

Here $G(x, y)$ is the classical Green's function of the problem

$$\begin{cases} -\Delta u + qu = f & (\tilde{\Omega}) \\ u = 0 & (\partial\tilde{\Omega}) \end{cases} .$$

This Poisson's formula is the simple consequence of Green's formula for the operator of problem (2.3) (pay attention that n_1 is an internal normal for domain $\tilde{\Omega}$ and n_2 is an external one). This formula implies explicit formulas for the operators F_{jk}. In particular, we have after closure for any u_1, $u_2 \in L_2(\Gamma_1) = L_2(\Gamma_2)$

$$\begin{aligned} F_{12}(u_2)(x) &= \textstyle\int_{\Gamma_2} \frac{\partial^2 G(y,x)}{\partial n_{1,x} \partial n_{2,y}}\, u_2(y)\, dy; \\ F_{21}(u_1)(x) &= \textstyle\int_{\Gamma_1} -\frac{\partial^2 G(y,x)}{\partial n_{2,x} \partial n_{1,y}}\, u_1(y)\, dy. \end{aligned} \tag{2.4}$$

Here $x \neq y$ (they belong to different parts of the boundary), so the kernels of this operators are continuous. It proves our proposition.

Proposition 2.1.2 $F_{12}^* = -F_{21}$, F_{11} *and* F_{22} *are symmetric operators,* F_{22} *is bounded from below,* F_{11} *is bounded from above.*

Proof. Let $u(x)$ and $v(x)$ be solutions of problem (2.1) and they are smooth enough (e.g. $u, v \in \mathbf{C}^2(\mathbf{R}^3)$). We apply second Green's formula for the operator $\Delta u - qu$ and the domain $\tilde{\Omega}$. From (2.1) $u = v = 0$ on $\partial\Omega$, n_1 is the internal normal for the domain $\tilde{\Omega}$, so we have

$$\int_{\Gamma_1} -u_1 \overline{\frac{\partial v}{\partial n_1}} + \frac{\partial u}{\partial n_1} \overline{v_1} = \int_{\Gamma_2} -u_2 \overline{\frac{\partial v}{\partial n_2}} + \frac{\partial u}{\partial n_2} \overline{v_2}. \tag{2.5}$$

Here u_1, u_2, v_1, v_2 are the traces of $u(x)$ and $v(x)$ on Γ_1 and Γ_2. But (2.3) has the unique solution, hence $u(x)$ and $v(x)$ are the solutions of problem (2.3) with boundary values $u = u_j$ on Γ_j, $v = v_j$ on Γ_j respectively. By taking $u_2 = v_1 = 0$ we have from (2.5)

$$\int_{\Gamma_1} -u_1 \overline{F_{12}(v_2)} = \int_{\Gamma_2} F_{21}(u_1) \overline{v_2},$$

and after closure with respect to the norm of L_2

$$\langle u_1, -F_{12}(v_2)\rangle \;=\; \langle F_{21}(u_1), v_2\rangle. \tag{2.6}$$

for all $u_1 \in L_2(\Gamma_1),\ v_2 \in L_2(\Gamma_2)$ (here $\langle\cdot,\cdot\rangle$ is the usual inner product in the space $L_2(\Gamma_1)$ (or, the same, $L_2(\Gamma_2)$).

Similarly, by the suitable choice of u and v, we have for all functions from the domains of operators F_{11} and F_{22}

$$\langle u_1, F_{11}(v_1)\rangle \;=\; \langle F_{11}(u_1), v_1\rangle \quad \text{and} \quad \langle u_2, F_{22}(v_2)\rangle \;=\; \langle F_{22}(u_2), v_2\rangle\,. \tag{2.7}$$

Equalities (2.6) and (2.7) prove the symmetry properties of the operators F_{11} and F_{22} and the formula $F_{12}^* \;=\; -F_{21}$. It still remains to prove the semi-boundedness for the operators F_{22} and F_{11}

First Green's formula for the Laplace operator and the domain $\tilde{\Omega}$ implies

$$\int_{\Gamma_1} -u_1\overline{\frac{\partial u}{\partial n_1}} + \int_{\Gamma_2} u_2\overline{\frac{\partial u}{\partial n_2}} \;=\; \int_{\tilde{\Omega}} u\,\overline{\Delta u} + \int_{\tilde{\Omega}} |\bigtriangledown u|^2 \;=\; \int_{\tilde{\Omega}} (q|u|^2 + |\bigtriangledown u|^2).$$

By taking $u_1 = 0$ we have after closure for all u_2 from the domain of the operator F_{22}

$$\langle F_{22}(u_2), u_2\rangle \;=\; \int_{\tilde{\Omega}} (q|u|^2 + |\bigtriangledown u|^2). \tag{2.8}$$

On the other hand for all solutions of (2.3) we have the standard estimate

$$\|u\|_{L_2(\tilde{\Omega})} \le c\|u_2\|_{L_2(\Gamma_2)}, \tag{2.9}$$

where c doesn't depend on u. In smooth domain this estimate follows from the classical regularity result for elliptic boundary value problem, namely: let u solve the Dirichlet problem for elliptic operator $Lu \;=\; 0$ on $\tilde{\Omega}$ and $u \;=\; f$ on $\partial\tilde{\Omega}$ then the operator $A : L_2(\partial\tilde{\Omega}) \;\to\; L_2(\tilde{\Omega}),\ A\,f \;=\; u$ is bounded. Our domain $\tilde{\Omega}$ has non-smooth boundary, however the regularity result for problem (2.3) holds true (see Remark 2 at the end of this section) and implies inequality (2.9). Let $q_0 = inf\ q(x),\ x \in \tilde{\Omega}$. Substitution (2.9) into (2.8) gives us

$$\langle F_{22}(u_2), u_2\rangle \ge q_0 \int_{\tilde{\Omega}} |u|^2 \ge q_0 c^2 \int_{\Gamma_2} |u_2|^2 \;=\; c_1\|u_2\|^2_{L_2(\Gamma_2)}$$

and F_{22} is bounded from below.

By taking $u_2 = 0$ we have from first Green's formula

$$\langle F_{11}(u_1), u_1 \rangle \; = \; - \int_{\tilde{\Omega}} (q|u|^2 + |\bigtriangledown u|^2)$$

instead of (2.8) and the same arguments prove now boundedness of F_{11} from above. The proposition is proved.

Proposition 2.1.3 *F_{11} and F_{22} are selfadjoint operators with compact resolvent.*

Proof. Let us prove this statement for the operator F_{22}. We prove it under assumption $Ker\ F_{22} = \{0\}$ (see arguments after (2.3) in behalf of this unessential restriction and see Remark 1 at the end of this section for it overcoming). Hence we have to prove F_{22}^{-1} is compact.

Consider the following problem:

$$\begin{cases} -\Delta u + qu = 0 & (\tilde{\Omega}) \\ u = 0 & (\partial\Omega \cup \Gamma_1) \\ \frac{\partial u}{\partial n} = \varphi & (\Gamma_2) \end{cases} . \tag{2.10}$$

The operator F_{22}^{-1} solves problem (2.10) and takes the trace of the solution on Γ_2, so $F_{22}^{-1}(\varphi) = u_2$. Problem (2.10) is a mixed problem with non-smooth boundary, there is a lot of papers devoted to regularity properties of such problems, we refer to the results of [28]. The main theorem of this paper implies boundedness of the operator T of problem (2.10) (so, $T\varphi = u$) as operator from $W_2^{-1/2}(\Gamma_2)$ to $W_2^1(\tilde{\Omega})$ (here W_2^s is a notation for the Sobolev space). We can present F_{22}^{-1} as

$$F_{22}^{-1} \; = \; V_2 \cdot \gamma \cdot T \cdot V_1,$$

where V_1 is the embedding operator from $L_2(\Gamma_2)$ to $W_2^{-1/2}(\Gamma_2)$, V_2 is the embedding operator from $W_2^{1/2}(\Gamma_2)$ to $L_2(\Gamma_2)$, γ is a trace operator, $\gamma u = u_2$, γ acts from $W_2^1(\tilde{\Omega})$ to $W_2^{1/2}(\Gamma_2)$. It follows from the classical embedding theorems that the operators V_1 and V_2 are compact and the operator γ is bounded. Hence F_{22}^{-1} is compact operator on $L_2(\Gamma_2)$. The first statement of this proposition (selfadjointness) follows from the second one proved above and from symmetry of F_{22} (symmetric operators with compact resolvent is selfadjoint, obviously).

The proposition is proved for F_{22}. The proof for F_{11} is the same.

Next proposition, the last one, follows immediately from the uniqueness theorem for Cauchy problem.

Proposition 2.1.4 *The operators F_{12} and F_{21} have trivial kernel.*

Indeed, if $F_{21}(u_1) = 0$, we get for the operator $-\Delta u + qu$ Cauchy problem with vanishing initial data on Γ_2, due to uniqueness its solution is trivial, so u_1 (which is the trace of this solution on Γ_1) is zero function. The same arguments are valid for the operator F_{12}.

3. Let us return to the spectral problem for monodromy operator (2.2). As we mentioned in the Introduction, to describe properties of the set of Floquet solutions and the set of Floquet multipliers one can study properties of eigenvalues and eigenfunctions of this operator, these two problems are equivalent (see Section 2.3 for details). So, we want to study the problem

$$U\begin{pmatrix} u_1 \\ u_{n_1} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_{n_1} \end{pmatrix} \quad \text{or, the same} \quad \begin{pmatrix} u_2 \\ u_{n_2} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_{n_1} \end{pmatrix}. \tag{2.11}$$

If we put $u_1 = u,\ u_{n_1} = v$, the last equation is equivalent to

$$F\begin{pmatrix} u \\ \lambda u \end{pmatrix} = \begin{pmatrix} v \\ \lambda v \end{pmatrix} \quad \text{or} \quad \begin{cases} F_{11}u + \lambda F_{12}u = v \\ F_{21}u + \lambda F_{22}u = \lambda v \end{cases}.$$

After multiplying the first equation by λ we have

$$\lambda F_{11}u + \lambda^2 F_{12}u = F_{21}u + \lambda F_{22}u.$$

Denote $F_{12} = A,\ (F_{22} - F_{11}) = B$. The last consideration and Propositions 2.1.1 - 2.1.4 prove the following result:

Theorem 2.1.5 *The spectral problem for monodromy operator (2.2) is equivalent to the spectral problem for the quadratic operator pencil*

$$\mathcal{L}(\lambda) = \lambda^2 A - \lambda B + A^*, \tag{2.12}$$

where A is a compact Hilbert-Schmidt operator with trivial kernel, B is a selfadjoint bounded from below operator with compact resolvent.

The equivalence of these problems means the following: operator (2.2) and pencil (2.12) have the same set of eigenvalues and u is an eigenfunction (associated function) corresponding to the eigenvalue λ of (2.12) if and only if it is a first component of the eigenfunction (associated function) corresponding to the same eigenvalue of (2.2).

For eigenvalues and eigenfunctions the statement is done above, for associated functions one can verify this statement by simple direct calculation, we omit it here.

The first conclusion on distribution of the Floquet multipliers follows from this theorem immediately:

Corollary 2.1.6 *The set of Floquet multipliers of problem (2.1) is or* $\mathbf{C}$ *(the whole complex plane) either a countable subset of* $\mathbf{C}$ *with accumulation points* 0 *and* ∞*. This set is symmetric with respect to the unit circle, that is if* λ *is the Floquet multiplier of problem (2.1) then* $\overline{\lambda}^{-1}$ *is too, and the multiplicity of* λ *is the same as the multiplicity of* $\overline{\lambda}^{-1}$*.*

Indeed, due to Theorem 2.1.5 the set of Floquet multipliers (or, the same, the set of eigenvalues of monodromy operator (2.2)) coincides with the set of eigenvalues of pencil (2.12). Proposition 2.1.3 implies existence of λ_0 such that $(B - \lambda_0 I)^{-1}$ is compact. Multiplying (2.12) by $\lambda^{-1}(B - \lambda_0 I)^{-1}$ (this operation does not change the set of nonzero eigenvalues) we get the analytic operator valued function on the domain $\mathbf{C} \setminus \{0\}$,

$$\mathcal{L}_1(\lambda) = \frac{1}{\lambda}\mathcal{L}(\lambda)(B-\lambda_0 I)^{-1} = \lambda A(B-\lambda_0 I)^{-1} - I - \lambda_0(B-\lambda_0 I)^{-1} + \frac{1}{\lambda}A^*(B-\lambda_0 I)^{-1},$$

and for every $\lambda \in \mathbf{C} \setminus \{0\}$ $\mathcal{L}_1(\lambda)$ is Fredholm operator of index zero. The first conclusion of the Corollary follows now from the classical result on spectrum of analytic Fredholm operator valued functions (see e.g. [12, theorem XI.8.2]) and Proposition 2.1.4 (to show that $\lambda = 0$ is not eigenvalue and so the set of different eigenvalues is infinite).

