TIME-DEPENDENT HEDGING POLICIES FOR OPTIMAL CONTROL OF A SYSTEM UNDER UNCERTAINTY

By

Gonen Singer

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Dr. Eugene Khmelnitsky
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Abstract

This work studies a problem of production control by means of perturbation analysis. An uncertain demand is modeled either as discrete-time jumps of inventory, or as a Wiener-type stochastic process. We consider a one-product-type system, and incorporate reputation of the system as a key factor influencing the distribution of future demands. We demonstrate the dynamics of inventory and reputation over a finite time horizon, and develop optimal strategies for production control. The strategies may depend on time, and at each time period depend on the inventory level and the reputation at the beginning of the period.

We believe such policies can be called discrete-time hedging policies, since (i) a target level of the state variable is found at each period, and (ii) when the length of a period tends to zero, the policy tends to a hedging-type policy known from continuous-time settings of the inventory control problems.

To solve the problems with discrete-time uncertainty, we use the perturbation analysis technique and derive necessary optimality conditions that the control function must satisfy. Then, we develop six possible forms the optimal feedback rule can have within each time interval between demand realizations. By that, structural properties of an optimal solution are characterized, and a specific method for state-costate forward-backward integration is developed. Finally, numerical procedures are built to approximately calculate the rules at each time period. For a continuous time demand, we prove the existence of a hedging curve, as buffer level vs. time. The complexity of the procedure is linear in the length of the time horizon. Numerical examples and a case study show the applicability of the results.
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1. Introduction

1.1. General review of the thesis

Most economic, production-distribution and various organizational systems operate under a great deal of uncertainty from many sources. The dynamic behavior of such systems can assume numerous, very complex forms. However, managers, economists and engineers have to plan and control these systems despite the severe uncertainty involved. Moreover, some of the systems have complex dynamics and are challenging to control even in a deterministic setting.

In the business world, the reputation of a firm is considered crucial for success in a competitive market. The reputation of the firm, its production-inventory policy, its service level, and customers purchasing decisions are mutually dependent (see Figure 1-1).

In this thesis, we study the influence of the two factors mentioned above, uncertainty and reputation, on an optimal marketing production strategy. We consider a manufacturing system, where production and inventory are two interviewed decisions designed to meet variable demand. The distribution of demand may depend on the reputation value of the firm. The importance of inventory arises from its ability to serve as a buffer between dynamic demands and the firm’s production, and maintaining sufficient inventory results in decreasing lost sales and harm of reputation.

The thesis formulates production inventory problems in the framework of stochastic optimal control, and applies a control-theoretic methodology to analyzing such problems. The methodology includes the derivation and analysis of optimality conditions, and the
development of computational methods for solving these problems. By means of the proposed methodology, we analyze cases of discrete-time and continuous-time stochastic demand, as well as finite and infinite time horizon, and develop analytical and numerical approaches to solve them. A case study shows applicability of the developed methods to control of service parts inventory in a computer company.

Figure 1-1: Reputation as a factor of decision making

1.2. Research methodology

The thesis makes use of a control-theoretic methodology which consists of three main stages. At the first stage, we mathematically formulate a general case and four particular cases of the one-dimensional, production-inventory control problem. The formulation includes an objective, system dynamics and constraints (Chapter 3). As an important part of the formulation, we describe uncertain demands as a set of all future scenarios in the discrete-time case and as a Wiener process in the continuous-time case. The problem is to find a control function such that the expected objective value is minimized. The proposed problem formulation allows:

- Considering discrete-time demand under continuous-time control.
• Considering continuous-time demand as a stochastic process of a general nature.
• Considering reputation and reputation-dependent demand.
• Applying the variational perturbation technique to analyze different cases of the problem.

At the second stage of the approach, we derive necessary optimality conditions that the control function must satisfy (Chapter 4). This is done in a way similar to deterministic optimal control, i.e., by taking a feasible variation of the control function. The optimality conditions are expressed in terms of costate functions. The information expressed by the optimality conditions is not enough for determining the optimal feedback policy. However, the obtained information assists in further analysis of the problem, and allows an optimal solution to be characterized and parameterized for the considered cases (Chapter 5).

At the third stage of the methodology, we develop numerical methods for finding the optimal strategies. The parameters that characterize feedback policies, such as target levels and state thresholds (hedging curves) are calculated by a numerical procedure. The procedure starts at the end of time horizon and moves backward in time by the steps of one period such that at each iteration the optimal control rule is extended in a subsequent time period. At this stage, specific methods for solving state-costate (forward-backward) differential equations are developed (Chapter 6).

Finally, we provide numerical examples (Chapter 7) that implement the proposed algorithms as well as a case study for inventory management of repairable service parts for personal computers (Chapter 8).
1.3. **Goals and significance of the research**

Various models of production problems have been studied using optimal control theory. The problems with stochastic demand have mostly dealt with a constant production environment, constant demand rate, and infinite planning horizon. As a result, stationary control policies have been obtained. However, quite often, a production manager is more interested in finding optimal or suboptimal strategies for a finite horizon, rather than for distant future. For example, production lines that produce products with short life cycles such as computers, which become obsolete when they are no longer near the technological frontier. Another example is a fashion product with time-dependent demand. One of the goals of this work is to develop nonstationary control policies, which are optimal in a nonstationary environment or over a finite time horizon.

Production control models commonly assume both the shortages are completely backlogged, or completely lost, and future demands do not depend on the current lost sales and the reputation of the firm. However, in a business world and competitive market, service level and reputation do influence future sales. Another goal of the research is to model and solve the problems, where backlogging, lost sales and reputation are key factors of the firm’s policy.

The main idea of the work is to develop computational procedures for stochastic optimal control problems. To this end, we follow a perturbation analysis technique, i.e.,

- aid of Formulate the necessary optimality conditions with the costate variables.
- Prove feedback control rules, following from the state-costate dynamics.
- Calculate parameters of the rules.
The methods developed in the thesis can be useful in applications from other management science areas, which involve methods of control theory. Such applications are marketing, financing, replacement of equipment and consumption of natural resources. In financing, for example, the objective could be to maximize the expected utility from wealth at a finite time horizon, when controlling the percentage of different market instruments. This problem is solved, subject to the dynamics and changeability of uncertain prices of stocks and other market instruments. In the marketing area, the objective could be to maximize the sales and profit of the firm, when controlling the optimal advertising expenditures over a finite time horizon. This control problem is solved, subject to the dynamics that defines the probabilistic effects of advertising on goodwill. These problems share many common features with the production control problem studied in the thesis.
2. Literature review

2.1 Modeling of dynamic demand in production problems

The theory of deterministic optimal control has been rapidly developing during the last several decades. The methodology, conditions of optimality, and various numerical methods have proven their efficiency in multiple applications in economics, management, and industrial engineering (Bensoussan et al., (1983), Maimon et al., (1998), Sethi and Thompson, (2000)).

Various models of manufacturing systems have been studied by using optimal control theory. In these models, production and inventory level are two interviewed decisions designed to meet uncertain demands and smooth the production process in an economic and efficient manner. The stochastic demand is modeled either as continuous in time or occurring at discrete-time periods.

A class of models in the discrete-time literature is the continuous-review stochastic inventory models with compound Poisson demand. In this scenario, the demands arrive at random epochs governed by a Poisson process. At any epoch, the demand size is independently and identically distributed (stationary demand). Inventory replenishment is allowed at discrete time points, and the optimal replenishment policy is proved to be of a \((s,S)\) type. An \((s,S)\) policy for this model means that the reorder point is \(s\) and the reorder quantity is \(S-s\). Richards, (1975), Thompstone and Silver, (1975), Archibald and Silver, (1978), Feldman, (1978), Federgruen and Schechner, (1983) and many others, studied such a scenario. In addition, Tijms, (1972), Sahin, (1979, 1983) and Federgruen and
Schechner, (1983) considered the compound renewal demands in which the jump epochs are assumed to follow a renewal process. Zipkin, (1986) used a compound counting process to model discrete-time demand. Some of these papers showed that an \((s, S)\) policy is optimal and others study the behavior of \((s, S)\) policies without showing their optimality.

A direct extension of the Poisson process models has been done by Presman and Sethi, (2004) who considered the demand to be the sum of a constant continuous-time term and a compound Poisson process. They developed a new approach to prove the optimality of an \((s, S)\) policy in the presence of a fixed ordering cost. This is also a unified approach in the sense that it deals with both the long-run average cost and the discounted cost criteria. Their method can be outlined as follows: It starts with an \((s, S)\) policy and then finds the corresponding discounted-cost formula and a closely related modified cost function. By minimizing the modified cost function, a candidate optimal \((s, S)\) policy is obtained. Using this candidate and the formula for the discounted-cost, a potential function is constructed, which is shown to satisfy the dynamic programming equation associated with the problem.

A further extension of continuous-time demand is to describe the demand as a Wiener process. Bensoussan and Tapiero, (1982) formulated a stochastic demand model consisting of a diffusion process and a pure Poisson process. This is a problem of impulse control, the theory of which has been developed by Bensoussan and Lions, (1984). The general idea is briefly as follows. Under the framework of impulse control, the Bellman equation for the inventory problem under consideration reduces to a quasi-variational inequality (QVI). They considered a finite horizon problem and derived the associated
QVI. While they give conditions for an \((s,S)\) policy to be optimal, they do not solve the problem explicitly.

Bensoussan et al., (2005) consider a continuous-review stochastic inventory model with demand consisting of a compound Poisson process and a diffusion process. They formulate the inventory problem as an impulse control problem, and prove the optimality of an \((s,S)\) policy in two cases: (i) when the demand is a mixture of a diffusion process and a compound Poisson process with exponentially distributed jump sizes, and (ii) when the demand is a mixture of a constant demand and a compound Poisson process.


In practice, demand is nonstationary, and usually fluctuates and depends on many exogenous factors such as economic conditions, natural disasters or strikes, or can depend on inward factors of the system such as initial or on-hand inventory levels, selling price, and the effectiveness of advertising. In these cases, a “state-dependent” demand model seems to be more appropriate to capture such randomly changing environmental factors.

There are certain types of items (like consumer goods, fashionable items, etc.) for which, according to market research, customers are motivated by the display of the items in the showroom i.e., the demand rate is dependent on the displayed inventory level. For these items, the consumption goes up if the inventory level is high and vice versa. As for incorporating inventory-level dependence into traditional inventory models, Johnson,

Another factor the demand of an item depends on is the selling price. It is a common practice that higher selling price of an item negates the demand of that item whereas lower prices have the reverse effect. Hence, it can be realistically assumed that the demand of an item is a function of selling price, time, current stock level, advertising expenditures, etc., jointly or separately. In this area, one may refer to the works of Urban, (1992), Goyal and Gunasekaran, (1995), and Abad, (2003).

Many inventoried items such as electronic products, fashionable clothes, tasty food products, and domestic goods generate increasing sales after gaining consumers’ acceptance. The sales for other products may decline drastically due to the introduction of more competitive products or to the change in consumers’ preferences. Therefore, the demand of the product, during its growth and decline phases, can be well approximated by time-dependent functions. Numerous papers have been written on the subject of lot sizing with time-dependent demand (Donaldson, (1977), Hariga, (1993), Silver, (1979), Phelps, (1980), Mitra \textit{et al.}, (1984) and Goyal \textit{et al.}, (1986)).

Another class of models is the so called world-dependent demands. The demand is driven by an underlying, exogenous continuous-time stochastic process called “state-of-the-world.” This process may model the economy or conditions in a particular industry. Song and Zipkin, (1993) considered a continuous-review model with world-dependent Poisson demands. They used a continuous-time Markov chain to model the “state-of-the-
world.” To solve the problem, they converted the continuous-time problem to a discrete-time problem, and then used the discrete-time dynamic programming to obtain a state-dependent \((s,S)\) policy.

In a continuous-time model, Browne and Zipkin, (1991) presented a diffusion term and assumed that the demand process is state-dependent and the underlying “state of the world” is modeled by either a continuous-time Markov chain or by a diffusion process. They considered an \((s,S)\) policy which typically is not optimal for their model.

Figure 2-1 sums up a classification of the production control problems associated with state dependent and state independent demand. Figure 2-2 sums up a classification of the production control problems associated with stochastic modeling.
Figure 2-1: Classification of state dependent demand in production control problems
In this thesis, we assume continuous-time inventory replenishment rate. In such a scenario, an optimal policy is proved to be of a hedging type in some cases. The next section reviews literature on the hedging-type policies.

### 2.2 Hedging policies in continuous-time production control

One of the most important results of stochastic optimal control has been the identification of the optimal feedback control structure, namely hedging point policy (Olsder et al.,
(1980), Boukas et al., (1991), Lehoczky et al., (1991), Perkins and Srikant, (1997), and Tan, (2002)). Under such a policy, a machine produces: a) at full capacity if the inventory level is lower than the hedging point; b) nothing if the inventory level is higher than the hedging point; and c) as much as the demand if the inventory level is equal to the hedging point. The value of the optimum hedging point for systems with one machine and one part-type was first obtained by Akella and Kumar, (1986), for the discounted cost problem and by Bielecki and Kumar, (1988), for the average cost problem.

Sharifnia, (1988), showed how the optimum hedging point in the case of one-part type, multiple machines can be calculated. Perkins and Srikant, (1997), obtained an optimal hedging policy based on linear switching curve for a problem of two-part type, single unreliable machine. Perkins and Srikant, (1998), extended their previous research to the multiple part type, single unreliable machine problem, where they presented new results on the structure of the optimal policy and provided bounds on the optimal hedging points.

Hu et al, (1994), studied both the necessary and sufficient conditions for the optimality of a hedging point policy for the problem of one part type manufacturing systems, in which the failure rate of machines depends on the rate of production. They showed that the hedging point policy is optimal when the failure rate function, as a function of the rate of production, is linear, i.e., $au+b$, where $u$ is the production rate and $a$ and $b$ are constants. This result generalizes the results of Akella and Kumar, (1986), which are derived for a constant failure rate.

Tan, (2002), proposed a hedging type policy with two hedging levels, depending on the level of the demand, for a continuous material flow manufacturing system and a
variable demand source which switches randomly between zero and maximum level. The problem of complete evaluation of the optimal production policy in a multiple part-type, multiple-machine problem is difficult because it requires solving either systems of partial differential equations or large dynamic programming problems that easily run into the “curse of dimensionality.” Therefore, approximation procedures to obtain near optimal controllers have been proposed (Caramanis and Liberopoulos, (1992) and Gershwin et al., (1985)).

Glasserman, (1995), extended the hedging point policy, so that demand may exceed capacity in some states. In that paper, the maximum production rate varies with the state of the machine. Assuming that the machine state is governed by a semi-Markov process, the paper evaluates average and discounted inventory costs for any hedging point, and thus provides a simple mechanism for identifying optimal hedging points.

In this thesis, we consider a one-dimensional system and focus on hedging (switching) points as major characteristics of optimal solutions both in discrete-time and continuous-time demand. In contrast to the existing literature, we study time-dependent hedging policies, i.e., the situations where the optimal feedback control rule changes in time. The research aims at developing a methodology for analyzing these kinds of optimal control problems. The methodology includes the development of numerical algorithms for calculation of the time-dependent hedging policies.

2.3 Holding and shortage costs estimation

Cost estimates are essential to the development and implementation of any production system modeling. The study of control problems in such systems is often divided into two
broad categories: The full cost model and the partial cost model. In the full cost model the objective is to find an inventory replenishment policy which minimizes the total cost associated with the system: the inventory holding cost, the production or ordering cost, and the cost of shortages. In the partial cost model, a service level constraint is used, which takes the shortage cost into account implicitly. Holding costs are incurred, when a product is stored in a warehouse. If an item is demanded and cannot be delivered, two kinds of shortage costs can occur. The first one relates to extra costs for administration, price discounts for late deliveries, material handling and transportation which are estimated overhead. The second one relates to lost sales and loss of goodwill. Because of the difficulties in estimating the second kind of shortage costs, partial cost modeling replaces them by a suitable service constraint.

Chen and Krass, (2001) have proved that shortage cost and service level constraints are not “duals” of each other. These authors show service level constrained models to be qualitatively different from their shortage costs counterpart.