To check the second conclusion of the Corollary we assume that λ is the Floquet multiplier of multiplicity r of problem (2.1). Then due to Theorem 2.1.5 λ is an eigenvalue of multiplicity r of the pencil $\mathcal{L}(\lambda) = \lambda^2 A - \lambda B + A^*$. But

$$\frac{(\mathcal{L}(\lambda))^*}{\overline{\lambda}^2} = A^* - \overline{\lambda}^{-1} B + \overline{\lambda}^{-2} A = \mathcal{L}(\overline{\lambda}^{-1}).$$

Therefore, $\overline{\lambda}^{-1}$ is an eigenvalue of multiplicity r of (2.12) and then it is Floquet multiplier of the same multiplicity of problem (2.1).

The Corollary is proved.

We note that the alternative mentioned above (the set of Floquet multipliers is or $\mathbf{C}$ either a countable subset of $\mathbf{C}$ with accumulation points 0 and ∞) is known for various elliptic periodic problems, see for example [21, theorems 5.4.4, 5.4.9], [9], [10]. It is possible to deduce the first conclusion of the Corollary from these results, but instead, we prefer to give the direct proof, much more simple.

4. Remark 1. There were two restrictions done in the previous considerations. As was mentioned, one can overcome these restrictions easily. The first one is the uniqueness of solution of boundary value problem (2.3). If it is not so, problem (2.3) has a finite dimensional (due to Fredholmness) kernel and all results obtained above are valid with two minor changes: firstly, the operator F takes normal derivatives (on Γ_1 and Γ_2) of the unique solution of (2.3), which has the boundary values u_1, u_2 and belongs to orthogonal complement to the kernel of problem (2.3), and secondly, to get Poisson's formula (see the proof of Proposition 2.1.1) we have to use so-called generalized Green's function instead of the classical one, see e.g. [7, section V.14]. The second restriction was done in the proof of Proposition 2.1.3. We assumed that F_{22} has the trivial kernel. As above, if it is not the case, the operator F_{22} has a finite dimensional (due to Fredholmness) kernel and $F_{22} = diag[PF_{22}P, (I - P)F_{22}(I - P)]$, where P is an orthogonal projection on this kernel. The operator $\tilde{F}_{22} = (I - P)F_{22}(I - P)$ is invertible on the complement to the kernel and we can apply all arguments of the proof of Proposition 2.1.3 to prove $\tilde{F}_{22}^{-1}$ is compact, hence (the kernel of F_{22} has finite dimension!) F_{22} has a compact resolvent and Proposition 2.1.3 is valid.

Remark 2. The following regularity result has been used in this section. Let u solve the Dirichlet problem for elliptic operator $Lu = 0$ on $\tilde{\Omega}$ and $u = f$ on $\partial\tilde{\Omega}$ (see Remark after (2.3)) then the operator $A : L_2(\partial\tilde{\Omega}) \to L_2(\tilde{\Omega})$, $A\, f = u$ is bounded. The domain $\tilde{\Omega}$ is Lipschitz domain and regularity properties for boundary value problems on Lipschitz domains generalize the classical results for smooth domains. The statement formulated above follows immediately from [18, pp.62-63].

2.2 The general case

1. The main goal of this section is to extend the results obtained in the previous one to a more general selfadjoint case. Namely, we consider a second order boundary problem of the following form:

$$\begin{cases} Lu = L(x,D)u = \sum\limits_{j,k=1}^{3} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) + a_0(x)\,u \;=\; 0 \quad (\Omega) \\ Tu = 0 \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad (\partial\Omega) \end{cases}, \tag{2.13}$$

where

(i) $\Omega \subset \mathbf{R}^3$ is an infinite and 1-periodic with respect to x_1 ($\mathbf{R}^3 \ni x = (x_1, x_2, x_3)$) domain with infinitely smooth boundary, such that any section of Ω by plane orthogonal to x_1 is bounded,

(ii) all coefficients of L and T are infinitely smooth 1-periodic with respect to x_1 functions,

(iii) L is an elliptic operator and the differential expression of L is formally selfadjoint, that is $a_{jk}(x) = \overline{a_{kj}(x)}$ for all x and $j,k = 1,2,3$ and $a_0(x)$ is a real valued function,

(iv) T is a boundary differential operator of the first order or of the order zero, T satisfies so-called normality and covering conditions (see e.g. [22, Chapter 2, Section 1]), T is "formally selfadjoint" with respect to Green's formula.

The last condition means that for any sections of Ω by smooth enough non intersecting surfaces α_1 and α_2 there exists boundary operator S, order$T$+order$S$ = 1, such that for any smooth enough functions $u(x)$ and $v(x)$ vanishing on α_1 and α_2 we have after integration by parts

$$\int_{\Omega'} (Lu)\overline{v} \;=\; -\int_{\Omega'} \sum_{j,k=1}^{3} \left(a_{jk}\frac{\partial u}{\partial x_j}\overline{\frac{\partial v}{\partial x_k}} - a_0 u\overline{v} \right) \;+\; \int_{\partial\Omega'} Su\overline{Tv}. \tag{2.14}$$

Here Ω' is the part of Ω between the sections α_1 and α_2 and $\partial\Omega'$ is the part of $\partial\Omega$ between these sections.

Thus (2.13) is a regular elliptic problem, periodic with respect to x_1 and selfadjoint. The definitions of the *Floquet solutions* and the *Floquet multipliers* for problem (2.13) are the same that in the model problem, see the beginning of Section 2.1

2. We define the monodromy operator in the same way as in Section 2.1. So, we take the sections of Ω by smooth enough simple surfaces Γ_1 and Γ_2 non-tangential to $\partial\Omega$, where Γ_2 is the shift of Γ_1 by 1 along x_1, and define the *elementary cell* $\tilde{\Omega}$ as the part of Ω between these sections. For functions $u(x)$ that satisfy the equation and the boundary condition from (2.13) on the closure of the domain $\tilde\Omega$ we define the *monodromy operator* as

$$U\begin{pmatrix} u_1 \\ u_{\nu_1}\end{pmatrix} = \begin{pmatrix} u_2 \\ u_{\nu_2}\end{pmatrix}. \tag{2.15}$$

Here u_j is the trace of $u(x)$ on Γ_j, $u_{\nu j} = \frac{\partial u}{\partial \nu}$ on Γ_j, $j = 1, 2$, and ν is so-called *conormal* for the operator L, that is by definition

$$\frac{\partial}{\partial \nu} = \sum_{j,k=1}^{3} a_{jk} N_j \frac{\partial}{\partial x_k},$$

where $N = (N_1, N_2, N_3)$ is a unit vector of an internal (with respect to $\tilde\Omega$) normal on Γ_1 and of an external normal on Γ_2. Thus ν_1 (on Γ_1) is an internal conormal for the domain $\tilde\Omega$ and ν_2 (on Γ_2) is an external one, and definition (2.15) coincides with definition (2.2) for the model case up to change the normals $n_1,\ n_2$ to conormals $\nu_1,\ \nu_2$.

By standard technique integration by parts one can obtain (using (2.14)) the following generalized Green's formulas for the operator L on the domain $\tilde\Omega$ (pay attention that ν_1 is an internal conormal for the domain $\tilde\Omega$ and ν_2 is an external one):

$$\int_{\tilde\Omega} (Lu)\overline{v} = a(u, v) + \int_{\Gamma_2} u_{\nu_2}\overline{v}_2 - \int_{\Gamma_1} u_{\nu_1}\overline{v}_1 + \int_{\partial\Omega'} Su\overline{Tv}, \tag{2.16a}$$

here

$$a(u, v) = -\int_{\tilde\Omega} \sum_{j,k=1}^{3} \left(a_{jk} \frac{\partial u}{\partial x_j} \overline{\frac{\partial v}{\partial x_k}} - a_0 u\overline{v}\right).$$

This formula implies immediately the second one

$$\begin{aligned} \textstyle\int_{\tilde\Omega} (Lu)\overline{v} - u(\overline{Lv}) &= \textstyle\int_{\Gamma_2} (u_{\nu_2}\overline{v}_2 - u_2\overline{v_{\nu_2}}) - \int_{\Gamma_1} (u_{\nu_1}\overline{v}_1 - u_1\overline{v_{\nu_1}}) \\ &\quad + \textstyle\int_{\partial\Omega'} (Su\overline{Tv} - Tu\overline{Sv}). \end{aligned} \tag{2.16b}$$

Here $\partial\Omega'$ is a part of $\partial\Omega$ between Γ_1 and Γ_2.

Repeating the arguments of Section 2.1 we replace Cauchy problem connected with definition (2.15) by the elliptic boundary value problem in the elementary cell $\tilde\Omega$ (see Remark after (2.3):

$$\begin{cases} Lu = 0 & (\tilde\Omega) \\ Tu = 0 & (\partial\Omega) \\ u = u_1 & (\Gamma_1) \\ u = u_2 & (\Gamma_2) \end{cases}. \tag{2.17}$$

3. Reasoning as in Section 2.1, we assume that the homogeneous problem has only the trivial solution[1] and consider the operator

$$F\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{\nu_1} \\ u_{\nu_2} \end{pmatrix},$$

here u is a solution of problem (2.17), F is 2×2 operator-matrix $F = (F_{jk})$, $j,k = 1,2$. Thus we just repeat the arguments of Section 2.1 with some technical complication. Producing in this manner let us prove the following

Proposition 2.2.1 *Propositions 2.1.1 — 2.1.4 are valid for the operators F_{jk} defined above.*

Proof. Properly speaking, we have to repeat all the arguments from the proofs of Section 2.1 with the following modifications and remarks. In the same way as was done in the proof of Proposition 2.1.1 we get Poisson's formula for solutions of (2.17)

$$u(y) = \int_{\Gamma_1} u_1 \overline{G_{\nu_1}}\, dx - \int_{\Gamma_2} u_2 \overline{G_{\nu_2}}\, dx. \tag{2.18}$$

Here $G(x,y)$ is Green's function of the problem

$$\begin{cases} Lu = f & (\tilde{\Omega}) \\ Tu = 0 & (\partial\Omega) \\ u = 0 & (\Gamma_1 \cup \Gamma_2) \end{cases}$$

and we get this Poisson's formula by the simple substitution of G instead of v in (2.16b) as u is the solution of (2.17). This formula implies explicit formulas for F_{jk}, just the same as (2.4) up to replacement of operator $\frac{\partial}{\partial n}$ by operator $\frac{\partial}{\partial \nu}$.

Next, if u and v are solutions of (2.17), we have from (2.16b)

$$\int_{\Gamma_1} u_{\nu_1}\overline{v}_1 - u_1\overline{v_{\nu_1}} = \int_{\Gamma_2} u_{\nu_2}\overline{v}_2 - u_2\overline{v_{\nu_2}}\,.$$

This formula plays the same role as (2.5) in the proof of Proposition 2.1.2. Instead of (2.8) we have from (2.16a) for $u = v$ (u is a solution of (2.17))

$$-a(u,u) = \int_{\tilde{\Omega}} \sum_{j,k=1}^{3} \left(a_{jk}\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}} - a_0|u|^2 \right) = \int_{\Gamma_2} u_{\nu_2}\overline{u}_2 - \int_{\Gamma_1} u_{\nu_1}\overline{u}_1$$

[1]one can overcome this restriction in the same manner as in the model case, see Remark 1 at the end of Section 2.1

and (by taking $u_1 = 0$)

$$\langle F_{22}(u_2), u_2\rangle \;=\; -a(u,u) \;\geq\; \int_{\tilde{\Omega}} -a_0|u|^2 \;\geq\; c_0 \int_{\tilde{\Omega}} |u|^2, \quad c_0 = inf_{x\in\tilde{\Omega}}\left(-a_0(x)\right).$$

Here we use the fact that ellipticity of L implies

$$\sum_{j,k=1}^{3} a_{jk} \frac{\partial u}{\partial x_j} \overline{\frac{\partial u}{\partial x_k}} \;\geq\; 0$$

for all u smooth enough, for all $x \in \tilde{\Omega}$. Now we can proceed in the same way as in the proof of Proposition 2.1.2.

Next, we replace problem (2.10) by the problem

$$\begin{cases} Lu = 0 & (\tilde{\Omega}) \\ Tu = 0 & (\partial\Omega) \\ u = 0 & (\Gamma_1) \\ \frac{\partial u}{\partial \nu} = \varphi & (\Gamma_2) \end{cases}.$$

It is the unique change that we have to do in the proofs of Propositions 2.1.3 and 2.1.4.

The proposition is proved.