Most researchers assume that the shortages are either completely backlogged or completely lost, since such formulations are much simpler to solve. In this thesis, we assume a partial backlogged model by defining the fraction of potential customers who choose not to order when there is a backlog. The greater the backlog, the more customers leave without making a purchase (lost sales). As result, we model the affect of these lost sales on the reputation of a firm. As the backlog being greater and closer, the value of the reputation is decreased. The distribution of demand is described as time- and reputation-dependent.
2.4 Customer behavior in queue models and production control problems

In the inventory management literature, most of the models assume that shortages are either completely lost or completely backlogged. However, recently, some studies incorporated partial backlogging, mainly in a supply chain configuration. Since the thesis deals with a partial defection of potential customers who are not willing to wait for their products (lost sales), we address the queuing literature which provides models that examine the behavior of a customer who has to wait for service.

2.4.1 Customer’s behavior in queuing systems

In the basic models, it is assumed that a customer stays in queuing system until she is served. The basic queuing models are extended to include reneging (abandoning the queue after waiting some time) and balking (not joining the queue if the server is not immediately available) as presented in Hall, (1991).

Whitt, (1999) proposed a method to predict waiting time in a multi-server exponential queue. The method utilizes information about the number of customers in the system ahead of the current customer. It is stated that the waiting time prediction may be used to decide when to add additional servers.

Cachon and Harker, (1999) analyzed competition between two firms that service time sensitive customers. In their study, the customers choose firms based on the firm’s prices, the firm’s expected waiting and service times, and the firm’s brands.

2.4.2 Customer’s behavior and inventory management

In a recent study, partial backlogging and service dependent sales are incorporated in a supply chain configuration by Rau et al., (2000). Chang and Dye, (1999) extended the basic economic order quantity model to include partial backlogging, where the backlog rate is inversely proportional to the waiting time for the next replenishment.

Tan and Gershwin, (2004) study a manufacturing firm and assume that the conservative estimates of the waiting time are given to the arriving customers. Having this information, the customer then decides to wait or leave the system. This process can be described by a defection function that maps the queue length into the probability with which the customer leaves. An example of how a defection function can be constructed is given in Appendix A. In this thesis we use the concept of the defection function proposed by Tan and Gershwin, (2004) for modeling lost sales.

2.5 Reputation as a key factor in the competitiveness of a firm

Reputation is a key factor in the competitiveness of the firm, since it affects present and future sales, and revenues. Firm’s reputation is influenced by various factors, including perceived value, pricing, quality, customer service, the image of the firm as a responsible member of society, and advertising. How the value of reputation translates into the company performance measures, however, is a complicated question (see, for example Ullmann, (1985); McGuire et al., (1988); Davidson and Worrell, (1992); Meznar et al., (1994); McWilliams and Siegel, (1997); Alexander and Buchholz, (1998)).

Nerlove and Arrow, (1962), treated advertising expenditure rate as a control variable that affects the goodwill evolution, according to a motion equation. Tapiero (1978),
extended the Nerlove-Arrow model by including the probabilistic effects of advertising and forgetting. Then, Tapiero (1979), provided a generalization of the Nerlove-Arrow model to a multi-firm advertising model under uncertainty. A multi-variable consumer behavior stochastic model that investigated buying behavior such as brand advertising and promotional effort was constructed in Nicosia (1966) and Tapiero (1982). Vidale and Wolfe, (1957), Palda, (1964) and others presented empirical evidence that the effects of advertising linger on but diminish over time.

Klein and Leffler, (1981), Shapiro, (1983), Rogerson, (1983) and Allen, (1984), considered the firm’s past performance and past customer’s experience as the source for reputation. The customers condition their today’s purchasing decisions on a firm’s reputation.

This thesis relates to the above literature by identifying the firm’s reputation with its past performance. Specifically, we model the reputation as a function of the service level (shortage), by using exponential smoothing of shortages in the past. Moreover, we study the influence of the reputation on firm’s present and future sales and develop a production control strategy sensitive to change in reputation.

### 2.6 Impact of shortage on reputation and future demand

There are four different approaches that analyze the effect of past shortages on future demands. The first approach models the demand in the next period as an explicit function of shortages in the previous period. It was Schwartz, ((1966), (1970)) who first considered the concept that shortage may not impose an immediate penalty on the retailer but may affect the distribution of its future demand. He used the following demand
model, \( \lambda = \frac{\lambda_0}{1 + \alpha I} \), where \( \lambda_0 \) is the expected demand rate that would prevail with no stockouts, \( \lambda \) is the demand rate that will prevail when stockouts occur, \( \alpha \) is the relative number of stockouts, (that is, the ratio of demand occurring when stock is exhausted to total demand), and \( I \) is a constant parameter of the model related to customer response. Robinson, (1990), showed that the mean of demand change over time in response to market changes depends on customer satisfaction. This paper assumes that there is an affine relationship between the mean of demand at time \( t+1 \), \( \mu_{t+1} \), and the mean of demand at time \( t \), \( \mu_t \), as:

\[
\mu_{t+1} = a + b \mu_t + r_s s_t - r_d d_t,
\]

where \( a \) and \( b \) reflect the underlying changes in the market, \( s_t \) and \( d_t \) are the number of satisfied and unsatisfied customers in period \( t \), respectively, and \( r_s \) and \( r_d \) are the response rates for satisfied and unsatisfied customers, respectively. An explicit calculation of the parameters of the models (\( I \) in Schwartz, ((1966), (1970)) and \( r_s \) and \( r_d \) in Robinson, (1990)) was not explained; their accurate values are not easy to obtain.

The second approach models the effect of shortages on future demand via an intermediate variable called service level. Ernst and Cohen, (1992) proposed a simple, flexible model that describes how the mean of long run demand changes as the retailer updates its service level. Practically, they utilize a demand formulation that relates the mean demand directly to the service level provided by the retailer, as follows:

\[
D(SL) = [1 + \eta(SL - SL_0)]D(SL_0) \quad \text{where} \quad D(SL_0) \quad \text{is the demand when the service level is} \quad SL_0, \quad D(SL) \quad \text{is the demand when the service level is} \quad SL, \quad \text{and} \quad \eta \quad \text{is a constant.}
\]

Ernst and Powell, ((1995), (1998)) developed their earlier model further in order to analyze optimal
order-up-to inventory policies when both the mean and variance of demand are sensitive to service levels. Their basic model for the mean and standard deviation of demand as a function of the service level takes the form:

\[
\mu(SL) = \left[1 + \alpha(SL - SL_0)\right] \mu_0
\]  

(2.2)
\[
\sigma(SL) = \left(1 + \beta^2 \alpha(SL - SL_0)\right)^{1/2} \sigma_0
\]  

(2.3)

where the parameter \( \alpha \) measures the increase in mean demand that results from a one percentage point increase in the service level, \( \beta \) measures the relative variability of the additional demand that is created by an increase in service level, and \( \mu_0 \) and \( \sigma_0 \) are the mean and standard deviation of demand at the service level \( SL_0 \). Merkuryev et al., (2003) numerically simulated the behavior of an inventory system under the service-sensitive demand, which is modeled similarly to Ernst and Powell, (1995). Most of the researches that followed this approach studied a demand model, which depends on the service level observed at the previous period, but did not discuss how the service level itself depends on shortages.

The third approach models the effect of shortages on future demand via an intermediate variable called goodwill, or reputation. Nerlove and Arrow, (1962), treated advertising expenditure rate as a control variable that affects the goodwill evolution, according to a motion equation. Vidale and Wolfe, (1957), Palda, (1964), and others present empirical evidence that the effects of advertising linger on but diminish over time. Klein and Leffler, (1981), Shapiro, (1983), Rogerson, (1983) and Allen, (1984), considered the firm’s past performance and past customer’s experience as the source for reputation. They showed that the customers link their today’s purchasing decisions with the firm’s reputation. Similar to the previous approach most of the researches in this
direction model demand as a function of the reputation value of the firm. However, no way of how to compute the reputation as a function of the previous performance of the firm is developed.

The forth approach models the effect of shortages on future demand by transition of unsatisfied customers of one firm to the other firm at the next period, and vice versa. Hall and Porteus, (2000) inserted the effect of shortages on future demand into a dynamic duopoly model. They modeled the expected number of customers of firm $i$ who switch to the other firm at the beginning of the next period as $\lambda_i \gamma_i h_i(y_{it})$, where $\gamma_i \in (0,1]$, $\lambda_i$ presents the fractional market share for firm $i$ in period $t$, $h_i(y_{it})$ presents the single measure of customer service. As a result, $\lambda_i h_i(y_{it})$ gives the expected number of firm $i$’s customers that experience service failures in period $t$ when firm $i$ has a normalized capacity of $y_{it}$. In that paper the market share of firm $i$ in period $t+1$ is expressed as,

$$E(\lambda_{i,t+1} | \lambda_i, \lambda_j) = \lambda_i - \lambda_i \gamma_i h_i(y_{it}) + \lambda_j \gamma_j h_j(y_{jt}),$$

(2.4)

i.e., the market share of firm $i$ in period $t+1$ consists of the market share of firm $i$ in period $t$ less the share lost to firm $j$ plus the share gained from firm $j$. Henig and Gerchak, (2004) modeled the effect of shortages on future demand in duopoly model, similar to Hall and Porteus, (2000), by transition of unsatisfied customers of one firm to the other firm at the next period. In their model, there are $N$ potential customers, of whom, at any given period, $n_A$ are affiliated with firm $A$ and $N-n_A$ with firm $B$. The random demands are the number of the affiliated customers which will show up at each of the firms. If in some period $A$ stocks $Q_A$ and $B$ stocks $Q_B$ and demands are $X_{n_A}$ and $X_{N-n_A}$, respectively, unsatisfied customers will switch their affiliation between the periods, so
$A$’s next period number of affiliated customer is 

$$n'_d = n_d - \left( X_{n_d} - Q_A \right)^+ + \left( X_{N-n_d} - Q_B \right)^+.$$ 

The researches in this direction mostly modeled the effect of shortages on future demand in dynamic duopoly and assumed that inventory is neither stored nor backlogged between periods.

Like the third approach discussed above, this thesis models the effect of shortages on future demand via reputation of the firm. We quantitatively define the reputation of the firm as a cumulative measure of the shortages in all past periods. We use exponential smoothing to express the difference between near past and far past influence. As the backlog being greater and more recent, the reputation of the firm decreases. Then, we link the next period demand distribution with the current reputation. As a result, an unambiguous feedback mechanism regulating demand fluctuations as a function of the past shortages is established.
3. Formulation of production control problems

3.1. Background

In this chapter we introduce a model of a stochastic control system influenced by both an uncertain discrete-time variable and a continuous-time stochastic process. Based on the general model we then present four particular problems under either discrete-time, or continuous-time uncertain factors.

3.2. General formulation

We consider a problem of controlling a one-dimensional system along a finite time horizon $\tau \in [0,T]$. The state of the system at $\tau$ is denoted by $x_\tau$. Without loss of generality, $T$ is assumed integer. The dynamics of the system are influenced by a Wiener process with drift $\mu$ and variance $\sigma^2$, and by random independent jumps, $d_\tau$, occurring at integer time points, $t = 0, 1, ..., T - 1$. The probability distribution of jumps can depend on $t$ (non-stationary distribution) and on the history of the system, which will be referred to as the system's reputation. The reputation at an integer $t$, $R_t$, is a cumulative measure of the system's performance in the past. We describe the dynamics of the reputation as an exponential smoothing with parameter $\gamma$, $0 \leq \gamma \leq 1$,

$$R_t = \gamma R_{t-1} + (1 - \gamma)f(x_t),$$

(3.1)

where $f(x_t)$ is a continuous function of the system state at $t$. The "better" the system states in the past, $x_t$, $x_{t-1}$, $x_{t-2}$, the higher the reputation $R_t$. Since the reputation is a
relative measure of performance, we assume it changes between 0 and 1, \(0 \leq R_{t} \leq 1\). The linear dependence of \(R_{t}\) on \(R_{t-1}\) assumed in (3.1) means that the rate of "forgetting" the system's performance in the previous time periods is constant and does not depend on time and on reputation value. Mathematically, the dynamic equation (3.1) can be generalized to arbitrary continuous and monotone dependence of \(R_{t}\) on \(R_{t-1}\) and on \(x_{t}\), leaving \(R_{t}\) between zero and one. Since perturbation analysis used in the subsequent chapters of the thesis takes into account only linear terms of the dynamic dependence, the optimality conditions and methods developed in the sequel do not significantly change. Section 3.5 below gives more details on this issue.

The probability density distribution of \(d_{t}\), \(\pi_{t, R_{t-1}}(\cdot)\), depends on \(t\) and \(R_{t-1}\), and is defined on a non-negative bounded continuous interval, i.e., \(\pi_{t, R_{t-1}}(\xi) > 0\) only if \(\alpha < \xi < \beta\), where \(\alpha\) and \(\beta\) are given non-negative parameters.

Let \(\omega=(\xi_{0}, \xi_{1}, ..., \xi_{T-1})\) be a scenario of jump realizations, then the state variable, \(x_{\tau}\), satisfies the stochastic equation:

\[
x_{\tau}(\omega) = x_{0} + \int_{0}^{\tau}(u_{r} - \mu(x_{r}, r))dr + \int_{0}^{\tau}\sigma(x_{r}, r)dW_{r} - S_{\tau}, \quad x_{0}\text{ is given,} \tag{3.2}
\]

where \(u_{\tau}\) defines the control effort at time \(\tau\); \(S_{\tau} = \sum_{t=0}^{T-1}s_{t}(\xi_{t}, x_{t-})\theta(\tau - t)\) is a cumulative impact of jumps on the state process; \(\theta(\tau)\) is the unit step function, \(\theta(\tau) = 1\) if \(\tau \geq 0\) and \(\theta(\tau) = 0\) if \(\tau < 0\); and \(s_{t}(\xi_{t}, x_{t-})\) is continuous w.r.t. its arguments.

We assume that control \(u_{\tau}\) is \(\mathcal{F}_{\tau}\)-predictable; \(\mathcal{F}_{\tau}\) is the \(\sigma\)-algebra generated by the scenarios of the stochastic processes in the interval \(0 \leq r < \tau\), and bounded as
where \( V \) is the maximal intensity of the control effort. The maximal intensity of the control effort can also depend on time. The functions \( u_t \) and \( x_t \) take values from a continuous range. If needed, a discretization of the resultant optimal policies can be done. In such a case, the policies become sub-optimal. Finding the optimal policies becomes a combinatorial NP-hard problem.

The objective is to maximize the total expected profit of the system defined as

\[
J(x) = E \left[ p S_{T-1} - \int_0^T (C(x_r) + c u_r) dr \bigg| x_0 = x \right],
\]

(3.4)

where \( p \geq 0 \) and \( c \geq 0 \) are constants, \( C(x) \) is a continuous and piecewise continuously differentiable convex cost function; \( x=0 \) is a unique minimum point of \( C(x) \). Our goal is to find an admissible control, which satisfies (3.3) and maximizes the performance measure (3.4).

In particular, the statement (3.1)-(3.4) can be illustrated by considering a reliable machine, which production is intended to track an external demand. The demand is either stochastic with the average rate \( \mu \) and variance \( \sigma^2 \), or random jumps occurring at integer time points; the control function is the production rate at time \( t \); the state variable is the inventory level. When \( x_t \) is positive, it is surplus; when \( x_t \) is negative, it is backlog. The objective is to maximize the difference between the revenue from the cumulative sales and the total cost of inventory, backlog and production.

In this thesis, we construct numerical methods for solving four particular cases originated from the problem (3.1)-(3.4), each emphasizing different features of the problem. In Case 1, we study a production control problem along a finite horizon, with
only discrete-time demand. In this case we assume that the orders are completely backlogged. In Case 2 we consider a partially backlogged model with lost sales. In Case 3 we model the effect of lost sales on the reputation value of the firm, which in turn, influences the demand distribution at future time periods. In Case 4, we consider a production control problem with continuous-time demand and assume that the orders are completely backlogged.

3.3. Formulation of a problem with discrete-time demand and complete backlogging (Case 1)

We consider the problem (3.1)-(3.4) with no diffusion, no impact of reputation on the demand distribution and no lost sales. No lost sales means that the sales at time \(t, s_t\) equals the demand realization at time \(t\), regardless of the inventory level,

\[
s_t(\xi_t, x_t) = \xi_t. \tag{3.5}
\]

In such a case, the cumulative sales are equal to cumulative demand,

\[
S_r = D_r = \sum_{i=0}^{T-1} \xi_i \theta(t - t), \quad \text{and, therefore, } E[S_{T-1}] \text{ in objective (3.4) is constant. Thus, the problem is}
\]

\[
J(x) = E \left[ \int_0^T \left( C(x_r) + cu_r \right) dr \right|_{x_0 = x} \rightarrow \min
\]

subject to

\[
x_r = x_0 + \int_0^r u_r dr - D_r, \tag{3.6}
\]

\[0 \leq u_r \leq V.\]
3.4. Formulation of a problem with discrete-time demand and lost sales (Case 2)

Similar to Case 1, we formulate a production control problem along a finite time horizon with discrete-time demand. However, the sales at each time $t$, $s_t(\xi_t, x_{t-})$ can be less than the realization of demand at $t$, $\xi_t$, that results from the lost sales effect.

When there is a backlog, after the demand realization at time $t$, $x_t = x_{t-} - \xi_t < 0$, a potential customer who has to wait to get a service, chooses not to order with probability $P(x_{t-} - \xi_t)$. Tan and Gershwin (2004) called $P(x)$, the defection function, and defined it as a non-increasing function of $x$,

$$
P(x) = \begin{cases} 
0, & x \geq 0 \\
(0,1] & x < 0
\end{cases}.
$$

(3.7)

The first line in (3.7) says that no potential customers are motivated to defect, when there is a surplus. The second line indicates that there are always some customers that refuse to wait, if there is a backlog. In this work, we use a linear defection function, (see also Figure 3-1a). However, the analysis conducted in the sequel does not significantly change for arbitrary continuous and non-increasing $P(x)$. 
Figure 3-1a: An example of the defection function $P(x)$

Since $P(x)$ is a non-increasing function, more customers are impatient, if there is a longer wait. The defection function can be reformulated in terms of waiting time. Since the waiting time of a customer, who orders at $t$, $WT_t$, is deterministic and proportional to the backlog value, $x_t$, with the coefficient $1/V$, $WT_t = \frac{x_t}{V}$, the function $P(x)$ in (3.8) can be re-formulated as (see also Figure 3-1b),

$$P(WT) = \begin{cases} 
0, & WT \leq 0 \\
-a \cdot WT, & 0 < WT \leq \frac{1}{aV} \\
1, & WT > \frac{1}{aV}
\end{cases}$$
In the sequel, we deal with the defection function in terms of backlog.

Now, the sales at time $t$ are defined as

$$s_t(x_t, x_{t-}) = \xi_t + P(x_{t-} - \xi_t) \cdot (x_t^+ - \xi_t),$$

(3.9)

where $x^+ = \max(x, 0)$.

Figure 3-2 presents a graph of sales, $s(\xi, x) = \xi + P(x - \xi) \cdot (x^+ - \xi)$ as a function of the inventory level $x$ before the demand realization $\xi$, $\xi < \frac{1}{a}$, while Figure 3-3 presents a similar graph, given $\xi > \frac{1}{a}$. Consider some specific scenarios of sales and lost sales.

**Scenario 1**: when $x > \xi > 0$, all customers make their orders, $P(x - \xi) = 0$. As a result, the sales are full, $s = \xi$.

**Scenario 2**: when $\xi > x > 0$ and $x - \xi > -\frac{1}{a}$, a part of the customers who have to wait because of shortage, decide not to order, $P(x - \xi) = -a(x - \xi)$. As a result, the sales are partial, $s = \xi - a(x - \xi)^2$. 

Figure 3-2b: An example of the defection function $P(WT)$
Scenario 3: when $\xi > x > 0$ and $x - \xi < -\frac{1}{a}$, all the customers who have to wait because of shortage, decide not to order, $P(x - \xi) = 1$. As a result, the sales are $s = x$. This scenario can hold only for $\xi > \frac{1}{a}$ (Figure 3-3).

Scenario 4: when $x < 0$ and $x - \xi > -\frac{1}{a}$, a part of the customers who have to wait because of shortage, decide not to order, $P(x - \xi) = -a(x - \xi)$. As a result, the sales are $s = \xi + a^2(x - \xi)$. This scenario can hold only for $\xi < \frac{1}{a}$ (Figure 3-2).

Scenario 5: when $x < 0$ and $x - \xi < -\frac{1}{a}$, all the customers who have to wait because of shortage, decide not to order, $P(x - \xi) = 1$. As a result, the sales are zero, $s = 0$.

\[ s(\xi, x) = \xi + P(x - \xi)(x^+ - \xi) \]

Figure 3-3: The sales function for a demand realization $\xi < \frac{1}{a}$.
Figure 3-4: The sales function for a demand realization \( \xi > \frac{1}{a} \)

Figure 3-4 presents a graph of sales, as a function of \( \xi \), given a positive level of inventory, \( x > 0 \), while Figure 3-5 presents a similar graph of the sales function, given a negative level of inventory, \( -\frac{1}{a} < x \leq 0 \). Figures 3-4 and 3-5 present the possible scenarios as follows:

**Scenario 1**: when \( \xi > x + \frac{1}{a} \), all the customers who have to wait because of shortage, decide not to order, \( P(x - \xi) = 1 \), and the sales are \( s = x^+ \).

**Scenario 2**: when \( x^+ < \xi < x + \frac{1}{a} \), part of the customers who have to wait because of shortage, decide not to order, \( P(x - \xi) = -a(x - \xi) \), and \( s = \xi - a(x - \xi)(x^+ - \xi) \).

**Scenario 3**: when \( 0 \leq \xi < x^+ \), all customers make their orders, \( P(x - \xi) = 0 \), and the sales are full, \( s = \xi \). This scenario can hold only for \( x > 0 \) (Figure 3-4).
\[ s(\xi, x) = \xi + P(x - \xi)(x^+ - \xi) \]

Figure 3-5: Sales function for a given inventory level, \( x > 0 \)

\[ s(\xi, x) = \xi + P(x - \xi)(x^+ - \xi) \]

Figure 3-6: Sales function for a given inventory level, \(-\frac{1}{a} \leq x < 0\)

Now, the cumulative sales process, \( S_\tau \), is not equivalent to the cumulative demand \( D_\tau \), because of the lost sales effect,

\[ D_\tau = \sum_{t=0}^{\tau-1} \xi_t \theta(\tau - t), \]
$$S_t = \sum_{i=0}^{T-1} \left[ \xi_i + P(x_{t-} - \xi_i) \cdot (x_{t-} - \xi_i) \right] \theta(t - t), \quad (3.10)$$

and the production control problem is,

$$J(x) = \mathbb{E} \left[ pS_{T-1} - \int_0^T (C(x_r) + cu_r) dr \bigg| x_0 = x \right] \rightarrow \text{max}$$

subject to

$$x_t = x_0 + \int_0^t u_r dr - S_t, \quad (3.11)$$

$$0 \leq u_t \leq V.$$

### 3.5. Formulation of a problem with lost sales and discrete-time, reputation-dependent demand (Case 3)

Case 3 considers the reputation of the firm $R_t$, as a factor that reflects the performance of the firm in the past. In (3.1) we defined the dynamics of $R_t$ as exponential smoothing. Now, we can be more specific and update the reputation value in terms of the defection function defined in the previous section. On the other hand, the reputation $R_{t-1}$ influences the distribution of demand in the current period, $\pi_{i,R_{t-1}}(\cdot)$, which now depends on $t$ and $R_{t-1}$. To specify this influence, we assume that the expected demand is proportional to $R_{t-1}$,

$$E(d_t) = \mu R_{t-1}, \quad (3.12)$$
where $\mu$ is the expected demand for the maximum possible reputation $R_t = 1$. The other moments of the demand distribution do not change when $R_t$ changes. In other words, the distribution $\pi_{t,R_t}(\cdot)$ shifts with no change of its form. The value of the shift is proportional to the value of the reputation change.

Let the distribution of demand $\pi_{t,R_t}(\cdot)$ be defined within the range $[\alpha, \beta]$, when $R_{t-1} = 1$ (see Figure 3-6). That is, realizations of demand greater than $\beta$ and smaller than $\alpha$ cannot occur. For $R_{t-1}$ less than one, the demand distribution shifts to the interval $[\alpha - \mu(1-R_{t-1}), \beta - \mu(1-R_{t-1})]$. To keep the demand non-negative, we have to ensure $R_t \geq \frac{\mu - \alpha}{\mu}$ for all $t$. Now, the dynamics of $R_t$ can be expressed as follows (see (3.1)),

$$R_t = R_{t-1} + (1 - \gamma) \left(1 - \frac{\alpha}{\mu} \mu \left(x_{t-1} - \xi_t\right)\right).$$

(3.13)

Expression (3.13) satisfies the properties the reputation dynamics must have:

- it is a particular case of (3.1);
- the “better” the system states in the past, $x_t, x_{t-1}, \ldots, x_1$, the higher the reputation $R_t$;
- the more the defection rate at $t$, the less the reputation at $t$;
- as more recent the defection, the less the reputation at $t$;
- $R_t \in \left[\frac{\mu - \alpha}{\mu}, 1\right]$ for all $t$, which ensures non-negative demand realizations.

We believe the expression (3.13) is not the only way to present the reputation update satisfying the above properties. Discussion of the other models of the reputation update is out of the scope of this work.
We wish to note that our modeling of demand resembles the modeling of Ernst and Powell, ((1995), (1998)) as discussed in Section 2.6 (see (2.2)). To make the similarity clearer, we can re-write (3.13) as,

\[
\mu R_t = \left( 1 + (1 - \gamma) \left[ \frac{1 - \frac{\alpha}{\mu} P(x_t - \xi_t) - R_{t-1}}{R_{t-1}} \right] \right) \mu R_{t-1} \quad \text{.} \tag{3.14}
\]

The left-hand side of (3.14) is the expected future demand (see the left-hand side of (2.2)). In the right-hand side of (3.14) the expression \( \frac{1 - \frac{\alpha}{\mu} P(x_t - \xi_t)}{R_{t-1}} \) can be related to the current service level, and the ratio \( \frac{1 - \frac{\alpha}{\mu} P(x_t - \xi_t) - R_{t-1}}{R_{t-1}} \) shows the relative improvement (deterioration) of the service level. The factor \((1 - \gamma)\) in our model is equivalent to the \(\alpha\) factor in (2.2).

The production control problem is,

\[
J(x) = E \left[ pS_{T-1} - \int_0^T (C(x) + cu) d\tau \bigg| x_0 = x \right] \rightarrow \max
\]

subject to

\[
x_t = x_0 + \int_0^t u_r dr - S_x(\omega) ,
\]

\[
R_t = \gamma R_{t-1} + (1 - \gamma) \left( 1 - \frac{\alpha}{\mu} P(x_t - \xi_t) \right) , \tag{3.15}
\]

\[0 \leq u_t \leq V , \]
3.6. Formulation of a problem with continuous-time demand and complete backlogging (Case 4)

In Case 4, we consider a production control problem along either infinite or finite time horizon with continuous-time demand, modeled by a Wiener process with drift $\mu$ and variance $\sigma^2$, assumed constant. The inventory level at time $t$, $x_t$, satisfies the stochastic differential equation:

$$dx_t = u_t dt - \mu dt + \sigma dW_t, \quad x_0 \text{ is given},$$

(3.16)

where $u_t$ defines the intensity of production at time $t$. Similar to the discrete-time cases, the production rate is bounded by its maximum value $V$,

$$0 \leq u_t \leq V.$$

(3.17)

For the sake of simplicity, we have assumed that $\mu$ and $\sigma$ are constants, as well as $0 \leq \mu \leq V$. In a case when $\mu$ and $\sigma$ depend on $t$ and $x_t$, more numerical work is
needed to approximate the optimal control policy. In Sections 5.5.2 and 5.5.3 we construct an optimal policy only for the simplified case. Thus, the problem is to minimize the total expected cost of production and inventory discounted by factor $\gamma$,

$$J(x) = E\left[\int_0^T (C(x_t) + cu_t) e^{-\gamma r} dr \bigg| x_0 = x\right] \rightarrow \min,$$  

(3.18)

for a finite horizon formulation and,

$$J(x) = E\left[\int_0^\infty (C(x_t) + cu_t) e^{-\gamma r} dr \bigg| x_0 = x\right] \rightarrow \min,$$  

(3.19)

for an infinite horizon formulation.
4. Optimality conditions for the stated problems

4.1 Background

In this section, we derive necessary optimality conditions for all cases stated in the previous chapter, by using perturbation analysis (see, e.g., Bryson and Ho, (1975)). Specifically, we consider a small variation of the optimal control and declaring that no such variation can improve the system performance.

4.2 Optimality conditions for the problem with discrete-time demand and complete backlogging (Case 1)

We make use of the fact that no stochastic event occurs between adjacent integer times, \( t \) and \( t + 1 \), and consider a small variation of the optimal control \( \delta u_t(\omega) \) for each \( \omega \in \mathcal{I}_t \), as follows

\[
\delta u_t(\omega) = \begin{cases} 
\Delta u, & \text{if } t < r < r + \Delta t < t + 1 \\
0, & \text{otherwise}
\end{cases}
\]

(4.1)

where \( \Delta t \) is a small interval within which the variation is non-zero and \( \tau \) is the point where the variation is placed. This is a needle-shaped variation (Sagan, (1969)). The influence of the variation on the state variable, \( x_r(\omega) \), is:

\[
\delta x_r(\omega) = \begin{cases} 
\Delta u \Delta t, & \text{if } r > \tau + \Delta t \\
O(\Delta t), & \text{if } \tau < r \leq \tau + \Delta t \\
0, & \text{if } r \leq \tau
\end{cases}
\]

(4.2)

The variation of the objective is:
\[ \delta J = E\left[ \int_0^\tau (C'(x_\tau) \delta x_\tau + c \delta u_\tau) \, dr \right] \bigg| \mathcal{F}_\tau. \] (4.3)

By substituting (4.1) and (4.2) into (4.3) and taking into account that no variation of the optimal control can improve the objective, we obtain

\[ \delta J = \left( -E\left[ \psi_\tau \bigg| \mathcal{F}_\tau \right] + c \right) \Delta t \Delta u \geq 0, \] (4.4)

where the costate variable \( \psi_\tau \) at time \( \tau \) is defined as,

\[ \psi_\tau(\omega) = -\int_\tau^\tau C'(x_\tau(\omega)) \, dr. \] (4.5)

From (4.4) it immediately follows that the optimal control can have three possible regimes. At the first one, where \( u_\tau = V \), a possible variation \( \Delta u \) can only be negative, therefore

\[ E\left[ \psi_\tau \bigg| \mathcal{F}_\tau \right] \geq c. \] (4.6)

At the second regime, where \( u_\tau = 0 \), a possible variation \( \Delta u \) can be only positive, therefore

\[ E\left[ \psi_\tau \bigg| \mathcal{F}_\tau \right] \leq c. \] (4.7)

Finally, at the third (singular) regime, where \( 0 < u_\tau < V \), a possible variation \( \Delta u \) can be either positive or negative, therefore

\[ E\left[ \psi_\tau \bigg| \mathcal{F}_\tau \right] = c. \] (4.8)

The necessary optimality conditions (4.6)-(4.8) are concisely re-written as
Recall that \( t \) in (4.9) is the largest integer less than \( \tau \). The conditions (4.9) express the optimal control as a function of the costate variable. Further analysis of (4.9), as well as of the state-costate dynamics is needed to find an equivalent feedback control rule, that is the rule that determines the optimal control as a function of the state variable. This is done in the next chapters of the thesis. Conditions similar to (4.9) have also been obtained and analyzed in (Khmelnitsky et al., 2004) for a production control problem of an unreliable machine and constant demand rate.

### 4.3 Optimality conditions for the problem with discrete-time demand and lost sales (Case 2)

Consider a control variation as in (4.1). Now, the variation of inventory differs from that of the previous case because of the lost sales effect,

\[
\delta x_r(\omega) = \begin{cases} 
  \Delta u \Delta t - \delta S_r, & \text{if } r > \tau + \Delta t \\
  O(\Delta t), & \text{if } \tau < r \leq \tau + \Delta t, \\
  0, & \text{if } r \leq \tau 
\end{cases}
\]  

where \( \delta S_r \) is the change in the cumulative sales up to time \( r \), resulting from the control variation (4.1). The objective variation is

\[
\delta J = E \left[ p \delta S_{\tau-1} - \int_0^\tau (C'(x_r) \delta x_r + c \delta u_r) \, dr \right]_{\mathcal{F}_r}.  
\]  

(4.11)
The change in the cumulative sales at \( t \) is zero, \( \delta S_t = 0 \), because the point \( t \) is earlier than the control variation (see (4.1)) takes place. At every integer point behind the control variation, the cumulative sales change, and the change value can be recursively calculated as follows,

\[
\delta S_t = 0
\]

\[
\delta S_q = \delta S_{q-1} + \frac{\partial S_q(\xi_q, x_{-q})}{\partial x_{-q}} (\Delta u \Delta t - \delta S_{q-1}), \quad q = t + 1, t + 2, ..., T - 1. \tag{4.12}
\]

Within the time interval between the adjacent integer points \( \delta S_t \) is constant and equals \( \delta S_t \), where \( t \) is the largest integer less than \( \tau \).