4. To study properties of the Floquet solutions and the Floquet multipliers we reduce the spectral problem for monodromy operator (2.15) to the spectral problem for a quadratic operator pencil. We do it in the same manner as in Section 2.1. Instead of spectral problem (2.11) we have

$$U\begin{pmatrix} u_1 \\ u_{\nu_1} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_{\nu_1} \end{pmatrix} \quad \text{or, the same} \quad \begin{pmatrix} u_2 \\ u_{\nu_2} \end{pmatrix} = \lambda \begin{pmatrix} u_1 \\ u_{\nu_1} \end{pmatrix}.$$

The last equation we can write in the form

$$F\begin{pmatrix} u \\ \lambda u \end{pmatrix} = \begin{pmatrix} v \\ \lambda v \end{pmatrix} \quad \text{where } u_1 = u, u_{\nu_1} = v,$$

or

$$\begin{cases} F_{11}u + \lambda F_{12}u = v \\ F_{21}u + \lambda F_{22}u = \lambda v \end{cases}.$$

After multiplying the first equation by λ we get once again quadratic pencil (2.12) with the properties described in Theorem 2.1.5. We summarize the consideration of this section by following

Theorem 2.2.2 *Theorem 2.1.5 is valid for monodromy operator (2.15), so the set of Floquet multipliers of problem (2.13) with properties* **(i)-(iv)** *coincides with the set of eigenvalues of pencil (2.12), the set of Floquet solutions corresponds in a very natural way to the set of eigenfunctions and associated functions of this pencil, namely $u(x_1, x_2, x_3)$ is a Floquet solution of (2.13) of order r that corresponds to a Floquet multiplier $\lambda = e^{\mu}$ if and only if $u_1 = u|_{\Gamma_1}$ is the first component of r-th associated function (eigenfunction, if $r = 0$) of monodromy operator (2.15) that corresponds to its eigenvalue λ or, the same, if and only if u_1 is an r-th associated function (eigenfunction, if $r = 0$) of operator pencil (2.12), that corresponds to its eigenvalue λ.*

Corollary 2.1.6 is also valid for problem (2.13).

2.3 Symmetric case and indefinite metric approach

1. In this section we study problem (2.13) with properties *(i)-(iv)* under one additional restriction, namely

$$\begin{gathered} \textit{we assume that the domain } \Omega \textit{ has a plane of symmetry} \\ \textit{orthogonal to } x_1 \textit{ and all coefficients of } L \textit{ and } T \\ \textit{are even functions with respect to this plane.} \end{gathered} \tag{A}$$

Let us choose the elementary cell $\tilde{\Omega}$ to be symmetric with respect to this plane (then Γ_1 and Γ_2 are planes orthogonal to the boundary $\partial\Omega$).

Under assumption (A) the functions G_{ν_1} and G_{ν_2} from (2.18) are equal so that $F_{12} = -F_{21}$ and $F_{11} = -F_{22}$. This fact together with Proposition 2.1.2 gives $F_{12} = F_{12}^*$ or, the same, the operator A from (2.12) is selfadjoint. Thus we have the pencil

$$\mathcal{L}(\lambda) = \lambda^2 A - \lambda B + A, \tag{2.19}$$

where A is a compact selfadjoint operator with trivial kernel on the space $L_2(\Gamma_1)$ or, the same, $L_2(\Gamma_2)$, $B = 2F_{22}$ is a selfadjoint bounded from below operator with compact resolvent on the same space.

The following theorem on spectral properties of pencil (2.19) is proved by indefinite metric technique.

Theorem 2.3.1

1) The spectrum of operator pencil (2.19) consists of two branches of its eigenvalues $\{\lambda_{1j}\}$ *and* $\{\lambda_{2j}\}$, $|\lambda_{1j}| \geq 1,\ |\lambda_{2j}| \leq 1,\ \lambda_{1j} \cdot \lambda_{2j} = 1,\ \{\lambda_{1j}\}$ *tends to infinity,* $\{\lambda_{2j}\}$ *tends to zero;*

2) if κ *is the number (with multiplicity) of non-positive eigenvalues of operator* B*, there are at most* 2κ *(with multiplicity) non-real eigenvalues of (2.19) inside the unit circle (and, so, at most* 2κ *outside the unit circle) and non-real spectrum is symmetric with respect to the real axis;*

3) there are at most 2κ *eigenvalues different from* ± 1 *and such that corresponding eigenfunctions have associated functions and the length of a chain of eigenfunction and associated functions cannot be grater than* $2\kappa+1$ *(if* $\lambda = \pm 1$ *is an eigenvalue of (2.19) than corresponding eigenfunctions always have associated functions and the length of a chain can reach* $4\kappa + 2$*);*

4) the same system of eigenfunctions and associated functions corresponds to each branch of eigenvalues, this system forms a Riesz basis in the energetic space of the operator $|B|$ *(see the proof).*

Proof. We shall first prove the theorem in the case $\mathrm{Ker}B = \{0\}$. From (2.19) the spectral problem $\mathcal{L}(\lambda)u = 0$ is equivalent to the problem $(\lambda^2 + 1)Au = \lambda Bu$. By the substitution $\mu = \lambda/(\lambda^2 + 1)$ we obtain the linear pencil

$$Au = \mu\, Bu. \tag{2.20}$$

If B is positive we get immediately the spectral problem for the compact selfadjoint operator $B^{-1/2}AB^{-1/2}\varphi = \mu\,\varphi,\ \varphi = B^{1/2}u$. In the general case B has a finite number of negative eigenvalues and we proceed in the following way.

Denote by H_+ (H_-) the linear span of all eigenfunctions of B corresponding to the positive (negative) eigenvalues. Since B is a selfadjoint bounded from below operator with compact inverse, we obtain the following decomposition:

$$L_2(\Gamma_1) = H_+ \oplus H_-, \quad \mathrm{dim}H_- = \kappa < \infty.$$

According to this decomposition

$$B = \begin{pmatrix} B_+ & 0 \\ 0 & -B_- \end{pmatrix}, \quad B_\pm > 0, \quad B_- \text{ is an operator of finite rank.}$$

Let us denote

$$J = P_+ \, - \, P_- \, = \begin{pmatrix} I & 0 \\ 0 & -I \end{pmatrix} (\text{ here } P_\pm \text{ are projections on } H_\pm),$$
$$\text{and } |B| = \begin{pmatrix} B_+ & 0 \\ 0 & B_- \end{pmatrix}.$$

Thus $B = |B|^{1/2} \, J \, |B|^{1/2}$ and after the substitution $u = |B|^{-1/2}\varphi$ and multiplication by $|B|^{-1/2}$ from the left we obtain from (2.20)

$$\hat{\mathcal{L}}(\mu)\varphi \, = \, (K - \mu J)\varphi \, = \, 0, \quad K = |B|^{-1/2} A |B|^{-1/2},$$

or, multiplying by J,

$$(JK)\varphi \, = \, \mu\varphi, \quad K \text{ is compact and selfadjoint.} \tag{2.21}$$

Problem (2.21) is a spectral problem for a compact selfadjoint operator on Pontrjagin space Π_κ, it is the well known problem in the theory of indefinite scalar product spaces and from the classical results (see e.g. [6, cor.VI.6.3, th.IX.4.6,4.8,4.9]) we obtain the following properties of the spectrum of (2.21):
the spectrum is discrete with one accumulation point zero, it is symmetric with respect to the real axis, total algebraic multiplicity of eigenvalues from the upper (lower) half-plane is at most κ, there are no more than κ eigenvalues such that corresponding eigenfunctions have associated functions and maximal length of a Jordan chain of a real eigenvalue is $2\kappa + 1$.

In addition, from [4, theorem IV.2.12, remark IV.2.13] follows that the system of eigenfunctions and associated functions of (2.21) forms an unconditional (Riesz) basis in the Hilbert space of all functions $\{\varphi = |B|^{1/2}u \;\big|\; u \in L_2(\Gamma_1) = L_2(\Gamma_2)\}$.

Now return to problem (2.19) by the "inverse" substitutions

$$\lambda_{1,2}(\mu) \, = \, \frac{1 \pm \sqrt{1 - 4\mu^2}}{2\mu}; \quad \varphi = |B|^{1/2}u.$$

It is obvious that $\lambda_1 \cdot \lambda_2 = 1$ and $\lambda_{1,2}(\overline{\mu}) = \overline{\lambda_{1,2}}(\mu)$.

Thus for each real eigenvalue μ of (2.21) such that $|\mu| < 1/2$ we have a pair of conjugate with respect to the unit circle real eigenvalues of (2.19), for each real eigenvalue μ of (2.21) such that $|\mu| > 1/2$ (there exists a finite number of such μ !) we have a pair of complex conjugate eigenvalues of (2.19) on the unit circle and for each pair of complex conjugate eigenvalues of (2.21) (there exist at most κ such pairs) we have 4 eigenvalues of (2.19) and they are symmetric with respect to the real axis and with respect to the unit circle.

Moreover, it is easy to see (e.g., from [25, lemma 11.3]) that the space of eigenfunctions and associated functions corresponding to eigenvalues of (2.21) different from $\pm 1/2$ is preserved under the substitution $\mu = \mu(\lambda)$, so the system of eigenfunctions and associated functions that corresponds to each of two branches of eigenvalues of (2.19) different from ± 1 (the same system for each branch) coincides with the system of eigenfunctions and associated functions of (2.21) (for $\mu \neq \pm 1/2$) up to the substitution $\varphi = |B|^{1/2}u$. The case $\mu = \pm 1/2$ is exceptional. The direct calculation shows that if $\mu = \pm 1/2$ is an eigenvalue of (2.21) of multiplicity m then $\lambda = \pm 1$ is an eigenvalue of (2.19) of multiplicity $2m$ and this eigenvalue has a “double” system of eigenfunctions and associated functions. It means that if u belongs to this system than u appears in this system twice up to multiplication by scalar and $\varphi = |B|^{1/2}u$ belongs to the root subspace of $\mu = \pm 1/2$. Thus we can split the eigenvalue $\lambda = \pm 1$ in such a way that it will belong to both branches of eigenvalues of (2.19) and corresponding subspace of eigenfunctions and associated functions, the same for each branch, coincides with the root subspace of the eigenvalue $\mu = \pm 1/2$ of (2.21) up to the substitution $\varphi = |B|^{1/2}u$.

Thus in any case the same system of eigenfunctions and associated functions corresponds to each of two branches of eigenvalues of (2.19) and it coincides with the system of eigenfunctions and associated functions of (2.21) up to the substitution $\varphi = |B|^{1/2}u$ and for this reason it forms a Riesz basis in the space with the norm $\|\cdot\| = \langle |B|^{1/2}\cdot, |B|^{1/2}\cdot\rangle$, that is in the so-called *energetic space* of the operator $|B|$. This completes the proof in the case $\mathrm{Ker}B = \{0\}$.

If $\mathrm{Ker}B \neq \{0\}$, denote by P_0 the projection on $\mathrm{Ker}B$, then $P_1 = I - P_0$ is the projection on $\mathrm{Ker}B^{\perp} = \mathrm{Im}B$. For $\lambda \neq \pm i$ problem (2.19) is equivalent to the problem

$$(\lambda^2 + 1)A_1 u = \lambda B_1 u, \tag{2.22}$$

where $A_1 = P_1 A P_1 - A_0$, A_0 is a selfadjoint operator of finite rank and $B_1 = P_1 B P_1$. The operators A_1 and B_1 have the same properties as A and B and $\mathrm{Ker}B_1 = \{0\}$, therefore all the arguments given above are valid for (2.22) on the space $\mathrm{Im}B$. To complete the proof we add to the set of eigenvalues of (2.22) $\lambda = \pm i$, its eigenspace from (2.19) is exactly $\mathrm{Ker}B$.

The theorem is proved.

Remark. If the spectrum of the pencil (2.12) does not coincide with the whole complex plane then the property *1)* of this Theorem is valid without symmetry condition *(A)*, so it is valid in the general case (see Corollary 2.1.6 and Theorem 2.2.2).

2. We are ready to prove now the main results of this paper. We recall that a solution of (2.13) of the form

$$u(x_1, x_2, x_3) = e^{\mu x_1} \sum_{k=0}^{r} x_1^k v_k(x_1, x_2, x_3),$$

$$v_k(x_1 + 1, x_2, x_3) = v_k(x_1, x_2, x_3) \text{ for all } \ x, k$$

is called *the Floquet solution of order* r *that corresponds to the Floquet multiplier* e^{μ}. The direct calculation and Theorem 2.2.2 show that $u(x_1, x_2, x_3)$ is a Floquet solution of (2.13) of order r that corresponds to a Floquet multiplier $\lambda = e^{\mu}$ if and only if $u_1 = u|_{\Gamma_1}$ is the first component of r-th associated function (eigenfunction, if $r = 0$) of monodromy operator (2.15) that corresponds to its eigenvalue λ or, the same, if and only if u_1 is an r-th associated function (eigenfunction, if $r = 0$) of operator pencil (2.12) (operator pencil (2.19), if assumption (A) is fulfilled), that corresponds to its eigenvalue λ.

Thus from Theorem 2.3.1 we obtain the following result on distribution of the Floquet multipliers of our problem:

Theorem 2.3.2

1) Under assumption (A) the set of Floquet multipliers of problem (2.13) is a discrete set with two accumulation points 0 *and* ∞*, this set has double symmetry with respect to the real axis and with respect to the unit circle;*

2) the set of non-real multipliers is finite, if κ *is the number (with multiplicity) of non-positive eigenvalues of operator* F_{22}*, then exist at most* 4κ *multipliers belonging to neither the real axis nor the unit circle;*

3) there are at most 2κ *multipliers different from* ± 1 *with Floquet solution of positive order and the order is not grater than* 2κ *(if* $\lambda = \pm 1$ *is a multiplier then its order can reach* $4\kappa + 1$*).*

Our last result deals with basis properties of the set of Floquet solutions. Let us denote by $W^s_{2,loc}(\Omega)$ the topological space of all functions that locally belong to Sobolev space $W_2^s(\Omega)$, that is belong to $W_2^s(\Omega')$ for any finite domain $\Omega' \subset \Omega$, convergence in this space means convergence in the sense of $W_2^s(\Omega')$ for all such Ω'. The notation $C_{loc}^\infty(\Omega)$ has the same sense.