For conciseness we denote the derivative \( \frac{\partial S_q(\xi_q, x_{-q})}{\partial x_{-q}} \) by \( s_q' \) and the derivative

\[
\lim_{\Delta \to 0} \frac{\delta S_q}{\Delta u \Delta t} \text{ by } S_q'.
\]

Then, the recursion (4.12) can be re-written as

\[
1 - S'_q = 1
\]

\[
1 - S'_q = (1 - S'_{q-1})(1 - s'_q), \quad q = t + 1, t + 2, ..., T - 1, \tag{4.13}
\]

or, equivalently

\[
1 - S'_q = \prod_{i=t}^{q} (1 - s'_i). \tag{4.14}
\]

Now, the objective variation (4.11) becomes

\[
\delta J = E \left[ p S'_{T-1} - \int_{\tau}^{T} C'(x_r)(1 - S'_q)dr - c \bigg| \mathcal{F}_t \right] \Delta u \Delta t,
\]

or, in terms of the costate variable

\[
\delta J = \left( E\left[ \psi_r \bigg| \mathcal{F}_t \right] - c \right) \Delta u \Delta t, \tag{4.15}
\]

where
\[ \psi_r = -\int_{\tau}^{T} C'(x_r)(1-S'_r)\,dr + pS'_{r-1}. \]  

(4.16)

Since the problem we are dealing with in Case 2 is a maximization one, from (4.15) it follows that the necessary optimality conditions should be written as in the previous section (see (4.9)). The difference is in the meaning of the costate variable. It is now influenced by the lost sales effect, i.e., by the function \( S'_r \), which was equal to zero in the previous section when all unfilled demand was backlogged.

4.4 Optimality conditions for the problem with lost sales and discrete-time, reputation-dependent demand (Case 3)

In this case, the system we are dealing with is characterized by two state variables, inventory level \( x_r, \ r \in [0, T] \), and reputation value \( R_t, \ t = 0, 1, ..., T-1 \). We alter the optimal control by the same variation (4.1) and obtain the same variation of \( x_r \) (see (4.10)). However, the meaning of \( \delta S_r \) is now different, because the demand distribution now depends on the reputation. Again, \( \delta S_r = 0 \), since the variation is placed after \( t \). But, at the subsequent integer points

\[ \delta S_q = \delta S_{q-1} + s'_q \left( \Delta u \Delta t - \delta S_{q-1} \right) + \dot{s}_q \mu \delta R_{q-1}, \quad q = t+1, t+2, ..., T-1, \]  

(4.17)

where \( \dot{s}_q \) is the derivative of the sales function (3.9) w.r.t. \( \xi_q \), \( \dot{s}_q = \frac{\partial s(\xi_q, x_q)}{\partial \xi_q} \), and \( \delta R_q \) is the variation of the reputation at \( q \).
The calculation of the reputation change is also recursive,

\[ \delta R_t = 0 \]

\[ \delta R_q = \gamma \delta R_{q-1} - \frac{\alpha(1 - \gamma)}{\mu} P_q' \cdot (\delta x_{q-1} - \delta \xi_{q-1}) \]  \hspace{1cm} \text{(4.18)}

\[ = \gamma \delta R_{q-1} - \frac{\alpha(1 - \gamma)}{\mu} P_q' \cdot (\Delta u \Delta t - \delta S_{q-1} - \mu \delta R_{q-1}), \quad q = t + 1, t + 2, \ldots, T - 1 \]

where \( P_q' \) is the derivative of the defection function at time \( q \). When \( \Delta t \to 0 \), we obtain

(4.17) and (4.18) in derivatives,

\[ 1 - S'_t = 1 \]

\[ 1 - S'_q = (1 - S'_{q-1})(1 - s'_q) - \dot{s}_q \mu R'_q, \quad q = t + 1, t + 2, \ldots, T - 1 \]  \hspace{1cm} \text{(4.19)}

and

\[ R'_t = 0 \]

\[ R'_q = (\gamma + \alpha(1 - \gamma)P_q')R_{q-1} - \frac{\alpha(1 - \gamma)}{\mu} P_q'(1 - S'_{q-1}), \quad q = t + 1, t + 2, \ldots, T - 1. \]  \hspace{1cm} \text{(4.20)}

The above system of the sales and reputation derivatives can be presented in a vector form,

\[ \begin{bmatrix} 1 - S'_t \\ R'_t \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \]

\[ \begin{bmatrix} 1 - S'_q \\ R'_q \end{bmatrix} = Z_q \begin{bmatrix} 1 - S'_{q-1} \\ R'_{q-1} \end{bmatrix}, \quad q = t + 1, \ldots, T - 1, \]  \hspace{1cm} \text{(4.21)}

where matrix

\[ Z_q = \begin{bmatrix} 1 - s'_q & -\dot{s}_q \mu \\ -\frac{\alpha(1 - \gamma)}{\mu} P_q' & \gamma + \alpha(1 - \gamma)P_q' \end{bmatrix}. \]  \hspace{1cm} \text{(4.22)}

Now, we can calculate the objective variation,
\[ \delta J = E \left[ pS'_{t-1} - \int_\tau^T C'(x_r)(1 - S'_r)dr - c \right] \Delta u \Delta t, \]

or, in terms of the costate variable

\[ \delta J = \left( E \left[ \psi_\tau \right] - c \right) \Delta u \Delta t, \]

where

\[ \psi_\tau = - \int_\tau^T C'(x_r)(1 - S'_r)dr + pS'_{t-1}. \quad (4.23) \]

Contrary to the previous section, the term \((1 - S'_r)\) in (4.23) is not independent. Its dynamics now depends on the dynamics of \(R'_r\), as follows from (4.21). Therefore, we need to introduce a second costate variable and update the costate dynamics in a matrix form.

Let

\[ \phi_\tau = - \int_\tau^T C'(x_r)R'_dr - pR'_{t-1} + p \quad (4.24) \]

be the second costate variable. Then, the joint costate dynamics is defined by combining (4.23) and (4.24),

\[ \begin{bmatrix} \psi_\tau \\ \phi_\tau \end{bmatrix} = - \int_\tau^T C'(x_r) \begin{bmatrix} 1 - S'_r \\ R'_r \end{bmatrix} dr - p \begin{bmatrix} 1 - S'_{t-1} \\ R'_{t-1} \end{bmatrix} + p \begin{bmatrix} 1 \\ 1 \end{bmatrix}. \quad (4.25) \]

With respect to the basic principles of optimal control, a two-dimensional problem has to have two costate variables, \(\psi_\tau\) and \(\phi_\tau\) in our case. In spite of the fact that the second costate variable \(\phi_\tau\) does not appear explicitly in the optimality conditions (4.9),
and does not explicitly influence the control function $u_t$, it has to be calculated, since it
explicitly influences the dynamics of $\psi_t$.

### 4.5 Optimality conditions for the problem with continuous-time
demand and complete backlogging (Case 4)

Following the main steps of the previous sections of this chapter, we derive the necessary
optimality conditions for the case of Wiener-type demand. The difference with regard to
the previous sections is the discount factor of the total cost. Now the conditions are

$$
\begin{align*}
&u_t = \begin{cases} 
V, & \mathbb{E}[\psi_t | \mathcal{F}_t] \geq ce^{-\gamma t} \\
0, & \mathbb{E}[\psi_t | \mathcal{F}_t] \leq ce^{-\gamma t}, \\
\in [0, V], & \mathbb{E}[\psi_t | \mathcal{F}_t] = ce^{-\gamma t}
\end{cases} \\
&\text{where} \\
&\psi_t = -\int_t^T C'(x_r)dr.
\end{align*}
$$

(4.26)

(4.27)
5. State-costate dynamics, properties of optimal policies and feedback control rules

5.1 Background

In this chapter, we continue studying the four cases of production control presented in the previous chapters, by means of analytical methods. For the first three cases (discrete-time demand) we determine the dynamics of the state and costate variables by forward-backward integration of the state and costate equations. This allows transforming the necessary optimality conditions derived in the previous chapter into an optimal feedback control rule. For the fourth case (continuous-time demand) we prove some monotonicity properties of the costate stochastic process. From those properties it follows that there exists a time-dependent hedging line that determines an optimal feedback control rule. The analysis conducted in this chapter is not sufficient to put down the optimal control rules explicitly. However, the obtained properties are quite constructive and the information about the optimal control rules is enough to lead us to developing efficient numerical procedures for approximating the rules. The numerical procedures will be presented in the next chapter.

To simplify the discussion and numerical work, and without loss of generality, we assume the production cost parameter $c$ is zero, and that the inventory cost function $C(x)$ is linear, i.e.,

$$C(x) = xsig(x),$$ (5.1)

where
\[
\text{sig}(x) = \begin{cases} 
  c^+, & \text{if } x > 0 \\
  0, & \text{if } x = 0, \\
  -c^-, & \text{if } x < 0 
\end{cases}
\] 

(5.2)

c^+ and c^- are holding inventory cost and backlog cost (per unit and per time unit) respectively (see Figure 5.1).

Figure 5-1: Linear cost function \( C(x) = x\text{sig}(x) \)

5.2 State-costate dynamics and feedback control rule for the problem with discrete-time demand and complete backlogging (Case 1)

Consider the time interval \((t, t+1)\), that is, the time between adjacent realizations of demand. No uncertain events occur within the interval. We wish to know what is the target inventory level at the end of the interval, \( x_{t+1-} \) as a function of the system state, \( x_t \), at the beginning of the interval? The second question is what is the optimal trajectory that leads from \( x_t \) to \( x_{t+1-} \)? The answer to these two questions defines the optimal feedback rule in the considered time interval.
Contrary to the well-known rules suggested in the inventory theory, such as \((S,s)\), \((Q,R)\) and up-to-a-given-stock policies, the optimal rule in our case is such that the target level is an explicit function of the initial inventory level. To illustrate this, we calculate the total cost of inventory for a simple example (see Figure 5-2). Let \(x\) and \(y\) be the beginning and target inventory in the interval \((t,t+1)\). Let \(x > 0\) and \(y - x < V\). In such a case, the optimal inventory behavior is such as given in Figure 5-2, namely no production at an initial part of the interval and full production toward the end of the interval, as will be shown in Section 5.2.1. Then, the total expected cost, \(J_t\), consists of the holding cost incurred at that interval plus the expected cost incurred beyond \(t+1\), i.e.,

\[
J_t = c^+ \left( x + \frac{(y-x)^2}{2V} \right) + J_{t+1}(y).
\]  
(5.3)

Due to the Markovian property, the cost beyond \(t+1\), \(J_{t+1}(y)\), depends only on \(y\). Given \(x\), we differentiate \(J_t\) with respect to \(y\) and obtain,

\[
\frac{\partial J_t}{\partial y} = \frac{c^+(y-x)}{V} + \frac{\partial J_{t+1}(y)}{\partial y} = 0.
\]  
(5.4)

The last equality shows that the optimal target level \(y\) is a function of the beginning level \(x\).

We will call this type of control policy a discrete-time hedging policy. This is for two reasons. First, as in continuous-time hedging policy, there is always a target to be reached. Second, when the length of the time interval between the adjacent realizations of demand tends to zero, the policy appears to converge to a hedging line policy as presented in Section 5.5.
Generally, we describe the state-costate dynamics separately at the integer points \( \tau = t, \ t = 0,1,...,T-1 \), and in the intervals between the jumps, \( \tau \in (t,t+1) \). First, we note that the dynamics within the intervals is deterministic, satisfying the state equation

\[
\frac{dx_{\tau}}{d\tau} = u_{\tau},
\]

(5.5)

the costate equation

\[
\frac{d\psi_{\tau}}{d\tau} = C'(x_{\tau}) = \text{sig}(x_{\tau}),
\]

(5.6)

and the optimality conditions

\[
u_{\tau} = \begin{cases} V, & E[\psi_{\tau}|x_{\tau}] > 0 \\ 0, & E[\psi_{\tau}|x_{\tau}] < 0 \\ \in [0,V], & E[\psi_{\tau}|x_{\tau}] = 0 \end{cases}
\]

(5.7)
A specific form of the trajectory, $x$, and $\psi$, $\tau \in [t, t+1)$, is obtained from (5.5)-(5.7) when boundary conditions, $x_i$ and $z_i = E[\psi_{t+1} | x_i]$ are given. The boundary condition for the state variable is given at the left limit of the time interval, while that of the costate variable is given at the right limit. Consequently, the integration of (5.5) is carried out from left to right, and of (5.6) is from right to left. As a result of the integration, the right-limit value of the state variable and the left-limit value of the costate variable are obtained. We denote them by $B_i(x_i, z_i)$ and $F(x_i, z_i)$ respectively, i.e., $x_{i, -} = B_i(x_i, z_i)$ and $E[\psi_i | x_i] = F(x_i, z_i)$.

### 5.2.1 State-costate dynamics

There are six different forms the state-costate dynamics can have, given boundary conditions $x_i = x$ and $z_i = z$ (see Figure 5.3).

**First form, (a) (Figure 5.3a):** For a sufficiently large $x$, the costate variable must be negative for the system not to produce waste inventory. This follows from the optimality conditions (5.7). From the costate dynamic equation (5.6), we have $\dot{\psi} = c^+$, that is, within the interval $\tau \in [t, t+1)$, $\psi$ increases with the constant rate $c^+$, as shown in Figure 5.3a). Thus, the state variable does not change, $B_i(x, z) = x$, and the costate value increases by $c^+$, $F(x, z) = z - c^+$. The first form works as long as the costate value at the right limit of the interval is negative, $z \leq 0$, and $x > 0$.

**Second form, (b) (Figure 5.3b):** The costate variable intersects the time axis ($t'$ is the intersection point), while the state variable remains positive along the interval. From the costate dynamic equation (5.6), we have $\dot{\psi} = c^+$, for all $\tau \in [t, t+1)$. Now, according to
(5.5) and (5.7), the state variable does not change before \( t' \), and increases with the rate of \( V \) after that. \( B_i(x,z)=x+\frac{zV}{c^+} \) and \( F(x,z)=z-c^+ \). The second form works as long as the costate value at the right limit of the interval is lower than \( c^+ \), \( z \leq c^+ \) and \( x > 0 \).

**Third form, (c) (Figure 5.3c):** Both state and costate variables are positive within the interval \( t \in [t,t+1] \). From the costate dynamic equation (5.6), as in the previous two forms, we have \( \dot{\psi}_t = c^+ \). Now, the state variable increases by \( V \). \( B_i(x,z)=x+V \) and \( F(x,z)=z-c^+ \). The third form works as far as \( z > c^+ \) and \( x > 0 \).

**Fourth form, (d) (Figure 5.3d):** The initial inventory is negative and a singular control regime (the third line of (5.7)) takes place in a part of \([t,t+1] \). Both state and costate variables must be zero along the singular regime. Before the singular regime, the costate decreases because of negative state. On the contrary, after exiting the singular regime, the costate increases allowing the state to increase and to reach a positive value optimal to start the next time period. \( B_i(x,z)=\frac{zV}{c^+} \) and \( F(x,z)=-c^-\frac{x}{V} \). The forth form works as far as the length of the singular regime is greater than zero, i.e., \( -\frac{x}{V} + \frac{z}{c^+} < 1 \), and \( x < 0 \).

**Fifth form, (e) (Figure 5.3e):** The initial inventory is too much negative for the singular regime can occur. The state increases all the time, and the costate, being positive, changes the slope at the point where the state changes the sign. \( B_i(x,z)=x+V \) and \( F(x,z)=-c^-\frac{x}{V} \). The fifth form works as far as \( -V < x < 0 \) and \( z > c^+\left(1+\frac{x}{V}\right) \). The former inequality follows from the state dynamics and the latter from the costate dynamics.
**Sixth form, (f) (Figure 5.3f):** The state variable is negative over the entire interval despite
full production. \( B_t(x, z) = x + V \) and \( F(x, z) = z + c^- \). The sixth case works as long as
\(-\infty < x \leq -V \) and \( z > 0 \).

It can be shown by contradiction that no other form of the state and costate dynamics
can satisfy (5.5)-(5.7).