Theorem 2.3.3 *Under assumption (A) an arbitrary solution of problem (2.13) can be expanded in a series over Floquet solutions, the series converges in the sense of $W_{2,loc}^s(\Omega)$ topology for all $s \geq 0$ and in the sense of $C_{loc}^\infty(\Omega)$ topology.*

Proof. First let us prove that the Floquet solutions of (2.13) form a basis in the space of all solutions of (2.13) with respect to the norm of $W_2^s(\Xi)$ for some $s \geq 0$ and for any finite union of elementary cells Ξ. We need the following definitions from spectral theory of operator pencils. If $u_0, u_1, \ldots, u_k$ is a chain of eigenfunction and associated functions that corresponds to eigenvalue λ_0 of a quadratic operator pencil, then the set

$$\left\{ \begin{pmatrix} u_j \\ \lambda_0 u_j + u_{j-1} \end{pmatrix} \; j = 0, 1, \ldots, k; \; u_{-1} = 0 \right\}$$

is called a *derive chain* of the operator pencil. The system of eigenfunctions and associated functions of quadratic operator pencil is called a *2-fold basis* in the space H if the system of all derive chains of this pencil forms a basis in the space $H^2 = H \oplus H$ (see for details [25]).

Let us verify that the last statement of Theorem 2.3.1 implies 2-fold basisness of the system of eigenfunctions and associated functions of pencil (2.19) in the energetic space of the operator $|B|$. To not complicate our reasonings by nonessential technical details, we will do it for the case when all Floquet solutions are of the order zero, so the monodromy operator has not associated functions. Suppose that f and g are functions from the energetic space of the operator $|B|$. Denote by $\{\phi_n\}$ the system of eigenfunctions of pencil (2.19) that corresponds to each branch of eigenvalues of the pencil and denote by $\{\lambda_n\}$ the branch of eigenvalues that tends to infinity. Due to Theorem 2.3.1 we have expansions

$$f = \sum a_n \phi_n \,, \quad g = \sum b_n \phi_n \tag{2.23}$$

and both series converge in the norm of the energetic space of the operator $|B|$. The system of derive chains of pencil (2.19) is

$$\left\{ \begin{pmatrix} \phi_n \\ \lambda_n \phi_n \end{pmatrix}, \, n = 1, 2, \ldots \right\} \cup \left\{ \begin{pmatrix} \phi_n \\ \frac{1}{\lambda_n} \phi_n \end{pmatrix}, \, n = 1, 2, \ldots \right\}$$

(it follows from Theorem 2.3.1 then any eigenfunction ϕ_n of (2.19) corresponds to two eigenvalues λ_n and $\frac{1}{\lambda n}$). To prove 2-fold basisness we have to obtain the following expansion:

$$\begin{pmatrix} f \\ g \end{pmatrix} = \sum c_n \begin{pmatrix} \phi_n \\ \lambda_n \phi_n \end{pmatrix} + \sum d_n \begin{pmatrix} \phi_n \\ \frac{1}{\lambda_n}\phi_n \end{pmatrix}. \tag{2.24}$$

From (2.23) and (2.24) we have

$$\begin{cases} c_n + d_n = a_n \\ c_n\lambda_n + \frac{d_n}{\lambda_n} = b_n \end{cases}$$

so that

$$\lambda_n c_n = \frac{\lambda_n b_n - a_n}{\lambda_n - \frac{1}{\lambda_n}}, \quad d_n = \frac{\lambda_n a_n - b_n}{\lambda_n - \frac{1}{\lambda_n}}. \tag{2.25}$$

Therefore, we have for all sufficiently large n:

$$|d_n|^2 = \left|\frac{\lambda_n a_n - b_n}{\lambda_n - \frac{1}{\lambda_n}}\right|^2 \le 2\left(\left|\frac{\lambda_n}{\lambda_n - \frac{1}{\lambda_n}}\right|^2 |a_n|^2 + \left|\frac{1}{\lambda_n - \frac{1}{\lambda_n}}\right|^2 |b_n|^2\right) \le 3\left(|a_n|^2 + |b_n|^2\right),$$

here we made use the fact that $\lambda_n \to \infty$ as $n \to \infty$. This estimation together with convergence of the series from (2.23) implies convergence of the series $\sum |d_n|^2$. In complete analogy with this result we get convergence of $\sum |\lambda_n c_n|^2$ from the first equality of (2.25). Obviously, the series $\sum |d_n/\lambda_n|^2$ and $\sum |c_n|^2$ also converge because $\lambda_n \to \infty$. Convergence of these four series proves that expansion (2.24) converges in the norm of the space H^2 where H is the energetic space of the operator $|B|$, therefore the system of eigenfunctions and associated functions of pencil (2.19) forms 2-fold basis in the energetic space of the operator $|B|$.

It follows from interpolation theory of Sobolev spaces (see e.g. [22]) that this space is $W_2^{1/2}$. As we mentioned above, ϕ is an eigenfunction of pencil (2.19) that corresponds to an eigenvalue λ if and only if ϕ is the first component of an eigenfunction of monodromy operator (2.15) that corresponds to an eigenvalue λ. Due to the definition of the monodromy operator we have $(\phi, \lambda\phi) = (u|_1, u|_2)$ where u is the corresponding eigenfunction of the monodromy operator and $u|_r$ is the trace of the function u on the Γ_r, $r = 1, 2$. Reasoning similarly we have $(\phi_j, \lambda\phi_j + \phi_{j-1}) = (u|_1, u|_2)$ for j-th associated function ϕ_j. Thus the set of derive chains of pencil (2.19) coincides with the set of pairs $(u|_1, u|_2)$ where u goes through the set of the first components of all eigenfunctions and associated functions of the monodromy operator or, the same, u goes through the set of Floquet solutions of (2.13). Therefore 2-fold basisness of the system of eigenfunctions and associated functions of pencil (2.19) in the space $W_2^{1/2}(\Gamma_1) = W_2^{1/2}(\Gamma_2)$ implies basisness of the set of all pairs $\{(\varphi|_1, \varphi|_2),\ \varphi$ is Floquet solution of (2.13) $\}$ in the space $W_2^{1/2}(\Gamma_1) \oplus W_2^{1/2}(\Gamma_2)$.

Thus, if u is a solution of (2.13), $\{\varphi_n\}$ is the set of Floquet solutions of (2.13), then the expansion (2.24) can be written in the form

$$\begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \sum_{n=0}^{\infty} c_n \begin{pmatrix} \varphi_{n1} \\ \varphi_{n2} \end{pmatrix}, \tag{2.26}$$

the series converges in the norm of $W_2^{1/2}(\Gamma_1) \oplus W_2^{1/2}(\Gamma_2)$ (all solutions of (2.13) are infinitely smooth, so $u|_r \in W_2^{1/2}(\Gamma_r),\ r = 1, 2)$. Let us denote

$$u_1^{(k)} = u_1 - \sum_{n=0}^{k} c_n \varphi_{n1}, \quad u_2^{(k)} = u_2 - \sum_{n=0}^{k} c_n \varphi_{n2}.$$

It follows from (2.26) that

$$\|u_1^{(k)}\|_{1/2} \to 0 \text{ and } \|u_2^{(k)}\|_{1/2} \to 0 \quad \text{as } k \to \infty.$$

It is obvious (due to linearity of problem (2.13)), that $u^{(k)} = u - \sum_{n=0}^{k} c_n \varphi_n$ is the solution of (2.13) with the boundary data $u_r^{(k)}$ on $\Gamma_r,\ r = 1, 2$.

As in Section 1, we can use the following regularity result. Let u solve the Dirichlet problem for elliptic operator $Lu = 0$ on $\tilde{\Omega}$ and $u = f$ on $\partial\tilde{\Omega}$ then the operator $A : W_2^{1/2}(\partial\tilde{\Omega}) \to W_2^{1/2}(\tilde{\Omega}),\ Af = u$ is bounded. The domain $\tilde{\Omega}$ is Lipschitz domain and regularity properties for boundary value problems on Lipschitz domains generalize the classical results for smooth domains. The statement formulated above follows immediately from [18, pp.62-63]. Due to this result we have

$$\|u^{(k)}\|_{W_2^{1/2}(\tilde{\Omega})} \le c(\|u_1^{(k)}\|_{W_2^{1/2}(\Gamma_1)} + \|u_2^{(k)}\|_{W_2^{1/2}(\Gamma_2)}) \to 0 \quad \text{as } k \to \infty,$$

so $u = \sum_{n=0}^{\infty} c_n \varphi_n$, the series converges in the sense of $W_2^{1/2}(\tilde{\Omega})$.

In complete analogy with this result one can prove convergence of the series in the sense of $W_2^{1/2}(\Xi)$ for arbitrary finite union of elementary cells Ξ. To obtain this result it is sufficient to change the elementary cell $\tilde{\Omega}$ for any union of m sequential elementary cells Ξ. The above reasoning holds true if we replace λ_n by $(\lambda_n)^m$.

Given an arbitrary finite domain $\Omega' \subset \Omega$ we take Ξ such that $\overline{\Omega'} \subset \Xi \cup \partial\Omega$. For any compact $K \subset \Xi \cup \partial\Omega$ with infinitely smooth boundary a solution of (2.13) on K can be represented by Poisson's formula and the kernel of this formula is infinitely smooth. Thus for any solution of (2.13) on K we have

$$u(x) = \int_{\partial K} G(x,t)u(t)dt$$

and therefore for any $s \geq 0$

$$u^{(s)}(x) = \int_{\partial K} \frac{\partial^s G(x,t)}{\partial x^s} u(t)dt\,.$$

Then $\|u_n(t)\|_{L_2(\partial K)} \to 0$ implies $\|u_n^{(s)}(x)\|_{L_2(K)} \to 0$, so that convergence in the sense of $W_2^{1/2}(\Xi)$ implies convergence in the sense of $W_2^s(K)$ on K for any $s \geq 0$ and hence in the sense of $W_2^s(\Omega')$ for any $s \geq 0$ and (due to embedding theorem for Sobolev spaces) in the sense of $C^\infty(\Omega')$ as well.

This completes the proof of the Theorem.

Remark. In the general case, that is without the symmetry assumption (A), we obtain by the same way the following result:

completeness (in the sense of $C^\infty_{loc}(\Omega)$ topology) of the set of Floquet solutions in the space of all solutions of problem (2.13) is equivalent to 2-fold completeness (in the sense of L_2) of the system of eigenfunctions and associated functions of pencil (2.12). If this system forms 2-fold basis in $L_2(\Gamma_1)$, then the conclusion of Theorem 2.3.3 (without assumption (A)) is valid.

Chapter 3

The properties of the monodromy operator

3.1 The basic properties of the monodromy operator

1. In Section 2.2 we introduced the monodromy operator for a general selfadjoint elliptic problem of second order periodic with respect to one of the variables. For convenience we recall here the definition of the monodromy operator. Then we establish some general properties of this operator.

We consider a second order boundary problem of the following form:

$$\begin{cases} Lu = L(x,D)u = \sum\limits_{j,k=1}^{3} \frac{\partial}{\partial x_j}\left(a_{jk}(x)\frac{\partial u}{\partial x_k}\right) + a_0(x)\,u \;=\; 0 & (\Omega) \\ Tu = 0 & (\partial\Omega) \end{cases}, \tag{3.1}$$

where

(i) $\Omega \subset \mathbf{R}^3$ is an infinite and 1-periodic with respect to x_1 ($\mathbf{R}^3 \ni x = (x_1, x_2, x_3)$) domain with infinitely smooth boundary, such that any section of Ω by a plane orthogonal to x_1 is bounded,

(ii) all coefficients of L and T are infinitely smooth 1-periodic with respect to x_1 functions,

(iii) L is an elliptic operator and the differential expression of L is formally selfadjoint, that is $a_{jk}(x) = \overline{a_{kj}(x)}$ for all x and $j,k = 1,2,3$ and $a_0(x)$

is a real valued function,

(iv) T is a boundary differential operator of the first order or of the order zero,

T satisfies so-called normality and covering conditions

(see e.g. [22, Chapter 2, Section 1]),

T is "formally selfadjoint" with respect to Green's formula.

The last condition means that for any sections of Ω by smooth enough non intersecting surfaces α_1 and α_2 there exists boundary operator S, order$T$+order$S$ = 1, such that for any smooth enough functions $u(x)$ and $v(x)$ vanishing on α_1 and α_2 we have after integration by parts

$$\int_{\Omega'}(Lu)\overline{v} = -\int_{\Omega'}\sum_{j,k=1}^{3}\left(a_{jk}\frac{\partial u}{\partial x_j}\overline{\frac{\partial v}{\partial x_k}} - a_0 u\overline{v}\right) + \int_{\partial\Omega'} Su\overline{Tv}. \tag{3.2}$$

Here Ω' is the part of Ω between the sections α_1 and α_2 and $\partial\Omega'$ is the part of $\partial\Omega$ between these sections.