At an integer time point, the state-costate dynamics are uncertain,

\[
x_t = x_{t-} - \xi_t, \quad (5.8)
\]

\[
z_t = \int_0^\infty \pi_{t+1}(\xi)E[\psi_{t+1} | x_{t+1} = B_t(x_t, z_t) - \xi_t]d\xi. \quad (5.9)
\]

Note that the state variable is calculated from left to right in (5.8) and the costate
variable calculated in the opposite direction in (5.9), as it was within the time interval.
5.2.2 Feedback rule

To convert the target inventory level in period \( t \), \( B_t(x,z) \), into a function of \( x_t \) only, \( \bar{B}_t(x_t) \), i.e., into a feedback rule, we need the following consideration. Let a function \( A_{t+1}(\cdot) \) be given for each integer \( t \). The function \( A_{t+1}(\cdot) \) is defined as
\[ A_{t+1}(y) = \int_0^\infty \pi_{t+1}(\xi) E[y_{t+1} | x_{t+1} = y - \xi] d\xi. \] (5.10)

It determines the costate value at the right limit of the time interval \([t, t+1)\) as a function of the state value at the same limit. Equation (5.10) basically repeats (5.9). Now, the right-side state limit (target inventory) can be found as a root of the following equation

\[ B_t(x_t, A_{t+1}(y)) = y. \] (5.11)

If \(y^*(x_t)\) is a solution of (5.11), then the target inventory level is a function of \(x_t\) only,

\[ \bar{B}_t(x_t) = B_t(x_t, A_{t+1}(y^*(x_t))). \] (5.12)

In the next chapter we show how the function \(A_{t+1}(\cdot)\) is recursively computed from \(t = T - 1\) down to \(t=0\).

A specific state-costate form (one out of the six discussed in the previous section), which leads the state trajectory from \(x_t\) to \(\bar{B}_t(x_t)\) is now determined by a feedback rule, too. Namely,

(a), if \(A_{t+1}^{-1}(0) < x < \infty\), \(\bar{B}_t(x) = x\) and \(F(x, z) = z - c^+\).

(b), if \(\max\{0, A_{t+1}^{-1}(c^+) - V\} < x \leq A_{t+1}^{-1}(0)\), \(\bar{B}_t(x) = y\), where \(y\) is the root of the equation

\[ \frac{y - x}{V} = \frac{A_{t+1}(y)}{c^+}, \] and \(F(x, z) = z - c^+\).

(c), if \(0 < x \leq \max\{0, A_{t+1}^{-1}(c^+) - V\}\), then \(\bar{B}_t(x) = x + V\) and \(F(x, z) = z - c^+\).

(d), if \(\min\{0, y - V\} < x \leq 0\), where \(y\) is the root of the equation

\[ \frac{y}{V} = \frac{A_{t+1}(y)}{c^+}. \] \(\bar{B}_t(x) = y\)

and \(F(x, z) = -c^- \frac{x}{V}\).
(e), if \(-V < x \leq \min \{0, y-V\}\), where \(y\) is the root of the equation \(\frac{y}{V} = \frac{A_{i+1}(y)}{c^+}\).

\(\overline{B}_i(x) = x + V\), and \(F(x, z) = z + c^+ + (c^+ - c^-) \frac{x}{V}\).

(f), if \(-\infty < x \leq -V\), then \(\overline{B}_i(x) = x + V\) and \(F(x, z) = z + c^-\).

The following lemma proves the existence of the inverse of \(A_i(\cdot)\).

**Lemma 5.1.** The function \(A_i(y) = \int_0^\infty \pi_i(\xi) E[\psi_i \mid x_i = y - \xi] d\xi\) is non-increasing.

**Proof:** Consider a scenario starting at \(t, \omega = (\ldots, \xi_t, \xi_{t+1}, \ldots, \xi_{T-1})\) and two trajectories corresponding to this scenario, \(x^{(1)}_r\) and \(x^{(2)}_r\), \(t \leq r \leq T\). The first trajectory starts with a lower inventory level than the other, \(x^{(1)}_r < x^{(2)}_r \leq x^{(2)}_{t-} = x_2\). Therefore, \(x^{(1)}_r = x_1 - \xi_t < x^{(2)}_r = x_2 - \xi_t\). In the sequel, the two trajectories retain the order, i.e., \(x^{(1)}_r < x^{(2)}_r\) for \(t \leq r \leq T\). Now, from (5.6) and the convexity of the \(C(x)\) function it follows that \(\psi^{(1)}_i \geq \psi^{(2)}_i\). Since the last inequality is true for each scenario, it is true also for the expected values, \(E[\psi_i \mid x_i = x_1 - \xi] \geq E[\psi_i \mid x_i = x_2 - \xi]\) for all \(\xi\). This proves the lemma. ■
The control function for the six cases of the state-costate dynamics is as follows,

For (a), \( u_\tau = 0, \tau \in [t, t+1) \).

\[
\begin{align*}
\text{For (b), } u_\tau &= \begin{cases} 
0, & \tau \in [t, t+1 - \frac{B_t(x_t)}{V} ) \\
V, & \tau \in [t+1 - \frac{B_t(x_t)}{V}, t+1) 
\end{cases} \\
\text{For (c), (d) and (e), } u_\tau = V, \tau \in [t, t+1). 
\end{align*}
\tag{5.13}
\]

The presented control rule satisfies the necessary optimality conditions. Therefore, it is optimal, since the problem is convex.

5.3 State-costate dynamics and feedback control rule for the problem with discrete-time demand and lost sales (Case 2)

Similar to the previous section, we describe the state-costate dynamics separately at the integer points \( \tau = t, t = 0, 1, \ldots, T-1 \), and in the intervals between the jumps, \( \tau \in (t, t+1) \).

The dynamics within the intervals satisfy the state equation

\[
\frac{dx_\tau}{d\tau} = u_\tau, 
\tag{5.14}
\]

the costate equation

\[
\frac{d\psi_\tau}{d\tau} = \text{sig}(x_\tau)(1 - S'_\tau) 
\tag{5.15}
\]
and the optimality conditions

\[
u_t = \begin{cases} 
V, & E[\psi_t | x_t] > 0 \\
x_t - s_t(\xi_t, x_t), & 0 < E[\psi_t | x_t] < 0 \\
[0, V], & E[\psi_t | x_t] = 0
\end{cases} \tag{5.16}
\]

From (4.13) it follows that \( S_t' = 0 \) for \( \tau \in (t, t+1) \). Therefore, equations (5.14)-(5.16) become equivalent to (5.5)-(5.7) and we obtain the same six forms of the state-costate dynamics within the time intervals.

At the integer time points, the state-costate dynamics differs from (5.8)-(5.9). Now,

\[ x_t = x_{t-} - s_t(\xi_t, x_t), \tag{5.17} \]

and the costate variable updates, when coming over from \( t^- \) to \( t^+ \) at each scenario, as follows,

\[
\psi_{t-} = -\int_{t-}^{t} C'(x_t)(1 - S_t')dr + pS_{t-1}' = -\int_{t-}^{t} C'(x_t)\prod_{i=t}^{r} (1 - s_i')dr + p\prod_{i=t}^{r-1} (1 - s_i') = \\
\psi_{t-} = -(1 - s_t')\int_{t-}^{t} C'(x_t)\prod_{i=t+1}^{r} (1 - s_i')dr + p(1 - s_t')\prod_{i=t+1}^{r-1} (1 - s_i') = (1 - s_t')\psi_t + ps_t'. \tag{5.18}
\]

By taking expected values of the both sides of (5.18), we obtain

\[
A_t(y) = \int_{0}^{\infty} \pi_t(\xi)\left((1 - s_t'(\xi, y)) E[\psi_t | x_t = y - s_t'(\xi, y)] + ps_t'(\xi, y)\right) d\xi. \tag{5.19}
\]

Given the above function \( A_t(\cdot) \), we obtain the feedback rule as we did in the previous section.
5.4 State-costate dynamics and feedback control rule for the problem with lost sales and discrete-time reputation-dependent demand (Case 3)

Contrary to the previous two cases, we are now dealing with two state and two costate variables. At the integer time points the state variables are updated as

\[ x_t = x_{t-} - s_t(x_{t-}, x_{t-}) , \]  

\[ R_t = \gamma R_{t-1} + (1 - \gamma) \left( 1 - \frac{a}{\mu} P(x_{t-} - \xi_t) \right) . \]  

To update the costate variables, we first introduce a new matrix

\[ H_t = \int_r^x C'(x_t) \prod_{i=t+1}^r Z_i dr - p \prod_{i=t+1}^{r-1} Z_i . \]  

When going from \( t \) to \( t^- \), the expected value of \( H_t \) is updated as

\[ E[H_{t-}|x_{t-} = y, R_{t-1} = R] = \int_0^\infty \pi_{t,R-1}(\xi) E[H_t|x_t = y - s_t(\xi, y), R_t = \gamma R + (1 - \gamma)(1 - a/\mu) P(y - \xi)] Z_i d\xi . \]  

The feedback rule determines the state dynamics as a function of initial inventory and reputation. Therefore, the costate function in terms of which the rule is formulated depends not only on \( y \) but also on \( R \),

\[ A_{t+1}(y, R) = \begin{bmatrix} E[\psi_{t+1}|x_{t+1} = y, R_t = R] \\ E[\varphi_{t+1}|x_{t+1} = y, R_t = R] \end{bmatrix} = p \begin{bmatrix} 1 \\ 0 \end{bmatrix} + E[H_{t+1}|x_{t+1} = y, R_t = R] \begin{bmatrix} 1 \\ 0 \end{bmatrix} . \]  

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5.5 State-costate dynamics and properties of the optimal policy for the problem with continuous-time demand and complete backlogging (Case 4)

Consider the costate stochastic process

\[ \theta_t = E[\psi_t(\omega) \mid \mathcal{F}_t]. \]  \hspace{1cm} (5.25)

Given a Markovian policy, the process \( x_t \) is uniquely defined, and

\[ \theta_t = \theta(x)e^{-\gamma t}, \]  \hspace{1cm} (5.26)

where

\[ \theta(x) = E[\psi_0 \mid x_0 = x] = E\left[ -\int_0^\infty C'(x_r)e^{-\gamma r}dr \mid x_0 = x \right]. \]  \hspace{1cm} (5.27)

In some extreme cases when the function \( C(x) \) rapidly increases for either \( x \to \infty \), or \( x \to -\infty \), the process \( \theta_t \) exists not for every policy, or cannot exist at all. The next lemma proves a monotonicity property of the function \( \theta(x) \).

**Lemma 5.2** If the costate process \( \theta_t \) exists, then the function \( \theta(x) \) is decreasing w.r.t. \( x \).

**Proof:** For each scenario, \( \omega \), we compare trajectories \( x_r^{(1)} \) and \( x_r^{(2)} \), such that \( x_0^{(1)} > x_0^{(2)} \).

Since both trajectories are continuous in time, the order \( x_r^{(1)} > x_r^{(2)} \) retains over at least a finite initial time interval. If the trajectories \( x_r^{(1)} \) and \( x_r^{(2)} \) intersect at some \( r \), they must coincide everywhere beyond \( r \), because of the chosen Markovian policy. As a result,

\[ \int_0^\infty C'(x_r^{(1)})e^{-\gamma r}dr \geq \int_0^\infty C'(x_r^{(2)})e^{-\gamma r}dr, \]  \hspace{1cm} (5.28)
since \( C'(x) \) is non-decreasing. Expression (5.28) is true for each scenario, therefore, it is true also for the expected values, i.e., \( \theta(x_0^{(1)}) \leq \theta(x_0^{(2)}) \). Moreover, the last inequality must be strong, since under a Wiener process there is a positive probability of the scenarios that go far up and far down with respect to \( x \), where \( C'(x_r^{(1)}) > C'(x_r^{(2)}) \) at the initial time interval before the trajectories \( x_r^{(1)} \) and \( x_r^{(2)} \) coincide. Thus, the function \( \theta(x) \) is strictly decreasing.

\[ \Box \]

### 5.5.1 Hedging policy

Based on the monotonicity property of the costate function proved in Lemma 5.2, in this section we prove that the optimal policy is of a threshold type (hedging policy).

**Lemma 5.3** Let \( \theta(x) \) be the costate function of the optimal policy. Then, the policy is

- if \( \sup_x c \theta(x) \geq 0 \), the policy is \( u_t = 0 \) identically;

- if \( \inf_x c \theta(x) \leq 0 \), the policy is \( u_t = V \);

- if \( \inf_x \theta(x) < c < \sup_x \theta(x) \), the policy is,

\[
    u_t = \begin{cases} 
        V, & \text{if } x_t < \bar{x}, \\
        \ 0, & \text{if } x_t > \bar{x}.
    \end{cases} 
\]

where the threshold value, \( \bar{x} \), is such that \( \theta(\bar{x}) = c \).

**Proof:** If \( c \geq \sup_x \theta(x) \), then \( \theta(x_t) < c \) regardless of \( x_t \). The last inequality is equivalent to the condition (4.26) that requires \( u_t = 0 \).
If \( c \leq \inf_{x} \theta(x) \), then \( \theta(x) > c \) regardless of \( x \). The last inequality is equivalent to the condition (4.26) that requires \( u_{\tau} = V \).

If \( \inf_{x} \theta(x) < c < \sup_{x} \theta(x) \), then from Lemma 5.2 we conclude that there exists a unique value of \( x, \bar{x} \), such that \( c = \theta(\bar{x}) \). The probability measure of the scenarios for which \( x_{\tau} = \bar{x} \) (i.e. constant) over a finite time interval, is zero, because of the Wiener process dynamics. Therefore, the singular control regime cannot be realized.

\[ \blacksquare \]

5.5.2 Calculation of the threshold level \( \bar{x} \)

The value of the optimal threshold satisfies

\[ \theta(\bar{x}) = c. \]

With respect to the definition of the costate function (5.27),

\[ \theta(\bar{x}) = -\int_{-\infty}^{\infty} \int_{0}^{\infty} \frac{\partial G(\kappa, \tau)}{\partial \kappa} C'(\kappa) e^{-\gamma \tau} d \tau d \kappa, \quad (5.30) \]

where \( G(\kappa, \tau) \) is the probability distribution function of \( x_{\tau} \), i.e.,

\( G(\kappa, \tau) = \Pr(x_{\tau} < \kappa \mid x_{0} = \bar{x}) \). From (5.30) we obtain

\[ \theta(\bar{x}) = -\int_{-\infty}^{\infty} \frac{\partial \text{Laplace}(G(\kappa, t))(\gamma)}{\partial \kappa} C'(\kappa) d \kappa, \quad (5.31) \]

where \( \text{Laplace}(G(\kappa, t))(\gamma) \) is the Laplace transform of the function \( G(\kappa, t) \) taken at \( \gamma \).

With respect to the definition, the probability distribution function \( G(\kappa, t) \) satisfies the following partial differential equation
For $x > \bar{x}$:
\[
\frac{\partial G}{\partial t} = \mu \frac{\partial G}{\partial x} + \frac{\sigma}{2} \frac{\partial^2 G}{\partial x^2}, \quad G(x, 0) = 1.
\] (5.32)

For $x < \bar{x}$:
\[
\frac{\partial G}{\partial t} = -(\mu - V) \frac{\partial G}{\partial x} + \frac{\sigma}{2} \frac{\partial^2 G}{\partial x^2}, \quad G(x, 0) = 0.
\] (5.33)

Boundary conditions:
\[
\lim_{x \to \bar{x}^+} G(x, t) = \lim_{x \to \bar{x}^-} G(x, t), \quad \lim_{x \to \bar{x}^+} \frac{\partial G(x, t)}{\partial x} = \lim_{x \to \bar{x}^-} \frac{\partial G(x, t)}{\partial x}.
\] (5.34)

A solution of (5.32)-(5.34) is found by the standard methods for solving diffusion equations, (Cannon (1984)). The Laplace transform of the solution of (5.32)-(5.34) is
\[
Laplace(G(x, t))(s) = \frac{1}{s} + \frac{\sigma}{2Vs} e^{-w_0(s)(x-\bar{x})} \left( \frac{V}{\sigma} + \frac{V(2\mu-V)}{\sigma^2 y_0(s)} + \frac{y_1(s)^2}{y_0(s)} - \frac{y_1(s)}{y_0(s)} \right), \quad x > \bar{x},
\] (5.35)

\[
Laplace(G(x, t))(s) = \frac{\sigma}{2Vs} e^{-w_0(s)(x-\bar{x})} \left( \frac{V}{\sigma} + y_0(s) - y_1(s) \right), \quad x < \bar{x},
\] (5.36)

where
\[
y_0(s) = \sqrt{\frac{2s}{\sigma} + \left( \frac{\mu}{\sigma} \right)^2}, \quad y_1(s) = \sqrt{\frac{2s}{\sigma} + \left( \frac{V - \mu}{\sigma} \right)^2}, \quad w_0(s) = y_0(s) + \frac{\mu}{\sigma},
\]

\[
w_1(s) = y_1(s) + \frac{V - \mu}{\sigma}.
\]

By substituting (5.35) and (5.36) into (5.31), we obtain
\[
\theta(\bar{x}) = -A_0 Laplace(C'(t + \bar{x}))(w_0(\gamma)) - A_1 Laplace(C'(-t + \bar{x}))(w_1(\gamma)),
\] (5.37)

where
\[
A_0 = -\frac{\sigma w_0(\gamma)}{2V} \left( \frac{V}{\sigma} + \frac{V(2\mu-V)}{\sigma^2 y_0(\gamma)} + \frac{y_1(\gamma)^2}{y_0(\gamma)} - \frac{y_1(\gamma)}{y_0(\gamma)} \right)
\]

and
\[
A_1 = \frac{\sigma w_1(\gamma)}{2V} \left( \frac{V}{\sigma} + y_0(\gamma) - y_1(\gamma) \right).
\]

Thus, $\bar{x}$ is found from
\[
c = -A_0 Laplace(C'(t + \bar{x}))(w_0(\gamma)) - A_1 Laplace(C'(-t + \bar{x}))(w_1(\gamma)).
\] (5.38)
If a solution of (5.38), $\bar{x}$, exists, then, according to Lemma 5.3, it is the threshold of the unique optimal policy. If a solution does not exist, the optimal policy is either $u_t = 0$, or $u_t = V$ as indicated in Lemma 5.3. More than one solution of (5.38) cannot exist for any $C(x)$.