Thus (3.1) is a regular elliptic problem, periodic with respect to x_1 and selfadjoint.

2. Let us take the section of Ω by a smooth enough simple surface Γ_1 non-tangential to $\partial\Omega$. Let Γ_2 be the shift of Γ_1 by 1 along x_1. We denote the part of Ω between these sections by $\tilde{\Omega}$. Thus $\tilde{\Omega}$ is the *elementary cell* of the domain Ω or the fundamental domain of the group of shifts by $z \in \mathbf{Z}$ on Ω.

Let function $u(x)$ satisfy the equation and the boundary condition from (3.1) on the closure of the domain $\tilde{\Omega}$. We define the *monodromy operator* as

$$U\begin{pmatrix} u_1 \\ u_{\nu_1} \end{pmatrix} = \begin{pmatrix} u_2 \\ u_{\nu_2} \end{pmatrix}. \tag{3.3}$$

Here u_j is the trace of $u(x)$ on Γ_j, $u_{\nu j} = \frac{\partial u}{\partial \nu}$ on Γ_j, $j = 1, 2$, and ν is so-called *conormal* for the operator L, that is by definition

$$\frac{\partial}{\partial \nu} = \sum_{j,k=1}^{3} a_{jk} N_j \frac{\partial}{\partial x_k},$$

where $N = (N_1, N_2, N_3)$ is a unit vector of an internal (with respect to $\tilde{\Omega}$) normal on Γ_1 and of an external normal on Γ_2. Thus ν_1 (on Γ_1) is an internal conormal for the domain $\tilde{\Omega}$ and ν_2 (on Γ_2) is an external one.

The monodromy operator (3.3) is an unbounded linear operator correctly defined due to uniqueness theorem for Cauchy problem for second order elliptic equations (see e.g. [16, theorem 17.2.6]). The surface Γ_2 is the shift of Γ_1 on the period, so it is convenient to identify these surfaces and consider functions on Γ_1 and Γ_2 as functions on the two-dimensional suffice with the standard Lebesgue measure. Henceforth we shall use the notation Γ instead of Γ_1 and Γ_2 if it doesn't imply misunderstanding.

Let the monodromy operator (3.3) act on the space $L_2(\Gamma) \oplus L_2(\Gamma)$ with the standard two-dimensional Lebesgue measure.

Recall also that to study properties of the monodromy operator we introduced the following elliptic boundary value problem in the elementary cell $\tilde{\Omega}$:

$$\begin{cases} Lu = 0 & (\tilde{\Omega}) \\ Tu = 0 & (\partial\Omega) \\ u = u_1 & (\Gamma_1) \\ u = u_2 & (\Gamma_2) \end{cases}, \tag{3.4}$$

In Sections 2.1 and 2.2 we defined the operator

$$F \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{\nu_1} \\ u_{\nu_2} \end{pmatrix}, \quad \text{or in the matrix form} \quad \begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{\nu_1} \\ u_{\nu_2} \end{pmatrix},$$

here u is a solution of problem (3.4), u_{ν_1} and u_{ν_2} are defined above. Obviously, the operator F is strongly connected with the monodromy operator U but F is more convenient because it is related to a boundary value problem whereas U is related to Cauchy problem. Thus it is naturally to study properties of U with help of properties of F. In this Chapter we use the results obtained in Sections 2.1 and 2.2.

3. We are now in a position to state and prove some basic properties of the monodromy operator.

Theorem 3.1.1 *The monodromy operator (3.3) is a closed operator with trivial kernel on the space* $L_2(\Gamma) \oplus L_2(\Gamma)$*, its domain and its range are dense sets in the space.*

Proof. Denote by $D(U)$ the domain of definition of the operator (3.3) and by H^2 the space $L_2(\Gamma) \oplus L_2(\Gamma)$.

Assume that the vector $(x,y) \in H^2$ is orthogonal to $D(U)$, so $(x,y) \perp (\varphi,\psi)$ for all $(\varphi,\psi) \in D(U)$. In particular, $(x,y) \perp (0,\psi)$ for all $(0,\psi) \in D(U)$ and hence $y \perp \psi$ for all ψ such that $(0,\psi) \in D(U)$. According to the definition of the operators F_{jk} the set $\{\psi : (0,\psi) \in D(U)\}$ is the range of F_{12}, thus $y \perp \mathrm{Im} F_{12}$ and hence $y \in \mathrm{Ker} F_{12}^*$. But $F_{12}^* = -F_{21}$, the operator F_{21} has the trivial kernel, therefore $y = 0$.

Thus if $(x,y) \perp (\varphi,\psi)$ for all $(\varphi,\psi) \in D(U)$ then $y = 0$. It implies $(x,0) \perp (\varphi,\psi)$ for all $(\varphi,\psi) \in D(U)$, so $x \perp \varphi$ for all φ such that there exists a vector in $D(U)$ with first component φ. Obviously, the set of all these φ is a dense set in $L_2(\Gamma)$ and hence $x = 0$. Therefore the only vector in H^2 orthogonal to $D(U)$ is the vector zero and density of $D(U)$ in H^2 is proved.

Precisely the same proof is valid for the range of the monodromy operator (denoted by $R(U)$) if we note that the set $\{\psi : (0,\psi) \in R(U)\}$ is the range of F_{21}.

We prove now that U is a closed operator. Let $(u_1^{(n)}, v_1^{(n)}) \in D(U)$ for all n. This means the following: $u_1^{(n)} \in L_2(\Gamma_1)$, $v_1^{(n)} \in L_2(\Gamma_1)$ and there exists $u^{(n)}$ such that

$$\begin{cases} Lu^{(n)} = 0 & (\tilde{\Omega}) \\ Tu^{(n)} = 0 & (\partial\Omega) \\ u^{(n)} = u_1^{(n)} & (\Gamma_1) \\ \frac{\partial u^{(n)}}{\partial \nu} = v_1^{(n)} & (\Gamma_1) \end{cases},$$

and $u_2^{(n)} \in L_2(\Gamma_2)$, $v_2^{(n)} \in L_2(\Gamma_2)$ where by $u_2^{(n)}$ and $v_2^{(n)}$ we denote the trace of $u^{(n)}$ and $\frac{\partial u^{(n)}}{\partial \nu}$ on Γ_2.

Suppose that

$$\begin{array}{c} (u_1^{(n)}, v_1^{(n)}) \to (u_1, v_1), \quad (u_2^{(n)}, v_2^{(n)}) \to (u_2, v_2) \\ \text{in the norm of } H^2 \end{array}. \tag{3.5}$$

We have to prove that $(u_1, v_1) \in D(U)$ and $U(u_1, v_1) = (u_2, v_2)$.

Therefore, it suffices to show that there exists u such that

$$\begin{cases} Lu = 0 & (\tilde{\Omega}) \\ Tu = 0 & (\partial\Omega) \\ u = u_j & (\Gamma_j),\ j = 1,2 \\ \frac{\partial u}{\partial \nu} = v_j & (\Gamma_j),\ j = 1,2 \end{cases}. \tag{3.6}$$

Let us denote by u the solution of the mixed problem

$$\begin{cases} Lu = 0 & (\tilde{\Omega}) \\ Tu = 0 & (\partial\Omega) \\ \frac{\partial u}{\partial \nu} = v_1 & (\Gamma_1) \\ \frac{\partial u}{\partial \nu} = v_2 & (\Gamma_2) \end{cases}.$$

To get (3.6), it remains to check that $u = u_1$ on Γ_1 and $u = u_2$ on Γ_2.

Due to (3.5) we have $(v_1^{(n)}, v_2^{(n)}) \to (v_1, v_2)$ in the norm of H^2, thus $u^{(n)} \to u$ in the norm of $W_2^1(\tilde{\Omega})$ (it follows from the regularity result for mixed problem with non-smooth boundary, see [28]). It implies convergence of the traces of $u^{(n)}$ on Γ_1 and Γ_2 to the traces of u in the norm of $W_2^{1/2}(\Gamma)$ and moreover in the norm of $L_2(\Gamma)$. But due to the definition of $u^{(n)}$ the trace of $u^{(n)}$ on Γ_j is $u_j^{(n)}$, $j = 1, 2$ and due to (3.5) $u_j^{(n)} \to u_j$, $j = 1, 2$ in the norm of $L_2(\Gamma)$. Therefore, the trace of u on Γ_1 is u_1 and the trace of u on Γ_2 is u_2. It is exactly that we wanted to check, so (3.6) is fulfilled. Hence, the operator U is a closed operator.

To complete the proof, we note that if $U(u_1, v_1) = (0, 0)$ then $u = 0$ (due to uniqueness of solution of Cauchy problem with initial data on Γ_2) and hence $u_1 = u_2 = 0$. Therefore the kernel of U is trivial.

The theorem is proved.

To complete this section we mention that the monodromy operator (3.3) indeed plays the role we have discussed in the Introduction. Actually, one can check by the simple direct calculation that the set of eigenvalues of this operator coincides with the set of Floquet multipliers of the problem (3.1). Moreover, $u(x_1, x_2, x_3)$ is a Floquet solution of (3.1) of order r that corresponds to a Floquet multiplier $\lambda = e^{\mu}$ if and only if $u_1 = u|_{\Gamma_1}$ is the first component of r-th associated function (eigenfunction, if $r = 0$) of monodromy operator (3.3) that corresponds to its eigenvalue λ.

3.2 Quasi-unitary property of the monodromy operator

1. We start this section with some simple technical results concerning the problem (3.1).

By standard integration by parts one can obtain (using (3.2)) the following generalized Green's formulas for the operator L on the domain $\tilde{\Omega}$ (pay attention that ν_1 is an internal conormal for the domain $\tilde{\Omega}$ and ν_2 is an external one):

$$\int_{\tilde{\Omega}} (Lu)\overline{v} \;=\; a(u,v) \;+\; \int_{\Gamma_2} u_{\nu_2}\overline{v}_2 \;-\; \int_{\Gamma_1} u_{\nu_1}\overline{v}_1 \;+\; \int_{\partial\Omega'} Su\overline{Tv}\,,$$

here

$$a(u,v) \;=\; -\int_{\tilde{\Omega}} \sum_{j,k=1}^{3} \left(a_{jk}\frac{\partial u}{\partial x_j}\overline{\frac{\partial v}{\partial x_k}} - a_0 u\overline{v} \right).$$

This formula implies immediately the second one

$$\begin{aligned} \textstyle\int_{\tilde{\Omega}} (Lu)\overline{v} \;-\; u(\overline{Lv}) \;\;&=\;\; \textstyle\int_{\Gamma_2} (u_{\nu_2}\overline{v}_2 - u_2\overline{v_{\nu_2}}) - \int_{\Gamma_1} (u_{\nu_1}\overline{v}_1 - u_1\overline{v_{\nu_1}}) \\ &\quad + \textstyle\int_{\partial\Omega'} (Su\overline{Tv} - Tu\overline{Sv}). \end{aligned} \tag{3.7}$$

Here $\partial\Omega'$ is a part of $\partial\Omega$ between Γ_1 and Γ_2.

Finally, if u and v are solutions of (3.4), we have from (3.7)

$$\int_{\Gamma_1} u_{\nu_1}\overline{v}_1 - u_1\overline{v_{\nu_1}} \;=\; \int_{\Gamma_2} u_{\nu_2}\overline{v}_2 - u_2\overline{v_{\nu_2}}\,. \tag{3.8}$$

2. We consider again the space $H^2 \;=\; L_2(\Gamma) \oplus L_2(\Gamma)$ of pairs of functions $f \;=\; (f_1, f_2)$ with the usual scalar product

$$\langle f, g\rangle \;=\; \int_\Gamma f_1\overline{g}_1 \;+\; f_2\overline{g}_2\,.$$

Let us introduce the operator J on H^2 such that

$$J \;=\; \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \quad \text{or, the same,} \quad J\begin{pmatrix} f_1 \\ f_2 \end{pmatrix} = \begin{pmatrix} f_2 \\ -f_1 \end{pmatrix}.$$

Proposition 3.2.1 *The operator J is an invertible bounded operator on H^2, $J^* = -J,\; J^2 = -I,\; J^{-1} = J^*$.*

One can check this proposition by a trivial direct calculation.

Now we define the following bilinear form:

$$[f,g] \;=\; \langle Jf, g\rangle \;=\; \int_\Gamma f_2\overline{g}_1 \;-\; f_1\overline{g}_2\,. \tag{3.9}$$

From this definition (or from Proposition 3.2.1) we have

$$[g, f] \;=\; -\overline{[f, g]}\,, \tag{3.10}$$

so that the bilinear form $[\cdot,\cdot]$ is a *complex skew-symmetric* bilinear form or, in another way, the scalar product $[\cdot,\cdot]$ is given by a complex skew-symmetric bilinear form and is called a *complex skew-scalar product.* Usual axioms of linearity hold true for $[\cdot,\cdot]$ but instead of symmetric property of the usual scalar product we have the skew-symmetric property (3.10) which implies

$$[f, f] \;=\; -\overline{[f, f]} \quad \text{and hence} \quad [f, f] \text{ is an imaginary number.}$$

From Proposition 3.2.1 $(iJ)^* = iJ$, so the operator $G = iJ$ is an invertible bounded selfadjoint operator. Thus $i[f, g] = \langle Gf, g\rangle$ and the skew-scalar product defined above becomes a particular case of so-called indefinite scalar product after multiplication by i.

3. After these preliminary considerations we return to the monodromy operator. Let f and g belong to $D(U)$ (the domain of definition of the operator U defined in (3.3)) and let $U(f_1, f_2) = (\hat{f}_1, \hat{f}_2)$. Substituting into (3.8) f_1 for u_1, f_2 for u_{ν_1}, $\hat{f}_1$ for u_2, $\hat{f}_2$ for u_{ν_2} and the same for g and v, we get

$$\int_{\Gamma_1} f_2\overline{g}_1 - f_1\overline{g}_2 \;=\; \int_{\Gamma_2} \hat{f}_2\overline{\hat{g}_1} - \hat{f}_1\overline{\hat{g}_2}\,.$$

Then by the definition (3.9) we have

$$[Uf, Ug] \;=\; [f, g] \quad \text{for all} \quad f, g \in D(U)\,. \tag{3.11}$$

Thus the monodromy operator preserves the skew-scalar product $[\cdot,\cdot] = \langle J\cdot,\cdot\rangle$. Such operators are called *quasi-unitary* or *J-unitary* operators (in a finite-dimensional case is also said about *symplectic transformations*, that is transformations preserving symplectic structure defined by a skew-scalar product).

Let us rewrite (3.11) using the usual scalar product:

$$\langle JUf, Ug\rangle \;=\; \langle Jf, g\rangle \quad \text{for all} \quad f, g \in D(U)\,. \tag{3.12}$$

Then

$$\Big|\langle Ug, JUf\rangle\Big| \;=\; \Big|\langle g, Jf\rangle\Big| \;\le\; \|g\|\,\|J\,f\| \;=\; \|g\|\,\|f\| \quad \text{for all} \quad f, g \in D(U)\,.$$

We recall that for an unbounded operator A on a Hilbert space X the domain of definition of an adjoint operator A^* is the set of all $v \in X$ for which there exists c such that $\left|\langle Au, v\rangle\right| \leq c\|u\|$ for all $u \in D(A)$. Then the previous formula implies $JUf \in D(U^*)$ for all $f \in D(U)$. The operator U^* is well-defined because $D(U)$ is a dense set in H^2 (see Theorem 3.1.1). From (3.12) we have $\langle Jf, g\rangle = \langle U^*JUf, g\rangle$ for all f and g belong to the dense in H^2 set $D(U)$, hence $Jf = U^*JUf$ for all $f \in D(U)$. Therefore, $J = U^*JU$ or, the same, $U^{-1} = J^{-1}U^*J$ (the operator U^{-1} exists due to Theorem 3.1.1).

Thus, we have proved the following result:

Theorem 3.2.2 *The monodromy operator U is a quasi-unitary (J-unitary) operator or, in another way, the operators U^{-1} and U^* are similar.*

The following statement is an immediate consequence of Theorem 3.2.2:

Corollary 3.2.3 *The operators U^{-1} and U^* are spectral equivalent, that is these operators have the same set of eigenvalues and the same structure of corresponding Jordan chains and root subspaces.*

Remark. It follows from this corollary that if λ is a Fredholm eigenvalue of U then $\overline{\lambda}^{-1}$ is too, that is the set of eigenvalues of the monodromy operator is symmetric with respect to the unit circle. This result has been obtained already in Chapter 2 (see Corollary 2.1.6 and Theorem 2.2.2).

We recall that $x \neq 0$ is an associated function of order $r \in \mathbf{N}$ of an operator A corresponding to its eigenvalue λ if $(A - \lambda I)^{r+1}x = 0$ and $(A - \lambda I)^r x \neq 0$. A subspace of all eigenfunctions and associated functions that correspond to an eigenvalue λ is called a root subspace corresponding to λ.

Corollary 3.2.4 *If λ and μ are eigenvalues of the monodromy operator and $\lambda\overline{\mu} \neq 1$ (that is λ and μ are not symmetric with respect to the unit circle) then the corresponding root subspaces are J-orthogonal, that is orthogonal with respect to the skew-scalar product $[\cdot,\cdot] = \langle J\cdot,\cdot\rangle$.*

This statement is the well-known consequence of the J-unitary property and it is possible to give references, but instead, we will prove it.

Proof. Let λ and μ be eigenvalues of the monodromy operator and $\lambda\overline{\mu} \neq 1$. Denote by $f^{(r)}$ and $g^{(s)}$ associated functions of order r and s that correspond to λ and μ respectively (if $r = 0$ or $s = 0$ the corresponding function is an eigenfunction). We will prove the statement by induction in $r + s$.

For $r + s = 0$ (so, $r = 0$ and $s = 0$, that is $f^{(0)}$ and $g^{(0)}$ are eigenfunctions) we have $Uf^{(0)} = \lambda f^{(0)}$ and $Ug^{(0)} = \mu g^{(0)}$. J-unitary property implies $[Uf, Ug] = [f, g]$ for all $f, g \in D(U)$, hence

$$[f^{(0)}, g^{(0)}] = [Uf^{(0)}, Ug^{(0)}] = [\lambda f^{(0)}, \mu g^{(0)}] = \lambda\overline{\mu}[f^{(0)}, g^{(0)}] .$$

Therefore, $[f^{(0)}, g^{(0)}] = 0$ (because $\lambda\overline{\mu} \neq 1$).

Assume that the statement is valid for all r and s such that $r + s \leq n$, that is $[f^{(r)}, g^{(s)}] = 0$ for all such r and s. Now let $r+s = n+1$. It follows from the definition of associated function that if x is an associated function of order r of an operator A corresponding to its eigenvalue λ, then $y = (A - \lambda I)x$ is an associated function of order $r - 1$ of A (corresponding to the same λ). Then $(U - \lambda I)f^{(r)} = f^{(r-1)}$ (if $r = 0$ we put $f^{(r-1)} = 0$) and $(U - \mu I)g^{(s)} = g^{(s-1)}$ (if $s = 0$ we put $g^{(s-1)} = 0$). Consequently,

$$[f^{(r)}, g^{(s)}] = [Uf^{(r)}, Ug^{(s)}] = [f^{(r-1)} + \lambda f^{(r)},\, g^{(s-1)} + \mu g^{(s)}] = \lambda\overline{\mu}[f^{(r)}, g^{(s)}] ,$$

because

$$[f^{(r-1)}, g^{(s-1)}] = [f^{(r-1)}, g^{(s)}] = [f^{(r)}, g^{(s-1)}] = 0$$

by the induction hypothesis.

Thus $[f^{(r)}, g^{(s)}] = \lambda\overline{\mu}[f^{(r)}, g^{(s)}]$, so that $[f^{(r)}, g^{(s)}] = 0$ and the statement is valid for all r and s such that $r + s \leq n + 1$.

The corollary is proved.

By taking $\mu = \lambda$, we get from Corollary 3.2.4 the following result:

Corollary 3.2.5 *Let* $f = (u_1, u_{\nu_1})$ *be an eigenfunction or an associated function of the monodromy operator and the corresponding eigenvalue* λ *does not belong to the unit circle. Then* $[f, f] = 0$ *or, the same,* $\langle u_1, u_{\nu_1}\rangle$ *is a real number.*

Indeed, the first statement ($[f, f] = 0$) is a direct consequence of Corollary 3.2.4

and the second one follows from the first by such a simple calculation:

$$\begin{aligned}[f,f] &= \left\langle J\begin{pmatrix} u_1 \\ u_{\nu_1}\end{pmatrix}, \begin{pmatrix} u_1 \\ u_{\nu_1}\end{pmatrix}\right\rangle = \left\langle \begin{pmatrix} u_{\nu_1} \\ -u_1\end{pmatrix}, \begin{pmatrix} u_1 \\ u_{\nu_1}\end{pmatrix}\right\rangle \\ &= \langle u_{\nu_1}, u_1\rangle - \langle u_1, u_{\nu_1}\rangle = \overline{\langle u_1, u_{\nu_1}\rangle} - \langle u_1, u_{\nu_1}\rangle . \end{aligned} \tag{3.13}$$

3.3 Classification of the spectrum

1. In this section we study properties of Floquet multipliers (that is, spectral properties of the monodromy operator) in a more delicate way. For this purpose we need to classify the multipliers. Such a classification is standard in the indefinite scalar product theory. This approach has been applied by M.G.Krein for periodic systems of linear differential equations (i.e., in a finite-dimensional case), see [20]. Our technique and results are not a direct generalization of this famous paper but this section and the next one are written in the spirit of the work of Krein.

As appears from Chapter 2 (see Theorem 2.1.5 and Theorem 2.2.2), the spectral problem for the monodromy operator

$$U\begin{pmatrix} u \\ v\end{pmatrix} = \lambda \begin{pmatrix} u \\ v\end{pmatrix}$$

is equivalent to the spectral problem for the quadratic operator pencil

$$\begin{aligned} &\mathcal{L}(\lambda)u = \lambda^2 Au - \lambda Bu + A^* u, \\ &\text{where} \quad A = F_{12},\ B = (F_{22} - F_{11}), \quad \text{and} \quad v = F_{11}u + \lambda F_{12}u. \end{aligned} \tag{3.14}$$

The operators F_{jk} are defined in subsection 1.2 (and in Chapter 2). It follows from the properties of these operators that A is a compact Hilbert-Schmidt operator with trivial kernel, B is a selfadjoint bounded from below operator with compact resolvent.

2. In Section 3.2 we have defined the skew-scalar product (3.9) on the space $H^2 = L_2(\Gamma) \oplus L_2(\Gamma)$. Let us classify functions of H^2 with respect to this indefinite scalar product. We recall that $[f, f] = -\overline{[f, f]}$, so $[f, f]$ is an imaginary number and hence $i[f, f]$ is a real number.

Definition 3.3.1 *A function f from H^2 is called* **positive** *(***negative** *or* **neutral***) if*
$i[f,f] > 0$ *(* $i[f,f] < 0$ *or* $[f,f] = 0$ *).*

The following classification of spectrum of the monodromy operator is based on the previous definition:

Definition 3.3.2 *An eigenvalue of the monodromy operator (the same, a Floquet multiplier) is called a multiplier of the* **first kind** *(***second kind** *or* **neutral***) if all corresponding eigenfunctions are positive (negative or neutral).*

It is possible that not all eigenfunctions corresponding to some eigenvalue λ have the same type, such a case is called **mixed**. We shall see later (Proposition 3.3.3 and Theorem 3.3.11) that this happens to be the case only if λ belongs to the unit circle and has a neutral eigenfunction.

Now we rewrite these definitions by means of the pencil (3.14) because such a form is more convenient for a further study.

Let $f = (u, v)$ be an eigenfunction of the monodromy operator corresponding to an eigenvalue λ. Then due to (3.13), (3.14) and properties of the operators F_{jk} we have

$$\begin{aligned} [f,f] = \; & \langle v,u\rangle - \langle u,v\rangle = \langle F_{11}u,u\rangle + \langle \lambda F_{12}u,u\rangle \\ & - \langle u,F_{11}u\rangle - \langle u,\lambda F_{12}u\rangle = \langle(\lambda A - \overline{\lambda}A^*)u,u\rangle \end{aligned} \tag{3.15}$$

The following simple propositions establish the general properties of non-unimodular multipliers.

Proposition 3.3.3 *Let λ be a multiplier (that is an eigenvalue of the monodromy operator) not belonging to the unit circle. Then λ is a neutral multiplier. Moreover, all associated functions corresponding to λ, if they exist, are also neutral.*

This proposition is just a reformulation of Corollary 3.2.5.

Proposition 3.3.4 *Let λ be a multiplier not belonging to the unit circle and u is a corresponding eigenfunction of the pencil (3.14) (or, the same, u is the first component of a corresponding eigenfunction of the monodromy operator). Then*

$$\lambda\langle Au,u\rangle \in \mathbf{R}.$$

Indeed, from the previous proposition and (3.15) we have $\langle(\lambda A - \overline{\lambda}A^*)u, u\rangle = 0$ or, the same, $\langle\lambda Au, u\rangle = \langle u, \lambda Au\rangle = \overline{\langle\lambda Au, u\rangle}$ and the statement is done.

Remark. Of special interest is the *symmetric case*, that is the problem (3.1) with the additional assumption that the domain Ω has a plane of symmetry orthogonal to x_1 and all coefficients of L and T are even functions with respect to this plane. It is shown in Section 2.3 that in this case the operator A is selfadjoint, hence $\langle Au, u\rangle \in \mathbf{R}$ and Proposition 3.3.4 implies that λ is a non-real and non-unimodular multiplier (that is, λ does not belong neither to the real axis nor to the unit circle) if and only if $\langle Au, u\rangle = \langle Bu, u\rangle = 0$, where u is defined in Proposition 3.3.4 (obviously, $\langle Au, u\rangle = 0$ implies $\langle Bu, u\rangle = 0$ because $\mathcal{L}(\lambda)u = 0$). In the particular case $B > 0$ such multipliers do not exist.