Generally, the value of $\bar{x}$ can be found from (5.38) either analytically or numerically. In a particular case of a linear cost function $C(x) = xsig(x)$ (see Figure 5-1), an analytical solution of $\bar{x}$ can be found. To find $\bar{x}$, we resolve (5.38) and obtain that if the cost of control, $c$, is small such that

$$c < \frac{A_0 c^-}{w_1(\lambda)} - \frac{A_0 c^+}{w_0(\gamma)},$$

then, the threshold is positive,

$$\bar{x} = \frac{1}{w_1(\gamma)} \ln \left( \frac{A_0 w_0(\gamma)(c^+ + c^-)}{w_0(\gamma)w_1(\gamma) p + A_0 c^+ w_1(\gamma) + A_0 c^- w_0(\gamma)} \right) . \quad (5.39)$$

Otherwise, the threshold is negative,

$$\bar{x} = -\frac{1}{w_0(\gamma)} \ln \left( \frac{A_0 w_1(\gamma)(c^+ + c^-)}{A_0 c^- w_1(\gamma) + A_0 c^- w_0(\gamma) - cw_0(\gamma)w_1(\gamma)} \right) . \quad (5.40)$$

5.5.3 The hedging policy over a finite time horizon

This section considers the problem on a finite time horizon $T$. In such a case, the costate variable is defined as

$$\psi_t(\omega) = -\int_t^T C'(x_\tau)e^{-\tau\gamma} d\tau . \quad (5.41)$$

From (5.41) it follows that $\psi_T(\omega) = 0$ at each scenario. The costate function now depends explicitly on $t$,
\[ \theta_t = \theta(x, t)e^{-\gamma t}, \quad (5.42) \]

where

\[ \theta(x, t) = E \left[ -\int_t^T C'(x_r)e^{-\gamma(r-t)}dr \mid x_t = x \right]. \quad (5.43) \]

Following the arguments of Lemmas 5.2 and 5.3, one can prove that \( \theta(x, t) \) is decreasing with respect to \( x \) for each \( t \), and that the optimal policy is of a threshold type with a threshold level that depends on time, \( \bar{x}_t \). The threshold curve is now obtained from the following equation

\[ \theta(\bar{x}_t, t) = c. \]

Remark 5.1. For \( c > 0 \) and linear cost case, the threshold curve does not exist in a neighborhood of \( T \). Indeed,

\[ \theta(\bar{x}_t, t) < c\int_t^T e^{-\gamma(r-t)}dr = \frac{c^-}{\gamma}(1 - e^{-\gamma(T-t)}). \]

The right-hand side of the last expression tends to zero when \( t \to T \). Therefore, for \( t \) sufficient close to \( T \), \( \theta(\bar{x}_t, t) \) cannot be equal to \( c \). The optimal policy in a neighborhood of \( T \) is \( u_t = 0 \).

Remark 5.2. For \( c = 0 \) and linear cost case, the threshold curve ends at zero, i.e., \( \bar{x}_T = 0 \).

Indeed, on one hand, \( \lim_{t \to T} \frac{\theta(\bar{x}_t, t)}{T-t} = 0 \), since \( \theta(\bar{x}_t, t) = 0 \) for all \( t \). On the other hand,
\[
\lim_{t \to T} \frac{\theta(\bar{x}_t, t)}{T - t} = -\text{sig}(\bar{x}_r) \lim_{t \to T} \frac{1}{T - t} \int_t^T e^{-\gamma(t - \tau)} d\tau = -\text{sig}(\bar{x}_r).
\]

Only \( \bar{x}_r = 0 \) avoids contradiction.

In Section 6.5 a numerical procedure is presented for computing the threshold curve for the case discussed in Remark 5.2.

Figure 5-4: Schematic threshold curves for the cases of Remark 5.1 (thin line) and Remark 5.2 (bold line)
6. Algorithms for calculating the optimal feedback control rules

6.1 Background

In this section, we suggest algorithms for calculating optimal feedback control rules for the four problems studied in the previous chapters. In cases 1-3 (discrete-time demand) we calculate the feedback rule in each period separately, beginning from the last period, $\tau \in [T-1,T)$, since the right-hand costate conditions are given. When crossing an integer time point, the costate value is updated as discussed in Chapter 5. This allows the constructing of the feedback rule in the preceding time period. Despite the fact that within each period the feedback rule takes on one of the six possible forms, the complexity of the algorithms is linear with $T$, because an appropriate form is determined separately for each period.

In the continuous-time case (Case 4), the numerical procedure approximates the threshold curve in the state space. The procedure starts at $t=T$, where the threshold curve equals zero. Then, the procedure moves backward, and at each iteration extends the curve over a small interval of length $\Delta$. At each $\Delta$-interval the curve is approximated by a straight line.
6.2 Algorithm for calculating the optimal feedback control rule for the problem with discrete-time demand and complete backlogging (Case 1)

From (4.5) it follows that $\psi_{T^-} = 0$ regardless of demand scenario. Therefore,

$$A_T(y) = E[\psi_{T^-} | x_{T^-} = y] = 0 \text{ for all } y.$$ This observation allows initiates the procedure.

Step 1. Set $t=T$ and $A_T(y) = 0$ for all $y$.

Step 2. Calculate $E[\psi_{T^-} | x_{T^-} = x] = F(x,0)$ and $\bar{B}_T(x)$ for all $x$ from the six forms in Section 5.2.1.

Step 3. Set $t = t-1$ and determine the optimal feedback control rule from Section 5.2.2

$$u_t = u_t(t,x,\tau), \tau \in [t,t+1].$$

Step 4. If $t=0$, stop, otherwise go to Step 5.

Step 5. Calculate the function $A_t(y)$ from equation (5.10).

Step 6. Calculate $E[\psi_{t^-} | x_{t^-} = x]$ and $\bar{B}_{t^-}(x)$ for all $x$ from the six forms in section (5.2.1). Go to Step 3.

The complexity of the algorithm is defined by the accuracy with which the function $A_t(y)$ in step 5, is calculated. If for each $t$, $A_t(y)$ is calculated in $N$ equally distributed mesh points and the integral in (5.10) is calculated by the trapezoid method with $K$ mesh intervals ($\pi_t(\xi)$ is pertinent in $K+1$ mesh points), the complexity of each loop of the algorithm is $O(KN)$. The total complexity is $O(KNT)$. 

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6.3 Algorithm for calculating the optimal feedback control rule for the problem with discrete-time demand and lost sales (Case 2)

In this case we make use of the defection function (3.8) and sales function (3.9), when updating the costate function $A_t(y)$ in (5.19).

Step 1. Set $t = T$ and $A_t(y) = 0$ for all $y$.

Step 2. Calculate $E[\psi_{T-1} | x_{T-1} = x] = F(x, 0)$ and $\overline{B}_t(x)$ for all $x$ from the six forms in Section 5.2.1.

Step 3. Set $t = t - 1$ and determine the optimal feedback control rule from Section 5.2.2

$$u_t = u_t(t, x, \tau), \quad \tau \in [t, t+1).$$

Step 4. If $t = 0$, stop, otherwise go to Step 5.

Step 5. Calculate the function $A_t(y)$ from equation (5.19).

Step 6. Calculate $E[\psi_{t-1} | x_{t-1} = x]$ and $\overline{B}_{t-1}(x)$ for all $x$ from the six forms in section 5.2.1. Go to Step 3.

Similar to the previous case, the complexity of the algorithm is $O(KN)$. The total complexity is $O(KNT)$. 
6.4 Algorithm for calculating the optimal feedback control rule for the problem with lost sales and discrete-time reputation-dependent demand (Case 3)

Step 1. Set \( t = T \), \( A_T(y,R) = \begin{bmatrix} E[\psi_{T-1} \mid x_{T-1} = y, R_{T-1} = R] \\ E[\varphi_{T-1} \mid x_{T-1} = y, R_{T-1} = R] \end{bmatrix} = \begin{bmatrix} 0 \\ p \end{bmatrix} \), and \( E[H_T \mid x_T = y, R_{T-1} = R] = 0 \) for all \( y \) and \( R \).

Step 2. Calculate
\[
\begin{bmatrix} E[\psi_{T-1} \mid x_{T-1} = x, R_{T-1} = R] \\ E[\varphi_{T-1} \mid x_{T-1} = x, R_{T-1} = R] \end{bmatrix}
\]
by integrating the costate equations (4.25) from \( \tau = T \) to \( \tau = T - 1 \).

Step 3. Calculate \( E[H_{T-1} \mid x_{T-1} = x, R_{T-1} = R] \) for all \( x \) and \( R \) from equation (5.22).

Step 4. Set \( t = t - 1 \) and determine the optimal feedback control rule from Section 5.2.2, \( u_t = u_c(t, x_t, \tau) , \tau \in [t, t+1] \).

Step 5. If \( t = 0 \), stop, otherwise go to Step 6.

Step 6. Calculate \( E[H_{t-1} \mid x_{t-1} = y, R_{t-1} = R] \) for all \( y \) and \( R \) from equation (5.23).

Step 7. Calculate the function \( A_t(y,R) \) from equation (5.24).

Step 8. Calculate
\[
\begin{bmatrix} E[\psi_{t-1} \mid x_{t-1} = x, R_{t-1} = R] \\ E[\varphi_{t-1} \mid x_{t-1} = x, R_{t-1} = R] \end{bmatrix}
\]
by integrating the costate equations (4.25) from \( \tau = t \) to \( \tau = t - 1 \).
Step 9. Calculate $E[H_{t-1} | x_{t-1} = x, R_{t-1} = R]$ for all $x$ and $R$ from equation (5.22). Go to Step 4.

The complexity of the algorithm is defined by the accuracy with which the function $E[H_{t-1} | x_{t-1} = y, R_{t-1} = R]$ in step 6 is calculated. If for each $t$, it is calculated in $M$ points of reputation and $N$ points of inventory level, and the integral in (5.23) is calculated by the trapezoid method with $K$ mesh intervals, the complexity of each loop of the algorithm is $O(KNM)$. The total complexity is $O(KNMT)$. 
6.5 Algorithm for calculating the optimal feedback control rule for the problem with continuous-time demand and complete backlogging (Case 4)

Step 1. Set $t=T$ and $\bar{x}_r = 0$.

Step 2. Set $t = t - \Delta$ and approximate the threshold curve within the interval $\tau \in [t, t + \Delta)$ by a straight line, $\bar{x}_r = b(\tau - t - \Delta) + x_{r,\Delta}$. The value of $b$ is found by solving the diffusion equation (5.32)-(5.34) with $\bar{x} := \bar{x}_{r,\Delta} - b\Delta$ and $\mu := \mu + b$, on the time interval $[0, \Delta]$. As a result the probability distribution of $x$, $G(\kappa, t), t \in [0, \Delta]$ is obtained. Then, we find the unknown $b$ from the optimality condition

$$
E \left[ \int_{0}^{\Delta} C'(x_\tau) e^{-\gamma \tau} \, dt \mid x_0 = -b\Delta \right] = \int_{0}^{\Delta} \left( c^+ (1 - G(0,t)) - c^- G(0,t) \right) e^{-\gamma \tau} \, dt = 0,
$$

or, equivalently,

$$
\int_{0}^{\Delta} G(0,t) e^{-\gamma \tau} \, dt = \frac{c^+}{\gamma (c^+ + c^-)} (1 - e^{-\gamma \Delta b}).
$$

Step 3. Calculate the value of $\bar{x}_r = \bar{x}_{r,\Delta} - b\Delta$.

Step 4. If $t=0$, stop, otherwise go to step 2.

The complexity of the algorithm is defined by the accuracy with which the threshold curve is to be calculated. For a given $\Delta$ and a given accuracy of computing $b$ in Step 2, $\Delta_b$, the complexity is $O\left( \frac{T}{\Delta \Delta_b} \right)$.
7. Implementation of the methods

The algorithms of cases 1-3 have been coded in C++ software (see Appendix B). In this chapter we discuss the results of several runs of the software, each emphasizing particular capabilities of the developed methods. The algorithm of Case 4 has not been coded.

7.1 Case 1 - Numerical results

Three examples have been calculated with the algorithm of Section 6.2. The first one (Section 7.1.1) assumes stationary demand distribution. We show the convergence of the feedback rule, when moving from the last time period to the beginning of the planning horizon. The convergence takes place when the expected demand is lower than the production capacity. The second example (Section 7.1.2) shows divergence of the feedback rule, when moving towards the beginning of the planning horizon. The divergence takes place when the expected demand is greater than the production capacity. In the third numerical example (Section 7.1.3), we show that the developed algorithm works for non-stationary demand as well.

7.1.1 Convergence of the feedback rule

For stationary demand, \( \pi_r(\cdot) = \pi(\cdot) \) for all \( t \), with the production capacity being greater than the expected demand, \( E[d_r] = E[d] < V \), the control policy converges. That is, there exists a limit function \( \bar{B}(x) \), such that
\[ \lim_{t \to \infty} \overline{B}_t(x) = \overline{B}(x) \]

for each \( t \) and \( x \).

We used the probability density function of demand, \( \pi(\xi) \), as follows (see also Figure 7-1),

\[
\pi(\xi) = \begin{cases} 
0, & \text{if } \xi < \alpha \text{ or } \xi \geq \beta \\
\frac{4(\xi - \alpha)}{(\beta - \alpha)^{2}}, & \text{if } \alpha \leq \xi < \frac{\alpha + \beta}{2} \\
\frac{4(\beta - \xi)}{(\beta - \alpha)^{2}}, & \text{if } \frac{\alpha + \beta}{2} \leq \xi < \beta
\end{cases}
\]

with the parameters \( \alpha = 0.1 \) and \( \beta = 1.7 \).

Figure 7-1: The probability density function of demand.

The other parameters are \( T=40, V=1, c^+ = 0.5, c^- = 1.5 \). Figures 7-2 and 7-3 present the function \( A_t(x) \) calculated by the developed algorithm. The function does not converge over the whole axis. For a very large positive \( x \), the costate function increases
by $c^+$ at each time period (see form (a) of the state-costate dynamics in Section 5.2.1), and for a very large negative $x$, the costate function decreases by $c^-$ at each time period (see form (f) in Section 5.2.1). Therefore,

$$\lim_{x \to \infty} A_i(x) = -c^+(T - t), \quad \lim_{x \to -\infty} A_i(x) = c^-(T - t).$$

However, the control strategy depends on the function $A_i(x)$ only within the area between $A_i^{-1}(c^+)$ and $A_i^{-1}(0)$ (see the feedback control rules in Section 5.2.2). Therefore, if the function $A_i(x)$ converges within the area between $A_i^{-1}(c^+)$ and $A_i^{-1}(0)$, the control strategy converges as well. Figure 7-4 shows the convergence of $A_i^{-1}(c^+)$ and $A_i^{-1}(0)$ as $t$ approaches zero. Figure 7-5 shows the target inventory in period $t$, $\bar{B}_i(x)$, as a function of the beginning inventory $x$. The volume of production within the period $(t, t+1)$ is $\bar{B}_i(x) - x$. It can be seen in Figure 7-5 that the graph of $\bar{B}_i(x)$ converges as $t$ approaches zero. For low initial inventory, the system produces with the maximum rate $V$ along the period, $\bar{B}_i(x) = x + V$ (see Forms 3, 5 and 6 in Section 5.2.1). For “intermediate” initial inventory, the production takes place on a part of the period, and $x < \bar{B}_i(x) < x + V$ (see Forms 2 and 4 in Section 5.2.1). For sufficient large initial inventory, no production regime is optimal, $\bar{B}_i(x) = x$ (see Form 1 in Section 5.2.1).
Figure 7-2: Function $A_t(x)$ for $t=39, 30, 20, 10, 0$: general scale.

Figure 7-3: Function $A_t(x)$ for $t=39, 30, 20, 10, 0$: the scale of $A_t^{-1}(0)$ and $A_t^{-1}(c^+)$.  

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Figure 7-4: Convergence of $A_t^{-1}(0)$ (bold line) and $A_t^{-1}(c^*)$ (thin line).
7.1.2 Divergence of the feedback rule

The next experiment shows the divergence of the control strategy. Here we used the following parameters of the problem: \( T = 40, V = 1, \ c^+ = 0.4, c^- = 1 \). We used the demand distribution from the previous experiment (see Figure 7-1) with the parameters \( \alpha = 0.5 \) and \( \beta = 1.9 \), i.e. \( E[d] > V \). Figures 7-6 to 7-9 are similar to Figures 7-2 to 7-5 of the previous experiment. Figures 7-6 and 7-7 show the divergence of \( A_i(x) \) in two scales, general and specifically within the area between \( A_i^{-1}(c^+) \) and \( A_i^{-1}(0) \). Figure 7-8 shows the divergence of the last two values. Figure 7-9 presents the divergence of the feedback rule \( \overline{B}_i(x) \).
Figure 7-6: Function $A_t(x)$ for $t=39, 30, 20, 10, 0$: general scale.

Figure 7-7: Function $A_t(x)$ for $t=39, 30, 20, 10, 0$: the scale of $A_t^{-1}(0)$ and $A_t^{-1}(c^+)$. 
Figure 7-8: Divergence of $A_i^{-1}(0)$ (bold line) and $A_i^{-1}(c^+)$ (thin line).

Figure 7-9: Divergence of $B_i(x)$. 
7.1.3 Time-dependent demand

The developed algorithm works for a non-stationary demand distribution as well. We used the demand distribution from the previous experiments (see Figure 7-1) with the parameters $\alpha = 0.1$ and $\beta = 0.8$ for $t=1,3,5,\ldots,39$, and $\alpha = 1.1$ and $\beta = 1.8$ for $t=0,2,4,\ldots,38$. The control policy converges to two different forms, one for the even time periods and the other for the odd periods. Figure 7-10 shows the limit policy approximated at $t=0$ and at $t=1$.

Figure 7-10: Non-stationary control rule $\overline{B}_t(x)$. 
7.2 Case 2 - Numerical results

We implemented the algorithm of Section 6.3 for the following parameters of the problem: \( T=15 \), \( V=1 \), \( c^+ = 0.5 \), \( c^- = 0.8 \), \( p = 1 \), \( P(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -0.5 \cdot x & \text{if } -2 \leq x < 0 \\ 1 & \text{if } x < -2 \end{cases} \). The demand distribution was as in the example in Section 7.1.1. Now, the backlog cost parameter \( c^- \) is lower than that in the previous example, since the lost sales phenomenon is now treated separately. The convergence (as \( t \) approaches zero) of the costate factors \( A_t^{-1}(c^+ \) and \( A_t^{-1}(0) \) that determine the feedback rule is shown in Figure 7-11. Figure 7-12 shows the target inventory \( B_t(x) \) at the end of period \( t \), when inventory at the beginning of period \( t \) is \( x \). This graph converges as \( t \) approaches zero.

![Figure 7-11: Convergence of \( A_t^{-1}(0) \) (bold line) and \( A_t^{-1}(c^+) \) (thin line).](image-url)
By comparing Figures 7-12 and 7-5, we conclude that for the considered parameters of the problem we have to store fewer inventories with the lost sales effect treated explicitly. For instance, for an inventory level of $x_0 = 0.3$ at the beginning of period $t=0$, we have to store $B_0(x) = 0.76$ at the end of the period, while in the example in Section 7.1.1 we have to store $B_0(x) = 1.11$. The difference is because on one hand, we lose $p = 1$, for each customer who leaves without purchasing (lost sale), while on the other hand we will not have to pay $c^- = 0.8$ during all future periods. As a result, fewer inventories are better, since future sales are not harmed and future shortage costs decrease.
7.3 Case 3 - Numerical results

We implemented the algorithm of Section 6.4 for the following parameters of the problem: \( T=15, \ V=1, \ c^+=0.5, \ c^-=0.2, \ p=1, \ P(x) = \begin{cases} 0 & \text{if } x \geq 0 \\ -0.5 \cdot x & \text{if } -2 \leq x < 0 \\ 1 & \text{if } x < -2 \end{cases} \), and

the demand distribution as in Figure 7-1 with \( \alpha = 0.7 \) and \( \beta = 1.7 \), when the reputation is \( R_t = 1 \). The reputation is updated in accordance with (3.13) with \( \gamma = 0.5 \),

\[
R_t = 0.5R_{t-1} + 0.5 \left( 1 - \frac{7P(x_t - \xi_t)}{12} \right).
\]

From the last equation (reputation dynamics) and the discussion in Section 3.5, we find that the values of the reputation move between \( 5/12 \) and \( 1 \), \( R_t \in [5/12,1], \forall t \). Figure 7-13 shows the demand distribution for the two limit cases, \( R = 1 \) and \( R = 5/12 \).

![Figure 7-13: Demand distribution for \( R = 1 \) and \( R = 5/12 \)](image)

We used the same values for the inventory cost, maximum production rate, price, horizon length and defection function as in the previous example. The backlog cost is now lower, \( c^- = 0.2 \), since it does not reflect the “loss of reputation” factor. The
reputation and its influence on demand distribution in future periods are treated explicitly.

Similar to Figure 7-11, Figure 7-14 shows the convergence of $A_t^{-1}(c^+, R = 1)$ and $A_t^{-1}(0, R = 1)$ as $t$ approaches zero. Similar to Figure 7-12, Figure 7-15 shows how much inventory is to be stored at the end of period $t$, $\bar{B}_t(x, R = 1)$, if inventory at the beginning of period $t$ is $x$ and the reputation is $R = 1$. Similar to Figure 7-12, this graph converge as $t$ approaches zero. The graphs of $A_t^{-1}(c^+, R)$, $A_t^{-1}(0, R)$ and $\bar{B}_t(x, R)$ converge for every value of $R$ as well. Therefore, the control strategy converges for each $R$.

Figure 7-14: Convergence of $A_t^{-1}(0, R = 1)$ (bold line) and $A_t^{-1}(c^+, R = 1)$ (thin line).
8. Inventory Management of Repairable Service Parts for Personal Computers: A Case Study

8.1 Introduction and motivation

In many manufacturing organizations the production process has changed drastically in recent years. Attention has shifted from increasing efficiency by means of economies of scale and internal specialization to meeting market conditions in terms of flexibility, delivery performance, and quality. Ideally, this trend towards JIT production implies working with absolute minimum work-in-process and finished good inventories.
However, not enough attention has been paid to the management of after sales activities. One of the important aspects is the management of (repairable) service parts inventory. Competition has forced consumer products industries to provide very short call service contracts in order to boost sales and this has resulted in large inventory of service parts in the after sales logistics chain. The methods developed in this thesis can improve management decisions in such situations.

Recent years have seen an increase of interest in the field of service parts inventory particularly in the computer industry. This increase can be attributed to the wide use of micro-computers. When a computer or a system fails, it is not always easy to trace the reason for its failure. It is frequently due to either a quality problem or the wrong way of using the computer or its component parts, or a combination of both. Whatever the reason for the failure is, the computer availability or generally system availability and the time needed for maintenance or repair are important to most users. For example, in a financial institution such as a bank, any short interruption may cost thousands of dollars. Here the goal is to reach 100 % system availability, or to come as close to it as possible. Without any doubt high service parts inventories can contribute to this goal.

As a whole, the computer industry is a highly competitive industry; products have to be repaired as quickly as possible, since a slow repair can lead to loss of future business to competitors with better service reputations. Therefore, high reputation is closely linked to the availability of spares on the market. For companies that operate worldwide the problem becomes even more complex and important.

Although in the design of computer systems, attention is already paid to reliability through careful selection of components, design sophistication, incorporating of various
types of redundancy and provision of back-ups, there is no doubt that efficient management of service parts inventory is of prime importance to many consumer companies, particularly the computer industry. We make use of the real-life case study presented by Ashayeri et al., (1996), and elaborate the management and control of service parts inventory by means of the methods developed in this work.

### 8.2 The Olivetti case study

Ashayeri et al., (1996) built a simulation model for the problem of service parts inventory with repairable items of Olivetti company.

Olivetti is an Italian company that produces and distributes computers, monitors, printers, etc. The sales of complete units in the Netherlands are coordinated by the Olivetti office in Leiden. Customer service calls are arriving at Olivetti Service in Nieuwegein, where the service activities are coordinated for the whole country. This department plays an important role in the management of service parts for the after sales activity of Olivetti in the Netherlands. The structure of inventories with repairable service parts for the case under study is comparable with the structure of a two-level distribution inventory problem. The major difference here is that there exist repair facilities in the system. The repair facilities are located in different countries, namely in the Netherlands and in France. Usually repair is performed locally in an electronic lab (e-lab) except for more complex or large repair batch that is performed in Paris (see Figure 8-1 for an overview of service parts logistics flow at Olivetti in the Netherlands). The non-Olivetti parts are sent to external repair facilities. The Olivetti service parts are divided into repairable modules and non-repairable components. These parts are separately stocked in
the inventory in Nieuwegein. The Nieuwegein facility is capable of storing over 10,000 different components. At any time, there is a positive inventory status for almost 50% of these components. The components on hand occupy about 20% of inventory space and are worth about 7% of the total value of inventory. The inventory management system for components is developed by Olivetti and is called SigerC. The system forecasts the demand for the components using double exponential smoothing. Based on this forecast and using the classical Economic Order Quantity (EOQ), the orders are generated for different components.

The current inventory control system of components has proved to be reliable enough. However, it does not control the flow of repairable modules. The repairable modules consist of 2300 different types from which 1600 types are frequently demanded. The stock of modules utilizes the 80% rest of inventory space and worth about 93% of the total value of inventory. Comparing the value of stock of components with those of modules, one can see that a good management and control of modules is a necessity.

Therefore, the study concentrated on the development of a solution approach for management of the modules. The logistics flow of repairable modules is elaborated as presented in Figure 8-1.
Demand for repairable modules are of two types, those generated by the customers who have no service contract (carry-in) arriving directly at Olivetti and those with a service contract. The latter constitutes a major portion of the demand for repairable modules (90%). As illustrated in Figure 8-1, telephone calls reporting a particular defect are all arriving at a central office in Nieuwegein where a rough diagnosis of the potential problem is made. Depending on the problem a field engineer is assigned to look after the service. The service contracts usually fall in one of the following categories of 2, 4, and 8 hours service delivery after receipt of phone calls. The service is provided from 8:30 to 17:00 (weekend is not included). Therefore, when a customer with a 4-hour service contract reports a defect at 16:00 on Monday, he should be serviced before 11:30 on Tuesday. As mentioned earlier respecting this “response-time” is crucial. In order to meet
the obligations and to reduce costs as far as possible the stock of modules are centralized in Nieuwegein. However, a small portion of the inventory lays maximum for one day with the service engineers to meet the short service contracts and then is returned to Nieuwegein.

The logistics process between Olivetti and service engineers and the repair facilities is as follows. When, for example, a number of modules are delivered on Monday evening to a service engineer for a diagnosed defect, on Tuesday the modules are taken by the engineer to the customer. Then, one or more good modules are swapped with the defective one(s). The good modules which are unused will stay with the service engineer until Wednesday evening and if not required for the day after, they are returned back together with defective modules to Nieuwegein on Thursday. Thus it takes a total of three days to re-book the good modules in inventory in Nieuwegein. The defective modules are then either repaired locally or are shipped to Paris for repairing. The repair facility in Paris usually delivers a repaired module in one day when enough inventory exists (Inventory in Circulation) or repairs the same module and returns that when there is enough time (Pure Repair). In the latter case the “Turn Around Time” is about 10 days. Exclusive repaired modules (more customer specific) are also available in very limited number in inventory and are lend temporary until the original module is repaired (Lend Inventory). From the three possible options, lending is the most expensive and pure repair is the cheapest. Given the three days delay in re-booking a good module sent by service engineers and the possible delay caused by the repair facility in Paris, currently a safety stock of 4 days is kept in Nieuwegein.
The current logistics flow of modules is managed by a simple information data base developed by Olivetti called OLBORD (OLivetti BOard Repair Decentral). However, all decisions concerning the modules are made manually. One of the important decisions made is batching, that is how many modules should be sent together to the repair facilities and when. A hurdle to this decision making process is the fact that the real defect is not known, only the customer complaint is noted. Therefore, a specialist, based on his experience and the back-order list, decides which modules should be sent to which repair facility for further examination. It is then that an estimation of repair time can be made. At this stage a fraction of defective modules is considered as scrap and therefore from time to time new modules should be procured. Thus the repair batch sizing decision and the decision concerning the number of new modules to be ordered and the timing of such an order is a complex task.

In order to develop a solution approach for the above decision making process, first an ABC analysis of different types of modules was performed (see figure 8.2). The analysis revealed that about 20% of the total number of module codes represent about 80% of the demand. Therefore, the attention was focused on only these modules. These modules roughly speaking are demanded from two times per month up to three times a day.
According to Ashayeri et al., (1996), the situation of inventory of repairable modules in Nieuwegein can be formulated in different ways; as single-echelon or a multi-echelon system. The inventory laying with the service engineers can be neglected since with a short delay it is registered back in the stock in Nieuwegein. The inventory of modules in Paris can also be overlooked since what counts is the repair time delay and that indirectly represents the inventory level in Paris. Thus the situation can be considered as a single-echelon inventory with repairable parts. Now the question is which modeling is most suitable for the Olivetti case.

### 8.3 Implementation issues

In this section we demonstrate how the case 3 method (Section 6.4) can be applied to the above environment. We assume:

- According to Ashayeri et al., (1996), the cost of defective modules held is less than the cost of Ready For Service (RFS) modules by at least the cost of repair labor and replacement components. Thus, if inventory is to be held in the system it would be
better held in defect condition than in RFS condition. Here we neglect the cost of defect modules.

- There are enough defect modules to repair at each time (a small portion of the modules are discarded as scrap) and 100 percent of demand supplied from repaired modules as presented at the second policy of Schrady model in Ashayeri et al., (1996).

- Demand transactions update the information system at night. That is, the length of time period is the working day.

Off-line implementation:

- The parameters of the Case 3 problem should be determined \((x_0, R_0, \text{demand distribution for } R = 1, \text{the defection function } P(x), \gamma, c^+, c^-, T, p)\).

- The algorithm from Section 6.4 is executed. The feedback control rule is obtained for each \(t, x_t, \text{and } R_t\).

On-line implementation

- The Service Engineers Planning Office observes the inventory level, \(x_{t-}\) at the end of day \(t\).