3. In this and the next subsections we shall study properties of multipliers belonging to the unit circle (*we denote the unit circle by* $\mathbf{T}$). This case is especially important (see Introduction). On the other hand, in this case it is possible to study the multipliers in greater detail.

First let us observe that if $\lambda \in \mathbf{T}$ then $\bar{\lambda} = 1/\lambda$ and we can rewrite (3.15) in the following way:

$$[f, f] \;=\; \langle(\lambda A - \frac{1}{\lambda}A^*)u, u\rangle.$$

But u is an eigenfunction of the pencil (3.14) corresponding to the eigenvalue λ, hence $-\frac{1}{\lambda}A^*u = \lambda Au - Bu$, therefore

$$\langle(\lambda A - \frac{1}{\lambda}A^*)u, u\rangle \;=\; \langle(2\lambda A - B)u, u\rangle \;=\; \langle\mathcal{L}'(\lambda)u, u\rangle. \tag{3.16}$$

Here $\mathcal{L}'$ is a derivative of $\mathcal{L}$ with respect to λ.

Finally, if $f = (u, v)$ is an eigenfunction of the monodromy operator corresponding to an eigenvalue $\lambda \in \mathbf{T}$, then

$$[f, f] \;=\; \langle\mathcal{L}'(\lambda)u, u\rangle. \tag{3.17}$$

Let us write $\lambda \in \mathbf{T}$ in the form $\lambda = exp(i\varphi)$ and denote

$$\widehat{\mathcal{L}}(\varphi) \;=\; \frac{1}{\lambda}\mathcal{L}(\lambda) \;=\; e^{i\varphi}A - B + e^{-i\varphi}A^*\,,$$

then $\lambda \in \mathbf{T}$ implies $\varphi \in \mathbf{R}$, thus

$$(\widehat{\mathcal{L}}(\varphi))^* \;=\; \widehat{\mathcal{L}}(\varphi). \tag{3.18}$$

From (3.16) we get

$$\begin{aligned} \langle \mathcal{L}'(\lambda)u, u\rangle &= \langle (\lambda A - \tfrac{1}{\lambda}A^*)u, u\rangle = \langle (e^{i\varphi}A - e^{-i\varphi}A^*)u, u\rangle \\ &= \tfrac{1}{i}\langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle. \end{aligned} \tag{3.19}$$

Substituting (3.19) into (3.17), we have

$$i[f, f] = \langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle,$$

therefore Definition 3.3.1 implies the following statement:

Proposition 3.3.5 *Let $f = (u, v)$ be an eigenfunction of the monodromy operator corresponding to an eigenvalue $\lambda = exp(i\varphi)$, $\varphi \in \mathbf{R}$ (so, λ belongs to the unit circle). Then f is a positive (negative or neutral) function if and only if $\langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle > 0$ ($\langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle < 0$ or $\langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle = 0$).*

The next statement is an immediately consequence of the previous one.

Proposition 3.3.6 *Under the conditions of Proposition 3.3.5 f is a neutral function if and only if*

$$\langle e^{i\varphi}Au, u\rangle = \langle e^{-i\varphi}A^*u, u\rangle \quad \textit{or, the same,} \quad \lambda\langle Au, u\rangle \in \mathbf{R}.$$

Observe that the last condition is the same as the condition for non-unimodular multiplier, see Proposition 3.3.4.

Remark. In the symmetric case (see Remark after Proposition 3.3.4) we have the following statement:

if $f = (u, v)$ is a neutral eigenfunction of the monodromy operator corresponding to an eigenvalue λ belonging to the unit circle, then either $\lambda = \pm 1$ or $\langle Au, u\rangle = \langle Bu, u\rangle = 0$.

4. Our next goal is finding the condition under which associated functions corresponding to a given multiplier exist. The following statement answers this question for multipliers belonging to the unit circle.

Before formulating the theorem we recall that x is an associated function of order $r \in \mathbf{N}$ for an eigenfunction f corresponding to an eigenvalue λ of the monodromy operator U if $(U - \lambda I)^r x = f$.

Theorem 3.3.7 *Let $\lambda = exp(i\varphi)$ be a multiplier (that is, an eigenvalue of the monodromy operator) belonging to the unit circle. Then a corresponding eigenfunction $f = (u, v)$ has associated functions if and only if $\langle \widehat{\mathcal{L}}'(\varphi)u, w\rangle = 0$ for all eigenfunctions w of the pencil (3.14) corresponding to λ, that is $\widehat{\mathcal{L}}'(\varphi)u$ is orthogonal to the eigenspace of λ (with respect to the usual scalar product of $L_2(\Gamma)$).*

Proof. It is shown in Chapter 2 that x is an associated function of U if and only if its first component is an associated function for the same eigenvalue of the pencil (3.14). Moreover, it is easy to see (e.g., from [25, lemma 11.3]) that the substitution $\lambda = exp(i\varphi)$ does not change the space of eigenfunctions and associated functions corresponding to unimodular eigenvalues of (3.14), therefore this space coincides with the space of eigenfunctions and associated functions corresponding to real eigenvalues of $\widehat{\mathcal{L}}(\varphi)$. Therefore, it is enough to get a necessary and sufficient condition for the existence of the associated function for the pencil $\widehat{\mathcal{L}}(\varphi)$.

By the definition of associated functions for operator pencils the eigenfunction u has an associated functions if and only if there exists the function u_1 such that $\widehat{\mathcal{L}}(\varphi)u_1 + \widehat{\mathcal{L}}'(\varphi)u = 0$ or, the same, $\widehat{\mathcal{L}}(\varphi)u_1 = -\widehat{\mathcal{L}}'(\varphi)u$. Therefore, the condition $\widehat{\mathcal{L}}'(\varphi)u \in \mathrm{Im}\widehat{\mathcal{L}}(\varphi)$ is the necessary and sufficient condition for the existence of the associated function. But the operator $\widehat{\mathcal{L}}(\varphi)$ is selfadjoint (it follows from (3.18)) and its range is closed due to the properties of the operators A and B, hence $\widehat{\mathcal{L}}'(\varphi)u \in \mathrm{Im}\widehat{\mathcal{L}}(\varphi)$ if and only if $\widehat{\mathcal{L}}'(\varphi)u \perp \mathrm{Ker}\widehat{\mathcal{L}}(\varphi)$, what had to be proved.

The theorem gives the necessary and sufficient condition for the existence of the associated function, but this condition is not convenient. The following corollaries that give separately only necessary and only sufficient condition are much more effective.

Corollary 3.3.8 *If an eigenfunction f of the monodromy operator has associated functions then f is neutral.*

The substitution of u for w in Theorem 3.3.7 together with Proposition 3.3.5 prove the corollary in the case of unimodular (that is belonging to the unit circle) multiplier. For non-unimodular multiplier the corollary follows from Proposition 3.3.3.

Corollary 3.3.9 *If a multiplier $\lambda \in \mathbf{T}$ is neutral then there exist associated functions corresponding to λ.*

Indeed, it follows from Definition 3.3.2 and Proposition 3.3.5 that the quadratic form $\langle \widehat{\mathcal{L}}'(\varphi)u, u\rangle$ vanishes on the eigenspace corresponding to λ. Hence, the bilinear form $\langle \widehat{\mathcal{L}}'(\varphi)u, w\rangle$ also vanishes on this subspace and Theorem 3.3.7 implies existence of associated functions for all eigenfunctions corresponding to λ.

The next corollary is an immediately consequence of the previous one.

Corollary 3.3.10 *Let λ be a neutral multiplier belonging to the unit circle. Then algebraic multiplicity (that is dimension of the corresponding subspace of eigenfunctions and associated functions) of λ is at least two.*

The proof is obvious: existence of the neutral multiplier of multiplicity one contradicts the previous corollary.

The last result of this section plays the important role in the next chapter.

We shall say that the multiplier λ is of **definite type** if it is the multiplier of the first kind or of the second kind, that is if the corresponding eigenspace consists of definite functions of the same sign (all eigenfunctions are positive or all eigenfunctions are negative). As appears from Corollary 3.3.8, a multiplier of definite type has no corresponding associated functions.

Theorem 3.3.11 *Let λ be a multiplier belonging to the unit circle. Then either λ is of definite type or there exists a neutral eigenfunction corresponding to λ.*

Proof. Let λ be a multiplier belonging to the unit circle and λ is not of definite type. We shall prove the theorem by contradiction.

Assume that all corresponding eigenfunctions are non-neutral. Denote by S the intersection of the unit ball of H^2 and the eigenspace corresponding to λ.Then there exist $f \in S$ and $g \in S$ such that f is positive and g is negative, so $i[f, f] > 0$ and $i[g, g] < 0$. Let z be the functional acting from S to $\mathbf{R}$ and defined as $z(f) = i[f, f] = i\langle Jf, f\rangle$. The operator J is bounded, so that the functional z is continuous. But $z(f) > 0$, $z(g) < 0$ and according to the assumption does not exist $h \in S$ such that $z(h) = 0$, contrary to continuity of z.

This contradiction prove the existence of a neutral eigenfunction corresponding to λ. The theorem is proved.

Chapter 4

The problem with a parameter. Motion of multipliers

1.In this chapter we consider periodic elliptic problems with a parameter. The main goal is to study a motion of multipliers of such problems. More precisely, we study behavior of multipliers belonging to the unit circle $\mathbf{T}$ under small selfadjoint perturbation. The case $\lambda \notin \mathbf{T}$ is not of great interest because the general properties of such multipliers (described above, see Propositions 3.3.3) do not depend on small perturbations.

Let all coefficients of the problem (3.1) depend on a complex parameter ϵ in such a way that the monodromy operator U is an analytic function of ϵ in a neighborhood of $\epsilon = 0$, the domain of definition of the operator $U(\epsilon)$ does not depend on ϵ and the conditions *(i)* - *(iv)* of (3.1) hold true for real values of ϵ. It follows from the classical results of the perturbation theory (see e.g. [19, theorem 7.1.8]) that in this case any finite system of eigenvalues of $U(\epsilon)$ consists of branches of one or several analytic functions which have at most algebraic singularities near $\epsilon = 0$. The same is true for the corresponding eigenprojections.

In Section 3.3 we introduced into consideration the operator function

$$\widehat{\mathcal{L}}(\varphi) = e^{i\varphi}A - B + e^{-i\varphi}A^* .$$

We used this function to study unimodular multipliers and corresponding eigenfunctions of the monodromy operator. For the present all coefficients of $\widehat{\mathcal{L}}(\varphi)$ are functions of the parameter ϵ.

Assume that in the non perturbed case, that is for $\epsilon = 0$, the problem has a multiplier $\lambda_0 = exp(i\varphi_0)$ on the unit circle and $f_0 = (u_0, v_0)$ is a corresponding eigenfunction of the monodromy operator. In order to study behavior of this multiplier under small perturbation we consider the equation

$$\widehat{\mathcal{L}}(\varphi)u \,=\, 0 \tag{4.1}$$

as equation with a parameter ϵ. We are interested in the case $\epsilon \in \mathbf{R}$.

As we mentioned in the Introduction, periodic differential problems with a parameter were studied thoroughly in the paper [20], but only in a finite-dimensional case (so, for systems of linear differential equations). This chapter is a generalization of [20] on an infinite-dimensional case of partial differential problems.

2. First let us assume that $\lambda_0 = exp(i\varphi_0)$ is a *simple multiplier*, that is an eigenvalue of algebraic multiplicity one. Due to the implicit function theorem the equation (4.1) defines implicitly φ and u as functions of ϵ. Let us consider the quadratic form $\langle\widehat{\mathcal{L}}(\varphi)u, u\rangle$ as function of ϵ. A perturbation of this function (denote it by $\Delta\langle\widehat{\mathcal{L}}(\varphi)u, u\rangle$) for small ϵ can be written in the form

$$\begin{aligned} \Delta\langle\widehat{\mathcal{L}}(\varphi)u, u\rangle \;&=\; \langle d\widehat{\mathcal{L}}\, u_0, u_0\rangle \,+\, \langle\widehat{\mathcal{L}}'_\varphi(\varphi_0)\, d\varphi\; u_0, u_0\rangle \\ &\quad + \langle\widehat{\mathcal{L}}(\varphi_0)\, du, u_0\rangle \,+\, \langle\widehat{\mathcal{L}}(\varphi_0)\, u_0, du\rangle \,+\, \alpha\,. \end{aligned} \tag{4.2}$$

Here $d\widehat{\mathcal{L}}$ means differential of the pencil $\widehat{\mathcal{L}}(\varphi_0)$ at the point $\epsilon = 0$ when the coefficients are functions of ϵ, that is

$$d\widehat{\mathcal{L}} \,=\, e^{i\varphi_0}dA - dB + e^{-i\varphi_0}dA^*\,,$$

and α is a function of ϵ.

Of course, (4.2) holds true only if all the differentials exist, so all the derivatives with respect to ϵ exist and are finite. In this case $\alpha = o(d\epsilon)$.

From (4.1) $\widehat{\mathcal{L}}(\varphi)u = 0$ for all ϵ, hence $\Delta\langle\widehat{\mathcal{L}}(\varphi)u, u\rangle = 0$. On the other hand, (4.1) is valid for $\epsilon = 0$, so that

$$\langle\widehat{\mathcal{L}}(\varphi_0)\, u_0, du\rangle \,=\, 0 \quad \text{and} \quad \langle\widehat{\mathcal{L}}(\varphi_0)\, du, u_0\rangle \,=\, \langle du, \widehat{\mathcal{L}}(\varphi_0)\, u_0\rangle \,=\, 0$$

(here we used (3.18)). Therefore, we have from (4.2)

$$\langle d\widehat{\mathcal{L}}\, u_0, u_0\rangle \,+\, \langle\widehat{\mathcal{L}}'(\varphi_0)\, u_0, u_0\rangle d\varphi \,+\, \alpha \,=\, 0 \tag{4.3}$$

at the point $\epsilon = 0$.

3. Let us remind that the multiplier λ_0 is simple, so that the eigenfunction $f_0 = (u_0, v_0)$ is non neutral (it follows immediately from Corollary 3.3.10).

Then according to Proposition 3.3.5 $\langle \widehat{\mathcal{L}}'(\varphi_0)\, u_0, u_0\rangle \neq 0$ and we have from (4.3)

$$\begin{aligned} \varphi'(0) &= \frac{d\varphi}{d\epsilon} = \frac{-\langle d\,\widehat{\mathcal{L}}\,u_0, u_0\rangle/d\epsilon}{\langle \widehat{\mathcal{L}}'(\varphi_0)\,u_0, u_0\rangle} - \lim_{d\epsilon\to 0} \frac{o(d\epsilon)/d\epsilon}{\langle \widehat{\mathcal{L}}'(\varphi_0)\,u_0, u_0\rangle} \\ &= -\frac{\langle \widehat{\mathcal{L}}_\epsilon\,u_0, u_0\rangle}{\langle \widehat{\mathcal{L}}'(\varphi_0)\,u_0, u_0\rangle}\,. \end{aligned} \tag{4.4}$$

Here $\widehat{\mathcal{L}}_\epsilon = e^{i\varphi_0}A'(0) - B'(0) + e^{-i\varphi_0}(A^*)'(0)$, when A, B, A^* are functions of a real parameter ϵ. Under the conditions of this chapter all the derivatives exist and $\widehat{\mathcal{L}}_\epsilon$ is a selfadjoint operator, so that $\varphi'(0) \in \mathbf{R}$. The set of multipliers is symmetric with respect to the unit circle, the multiplier λ_0 preserves the multiplicity (so, remains simple) under small analytic perturbation. For this reason there exists a neighborhood of zero such that $\varphi(\epsilon) \in \mathbf{R}$ for real ϵ from this neighborhood, that is a perturbed multiplier remains on the unit circle (it follows also from Proposition 3.3.3 since the form $[f, f]$ is a continuous function of ϵ and the non perturbed eigenfunction f_0 is non neutral).

Now we obtain more precise result about motion of such multipliers. To this end, let us clarify a nature of the quadratic form $\langle \widehat{\mathcal{L}}_\epsilon\,u_0, u_0\rangle$ from (4.4).

As in Section 3.1, let function $u(x)$ satisfy the equation and the boundary condition from (3.1) on the closure of the domain $\tilde{\Omega}$, u_j is the trace of $u(x)$ on Γ_j, $u_{\nu j} = \frac{\partial u}{\partial \nu}$ on Γ_j, $j = 1, 2$, and

$$\begin{pmatrix} F_{11} & F_{12} \\ F_{21} & F_{22} \end{pmatrix} \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} = \begin{pmatrix} u_{\nu_1} \\ u_{\nu_2} \end{pmatrix}, \quad \text{that is} \quad \begin{aligned} F_{11}u_1 + F_{12}u_2 &= u_{\nu_1} \\ F_{21}u_1 + F_{22}u_2 &= u_{\nu_2} \end{aligned}\,.$$

Multiplying the first equation by u_1 and the second one by u_2 and then subtracting the first equation from the second one, we have

$$\langle F_{22}u_2, u_2\rangle - \langle F_{11}u_1, u_1\rangle - \langle F_{12}u_2, u_1\rangle + \langle F_{21}u_1, u_2\rangle = \langle u_{\nu 2}, u_2\rangle - \langle u_{\nu 1}, u_1\rangle\,.$$

From (3.7a)

$$\langle u_{\nu 2}, u_2\rangle \;-\; \langle u_{\nu 1}, u_1\rangle \;=\; -a(u,u) \;=\; \int_{\tilde\Omega} \sum_{j,k=1}^{3} \left(a_{jk} \frac{\partial u}{\partial x_j} \overline{\frac{\partial u}{\partial x_k}} - a_0 u \overline{u} \right),$$

therefore

$$\langle F_{22}u_2, u_2\rangle \;-\; \langle F_{11}u_1, u_1\rangle \;-\; \langle F_{12}u_2, u_1\rangle \;+\; \langle F_{21}u_1, u_2\rangle \;=\; -a(u,u)\,.$$

Substituting $u_2 = e^{i\varphi}u_1,\ \varphi \in \mathbf{R}$, we get

$$e^{i\varphi}e^{-i\varphi}\langle F_{22}u_1, u_1\rangle \;-\; \langle F_{11}u_1, u_1\rangle \;-\; e^{i\varphi}\langle F_{12}u_1, u_1\rangle \;+\; e^{-i\varphi}\langle F_{21}u_1, u_1\rangle \;=\; -a(u,u)\,.$$

But the expression on the left-hand side is $-\langle \widehat{\mathcal{L}}(\varphi)u_1, u_1\rangle$, thus finally we have for all u_1 from the domain of $\widehat{\mathcal{L}}(\varphi)$ and for all $\varphi \in \mathbf{R}$:

$$\langle \widehat{\mathcal{L}}(\varphi)u_1, u_1\rangle \;=\; a(u,u)\,, \tag{4.5}$$

where $a(u,v)$ is a bilinear form defined in (3.7a) and u is a solution of (3.4) with a boundary data $u = u_1$ on Γ_1, $u = e^{i\varphi}u_1$ on Γ_2. From the physical point of view the quadratic form $a(u,u)$ has the energy sense, it enables us to call $\langle \widehat{\mathcal{L}}(\varphi)u, u\rangle$ **the energy** of problem (3.1). Therefore $\langle \widehat{\mathcal{L}}_\epsilon\, u_0, u_0\rangle$ is an increment of the energy as the function of ϵ at the point $\epsilon = 0$ when the coefficients of problem (3.1) are functions of ϵ and the boundary data $u = u_0$ on Γ_1, $u = e^{i\varphi}u_0$ on Γ_2 does not depend on $\epsilon = 0$.

The following proposition concludes our previous considerations:

Proposition 4.1 *Let $\lambda_0 = exp(i\varphi_0)$ be a simple multiplier belonging to the unit circle in the non perturbed case (that is, for $\epsilon = 0$) and $f_0 = (u_0, v_0)$ is a corresponding eigenfunction of the monodromy operator. Assume that for φ_0 and u_0 the energy (4.5) is a strictly increasing function of the parameter at the point $\epsilon = 0$. Then there exists a real neighborhood of zero such that for ϵ from this neighborhood a perturbed multiplier remains on the unit circle and moves clockwise or counter-clockwise on the unit circle as ϵ increases, depending on whether λ_0 is a multiplier of the first or of the second kind.*

Indeed, under the conditions of the proposition $\langle \widehat{\mathcal{L}}_\epsilon\, u_0, u_0\rangle > 0$, thus from (4.4) $\varphi'(0) > 0$ if λ_0 is a multiplier of the first kind and $\varphi'(0) > 0$ if λ_0 is a multiplier of the second kind (λ_0 is of definite type due to Corollary 3.3.10). Therefore, $\varphi(\epsilon)$

is decreasing function at the point $\epsilon = 0$ in the first case and $\varphi(\epsilon)$ is increasing function in the second one. This statement is equivalent to the conclusion of the proposition.

4. Show now that the property established above holds true in the more general case, namely for all multipliers of definite type.

Theorem 4.2 *Let the following assumptions be valid:*

(i) all coefficients of the problem (3.1) depend on a complex parameter ϵ in such a way that the monodromy operator U is an analytic function of ϵ in a neighborhood of $\epsilon = 0$, the domain of definition of the operator $U(\epsilon)$ does not depend on ϵ and the conditions (i) - (iv) of (3.1) hold true for real values of ϵ,

(ii) $\lambda_0 = exp(i\varphi_0)$ is an unimodular multiplier of the first kind in the non perturbed case (that is, for $\epsilon = 0$),

(iii) for φ_0 and any u_1 from the eigenspace of λ_0 the energy (4.5) is an increasing function of the parameter at the point $\epsilon = 0$.

Then there exists a real neighborhood of $\epsilon = 0$ such that for all ϵ from this neighborhood all perturbed multipliers (that is, all branches $\lambda(\epsilon)$ of multipliers of the perturbed problem for which $\lambda(0) = \lambda_0$) remain on the unit circle and move clockwise on the unit circle as ϵ increases.

The same result holds true for multipliers of the second kind if we replace the word "clockwise" to "counterclockwise".

Proof. As we mentioned at the beginning of this chapter, it follows from the assumption *(i)* that a perturbed eigenvalue consists of branches of one or several analytic functions which have at most algebraic singularities at the point $\epsilon = 0$. Let $\lambda(\epsilon)$ be one of such branches and $u(\epsilon)$ is a corresponding eigenfunction (an eigenvalue from each branch is simple). The function $u(\epsilon)$ is a continuous function, denote by u_0 the limit of $u(\epsilon)$ as $\epsilon \to 0$, then u_0 is an eigenfunction of $\widehat{\mathcal{L}}(\varphi)$ corresponding to φ_0. All considerations of two previous subsections are valid now for this chosen branch, thus the statement of the theorem is a direct consequence of Proposition 4.1.

Remark. Speed of rotation of multipliers is defined by the formula (4.4). It follows from (4.4) that for all branches $\varphi'(0)$ is finite, so that $\varphi(\epsilon)$ has not algebraic singularity at the point $\epsilon = 0$. Thus each branch of perturbed multiplier of definite type is an analytic function in a neighborhood of $\epsilon = 0$.

Due to Theorem 3.3.11 λ is an eigenvalue of definite type if and only if all corresponding eigenfunctions are non neutral. Therefore, we obtain the following result:

Corollary 4.3 *Let the assumption (i) from Theorem 4.2 be valid and the energy (4.5) be an increasing function of the parameter at the point* $\epsilon = 0$ *for all* u_1 *from the domain of* $\widehat{\mathcal{L}}(\varphi)$ *and for all* $\varphi \in \mathbf{R}$.

Then all unimodular multipliers which have not neutral eigenfunctions move under small perturbation clockwise or counterclockwise on the unit circle as ϵ *increases and direction of rotation is defined by the "sign" of any corresponding eigenfunction.*

The corollary gives us the complete local description of motion of all multipliers which have not neutral eigenfunctions.

5. Let us restate Corollary 4.3 in the following way:

Corollary 4.4 *Let the assumption (i) from Theorem 4.2 be valid and the energy (4.5) be an increasing function of the parameter at the point* $\epsilon = 0$ *for all* u_1 *from the domain of* $\widehat{\mathcal{L}}(\varphi)$ *and for all* $\varphi \in \mathbf{R}$.

Then a multiplier can jump off the unit circle only if it has a neutral eigenfunction.

The behavior of unimodular multipliers which have neutral eigenfunctions, so the motion of such multipliers under small perturbation, is more complicated. We don't study this case and restrict ourselves to one simple remark.

Let $\lambda_0 = exp(i\varphi_0)$ be a mixed or neutral multiplier in the non perturbed case (that is, for $\epsilon = 0$). We assume that the conditions *(i)* and *(iii)* of Theorem 4.2 are fulfilled. Producing in the manner of the proof of this theorem we note that if limit values of all branches of $u(\epsilon)$ (as $\epsilon \to 0$) are the first components of positive or negative eigenfunctions of the monodromy operator then the conclusions of Theorem 4.2 are valid, that is all perturbed multipliers remain on the unit circle and move under small perturbation clockwise or counterclockwise on the unit circle as ϵ increases and direction of rotation is defined by the "sign" of corresponding eigenfunction. In this case each branch of perturbed multiplier is an analytic function in a neighborhood of $\epsilon = 0$.

This conclusion is the same as in Corollary 4.3, therefore the case of mixed or neutral multiplier described here does not differ from the case of multiplier of definite type. In order that a multiplier jumps off the unit circle as ϵ increases, it is necessary that the multiplier has a neutral eigenfunction such that its first component is a limit value of some branch of corresponding eigenfunction $u(\epsilon)$.

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