- The Service Engineers Planning Office updates each customer regarding the value of \(x_{t-} - \xi_t\). The customers choose to make their orders or leave without purchasing. If \(x_{t-} - \xi_t\) is positive, all customers are satisfied.
8.4 Numerical example for the off-line implementation

We assume the following normalized data regarding a specific module: \( T = 15 \) days,

\[
V = 150 \text{ units/day}, \quad c^+ = 0.7 \text{ $/unit\cdot day}, \quad c^- = 0.05 \text{ $/unit\cdot day}, \quad p = 1.15 \text{ $/unit},
\]

\[
P(x) = \begin{cases} 
0 & \text{if } x \geq 0 \\
- x/100 & \text{if } 100 < x < 0 \\
1 & \text{if } x < -100 
\end{cases}
\]

and the probability density function of demand, \( \pi(\xi) \), when the reputation is \( R_t = 1 \), as follows (see also Figure 8-3),

\[
\pi(\xi) = \begin{cases} 
0, & \text{if } \xi < \alpha \text{ or } \xi \geq \beta \\
\frac{4(\xi - \alpha)}{(\beta - \alpha)^2}, & \text{if } \alpha \leq \xi < \frac{\alpha + \beta}{2} \\
\frac{4(\beta - \xi)}{(\beta - \alpha)^2}, & \text{if } \frac{\alpha + \beta}{2} \leq \xi < \beta
\end{cases}
\]

with the parameters \( \alpha = 100 \) units and \( \beta = 200 \) units. The reputation is updated by

\[
R_t = 0.5R_{t-1} + 0.5 \left( 1 - \frac{P(x_{t-1} - \xi)}{2} \right).
\]

Figure 8-3: The probability density function of demand.
Figure 8-4 shows the values of $A_t^{-1}(c^+, R)$ and $A_t^{-1}(0, R)$ calculated by the developed algorithm for different values of reputation. Since these two previous values converge, the control strategy converges as well. Figure 8-5 shows how much inventory is to be stored at the end of period $t$, $\overline{B}_t(x, R)$, if inventory at the beginning of period $t$ is $x$. It can be seen that the function converges as $t$ approaches zero for any value of $R$. It can be seen from the graph that as long as the value of reputation larger the inventory to be stored at the end of each period is larger as well. This phenomenon holds, since the greater the reputation more demands can occur and as result more potential decrease its value.
Figure 8-4: Convergence of $A_t^{-1}(0, R = 0.8)$ (bold line) and $A_t^{-1}(c^+, R = 0.8)$ (thin line).

Figure 8-5: Convergence of $A_t^{-1}(0, R = 0.9)$ (bold line) and $A_t^{-1}(c^+, R = 0.9)$ (thin line).

Figure 8-6: Convergence of $A_t^{-1}(0, R = 1)$ (bold line) and $A_t^{-1}(c^+, R = 1)$ (thin line).
Figure 8-5: Convergence of $\bar{B}_y(x, R)$. 

\[ \bar{B}_y(x, R = 0.8) \] (units) 

\[ \bar{B}_y(x, R = 0.9) \] (units) 

\[ \bar{B}_y(x, R = 1) \] (units)
9. Conclusions

9.1 Main results

- This work applies a perturbation analysis technique to cases of a production control problem. A method of state-costate forward-backward integration is developed, which allows characterizing optimal feedback control rules. Analysis of information contained in the optimality conditions results in efficient numerical procedures, computing the feedback rules. For a given accuracy, the complexity of the rules is linear with the length of the time horizon. The more the accuracy is required, the greater the complexity of the rules. The developed method is quite general, it allows additional features of the problem, such as lost sales and reputation effects, to be incorporated, analyzed and computed. As discussed, the algorithms and results presented in this thesis also may be utilized in some non-stationary environments.

- For discrete-time demand, we prove the optimality of discrete-time hedging policies. They are characterized by a target inventory level, computed at each time period as a function of inventory and reputation at the beginning of the period. Such a policy differs from the periodic inventory review policies discussed in the inventory theory literature. Time-dependent policies are developed to control in a non-stationary environment.

- For continuous-time demand, a hedging-type policy is proved to be optimal, where the hedging curve (threshold) changes with time (see Lemma 5.3). A computational algorithm for calculating the line in a case with no lost sales and no reputation effects is developed.
9.2 Additional points of interest

- Incorporating lost sales and reputation effects into the continuous-time model is important from both a theoretical and a practical point of view. In such a case, the diffusion of inventory and reputation probabilities is mutually dependent, and much more numerical work will be required to approximate two-dimensional thresholds.

- Controlling a combination of stochastic processes of different types is a natural generalization of this work. For example, production capacity described by a multi-state Markov chain, and demand and reputation by diffusion, result in a challenging model, the solution of which may lead to a large scope of applications.

- Modifying the method to be applied to optimization problems with constraints that arise in various areas such as marketing and finance is of interest.
References

Books


**Journal Articles**


[50] Palda


Conferences


Appendix A – Effect of Customer Behavior on the Defection Function

In this appendix, we discuss the effect of customer response to waiting on the customer defection realized by servers. In order to analyze and estimate the defection function, we employ a simple queuing model with two exponential servers and two queues. The arrival times are Poisson. When a customer arrives, she joins the shortest queue. Under this behavior, each server will see a customer defection rate that depends on the queue length.

Let $N_1$ and $N_2$ denote the number of customers in the first and second queue respectively. The steady-state joint probability function is defined as

$$ p(n_1, n_2) = \text{prob}[N_1 = n_1, N_2 = n_2] \quad (A1) $$

The steady state dynamics of the system is given by the following equations:

$$
\begin{align*}
\dot{\lambda}p(0,0) &= \mu_1 p(1,0) + \mu_2 p(0,1) \\
(\mu_1 + \lambda)p(1,0) &= \lambda / 2 \cdot p(0,0) + \mu_1 p(2,0) + \mu_2 p(1,1) \\
(\mu_1 + \lambda)p(n_1,0) &= \mu_1 p(n_1 + 1,0) + \mu_2 p(n_1,1) \quad n_1 \geq 2 \\
(\mu_2 + \lambda)p(0,1) &= \lambda / 2 \cdot p(0,0) + \mu_1 p(1,1) + \mu_2 p(0,2) \\
(\mu_2 + \lambda)p(0,n_2) &= \mu_1 p(1,n_2) + \mu_2 p(0,n_2 + 1) \quad n_2 \geq 2 \\
(\mu_1 + \mu_2 + \lambda)p(n_1, n_1) &= \lambda p(n_1, n_1 - 1) + \lambda p(n_1 - 1, n_1) + \mu_1 p(n_1 + 1, n_1) + \mu_2 p(n_1, n_1 + 1), n_1 \geq 1 \\
(\mu_1 + \mu_2 + \lambda)p(n_1, n_1) &= \lambda / 2 \cdot p(n_1, n_1) + \mu_1 p(n_1 + 2, n_1) + \mu_2 p(n_1 + 1, n_1 + 1), n_1 \geq 1 \\
(\mu_1 + \mu_2 + \lambda)p(n_1, n_1 + 1) &= \lambda / 2 \cdot p(n_1, n_1) + \mu_1 p(n_1 + 1, n_1 + 1) + \mu_2 p(n_1, n_1 + 2), n_1 \geq 1 \\
(\mu_1 + \mu_2 + \lambda)p(n_1, n_2) &= \lambda \cdot p(n_1, n_2 - 1) + \mu_1 p(n_1 + 1, n_2) + \mu_2 p(n_1, n_2 + 1), n_1 \geq 3, n_1 - 2 \geq n_2 \geq 1 \\
(\mu_1 + \mu_2 + \lambda)p(n_1, n_2) &= \lambda \cdot p(n_1, n_2 + 1) + \mu_1 p(n_1 + 1, n_2) + \mu_2 p(n_1, n_2 + 1), n_2 - 2 \geq n_1 \geq 1, n_2 \geq 3
\end{align*}
$$
The solution of the above equations with \( \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} p(n_1,n_2) = 1 \) yields the steady state joint probability function.

When \( n_1 > n_2 \), the first server loses all the arriving customers. When \( n_1 = n_2 \), an arriving customer defects with 50% probability. Therefore, the conditional probability that an arriving customer defects when there are \( n_1 \) customers waiting in the system gives the customer defection function. By denoting the number of customers waiting at the first queue as a negative number, \( x \), the defection function of the first server, \( P_1(x) \), is determined as,

\[
P_1(x) = \text{Prob}[N_1 > N_2|N_1 = -x] + \frac{1}{2} \text{Prob}[N_1 = N_2|N_1 = -x] \quad \text{(A3)}
\]
or, equivalently,

\[
P_1(x) = \sum_{n_2=0}^{x-1} p(-x,n_2) + \frac{1}{2} p(-x,-x) \quad \text{(A4)}
\]

where \( \sum_{n_2=0}^{x-1} p(-x,n_2) \) and \( p(-x,-x) \) are found from (A2).

Similarly, the defection function of the second server, \( P_2(x) \), is determined as,

\[
P_2(x) = \sum_{n_1=0}^{x-1} p(n_1,-x) + \frac{1}{2} p(-x,-x) . \quad \text{(A5)}
\]

Customer behavior and the competitiveness of the company in the market, determine the customer defection rate. In the previous example, the combined effect of customer’s choice of joining the shortest queue, availability of an alternative server, and the difference between the service rates of the servers determine the customer defection function.
Appendix B – C code of the Case 1 algorithm

#include <stdio.h>
#include <math.h>
#include <stdlib.h>
#include <time.h>

double V=1.; /* machine capacity */
int T=40;  /* planning horizon */
double cp=0.4;
double cm=1.;
double A[20001]; /* A[1000] is A(0); A[0]=A(-10); A[2000]=A(10) */
double A_new[20001];
double CC[5][20001];
double delta=0.01;
double psi_0;
double yy0,yyp; /* yy0=A^(-1)(0), yyp=A^(-1)(cp) */

double pai(double s,double a0,double b0)
{
    double o,h;
    h=2./(b0-a0);
    if (s<a0) o=0.;
    else {
        if (s<(a0+b0)/2.) o=2.*h*(s-a0)/(b0-a0);
        else {
            if (s<b0) o=2.*h*(b0-s)/(b0-a0);
            else o=0.;
        }
    }
    return o;
}

double root2(double xx)
{
    double o,as,bs;
    int i;
    if (A[10000]/cp < -xx/V) printf("strange thing\n");
    if (A[20000]/cp > (100.-xx)/V) printf("strange thing\n");
    else {
        for (i=10001;i<20001;i++) {
            if (A[i]/cp < ((i-10000)*delta-xx)/V) {
                as=-xx/V-A[i-1]/cp+(i-10000)*(A[i]-A[i-1])/cp;
                bs=-1/V+(A[i]-A[i-1])/(delta*cp);
                o=as-bs;
            }
        }
    }
}
```c

\begin{verbatim}

double inverse()
{
    // This function calculates A^(-1)(0) and A^(-1)(cp)
    // yy0=A^(-1)(0),  yyp=A^(-1)(cp)
    int i;
    yy0=0;
    yyp=0;
    for (i=9998;i<10010;i++) {
        printf("A= %f\n",A[i]);
    }
    if (A[10000]<0) printf("error #1\n");
    else {
        for (i=10000;i<20000;i++) {
            if (A[i]<=0.) {
                yy0=delta*(i-10001.+A[i-1]/(A[i-1]-A[i]));
                break;
            }
        }
    }

    if (A[10000]<cp) {
        for (i=10000;i>0;i--) {
            if (A[i]>=cp) {
                yyp=delta*(i-10001.+A[i]-cp)/(A[i]-A[i-1]));
                break;
            }
        }
    }
    else {
        for (i=10000;i<20000;i++) {
            if (A[i]<=cp) {
                yyp=delta*(i-10001.+A[i]-cp)/(A[i]-A[i+1]));
                break;
            }
        }
    }
    return 1.;
}
\end{verbatim}

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double B(double x)
// This function calculates x at t+1- as a function of x at t.
{
    int j;
    double aa,y;

    if (x>yy0) {
        y=x;
        j=(int)(y/delta+10000);
        psi_0=A[j]-cp;
    }
    else {
        aa=yyp-V;
        if (aa<0.) aa=0.;
        if (x>aa) {
            y=root2(x);
            j=(int)(y/delta+10000);
            psi_0=A[j]-cp;
        }
        else {
            if (x>0.) {
                y=x+V;
                j=(int)(y/delta+10000);
                psi_0=A[j]-cp;
            }
            else {
                y=root2(0.);
                aa=y-V;
                if (aa<-V) aa=-V;
                if (x>aa) {
                    psi_0=-cm*x/V;
                }
                else {
                    if (x>-V) {
                        y=x+V;
                        j=(int)(y/delta+10000);
                        psi_0=A[j]-cp-(cp+cm)*x/V;
                    }
                    else {
                        y=x+V;
                        j=(int)(y/delta+10000);
                        psi_0=A[j]+cm;
                    }
                }
            }
        }
    }
}
return y;
}

double BT(double x) /* valid for t=T only; returns psi_0 = C(x,0) */
{
    double y;
    if (x>0) { psi_0=-cp; y=x; }
    else {
        if (x==0) { psi_0=0.; y=x; }
        else {
            if (x>-V) { psi_0=-cm*x/V; y=0.; }
            else {
                psi_0=cm; y=x+V;
            }
        }
    }
    return y;
}

void main (void)
{
    FILE *stream;
    int i,t,flag;
    double x,y,s,am,sum,a,b,a01,a02,b01,b02;
    a01=0.1;
    b01=1.7;
    a02=0.1;
    b02=1.7;
    stream=fopen("AA10.dat","w+");
    for (i=0;i<20001;i++) {
        A[i]=0.;
        A_new[i]=0.;
        CC[0][i]=0.;
        CC[1][i]=0.;
        CC[2][i]=0.;
        CC[3][i]=0.;
        CC[4][i]=0.;
    }

    // Calculation of A(y) at t=T-1
    for (i=0;i<20001;i++) {
        y=delta*(i-10000.);
        a=a01;
        b=b01;
        flag=1;
        A_new[i]=0.;
        sum=0.;
for (s=a; s<=b; s=s+delta) {
    x=y-s;
    am=BT(x);
    if (s==a || s==b) {
        A_new[i]=A_new[i]+0.5*delta*pai(s,a,b)*psi_0;
        sum=sum+0.5*delta*pai(s,a,b);
    } else {
        A_new[i]=A_new[i]+delta*pai(s,a,b)*psi_0;
        sum=sum+delta*pai(s,a,b);
    }
}
for (i=0; i<20001; i++) {
    x=delta*(i-10000.);
    CC[0][i]=BT(x);
}
printf("sum= %f\n",sum);
for (i=0; i<20001; i++) {
    A[i]=A_new[i];
}
iinverse();
printf("t= %d   A^-1(0)= %f   A^-1(cp)= %f \n",T-1,yy0,yyp);
for (t=T-2; t>=0; t--) {
    if (flag==1) {
        flag=2;
        a=a02;
        b=b02;
    } else {
        flag=1;
        a=a01;
        b=b01;
    }
    for (i=0; i<20001; i++) {
        y=delta*(i-10000.);
        A_new[i]=0.;
        for (s=a; s<=b; s=s+delta) {
            x=y-s;
            am=B(x);
            if (s==a || s==b)
                A_new[i]=A_new[i]+0.5*delta*pai(s,a,b)*psi_0;
            else A_new[i]=A_new[i]+delta*pai(s,a,b)*psi_0;
        }
    }
}
for (i=0;i<20001;i++) {
    A[i]=A_new[i];
}
inverse();
printf("t= %d    A^-1(0)= %f    A^-1(cp)= %f\n",t,yy0,yyp);
if (t==59) {
    for (i=0;i<20001;i++) {
        x=delta*(i-10000.);
        CC[1][i]=B(x);
    }
}
if (t==39) {
    for (i=0;i<20001;i++) {
        x=delta*(i-10000.);
        CC[2][i]=B(x);
    }
}
if (t==1) {
    for (i=0;i<20001;i++) {
        x=delta*(i-10000.);
        CC[3][i]=B(x);
    }
}
if (t==0) {
    for (i=0;i<20001;i++) {
        x=delta*(i-10000.);
        CC[4][i]=B(x);
    }
}
for (i=9800;i<10201;i++) {
    fprintf(stream,"%2.2f  %f  %f  %f\n",(i-10000)*delta,CC[0][i],CC[3][i],CC[4][i]);
}
fclose(stream);
x=x;
תוכן העניינים

1 תקציר העבדה
2 תום העניינים
1. מבוא
1.1 המבנה העבורה
1.2 שיטות המחקר
1.3فشיבויות ותרומות העבורה
2 סקר ספרות
2.1 מידוי דינמיות ביקוש ובוית ציור
2.2אמרות תיאורカラー ציוף בדמנה
2.3ערוך עליון מעליי וווסר
2.4ההנגנות של לקוות במערך תורם ובוית בברכיו ציור
2.4.1ההנגנות לקוות תורם במערך תורם
2.4.2ההנגנות לקוות יוזם מחול
2.5سكنיני גורם מאפיון ציור ציור תרבותי לשברב
2.6השימוש חיסר מעליי לחדר ובוית ציור
3 ניסוח ביוונית בברכיו Цיאור
3.1ركز
3.2נסוח כללי
3.3ניסוח ביוון ציור עמו ביקוש בנוכחות צאן בינידות שללא הפוספי מ ראוי (מקרה 1)
3.4ניסוח ביוון ציור עמו ביקוש בנוכחות צאן בינידות הפוספי מ入りי (מקרה 2)
3.5ניסוח ביוון ציור עמו ביקוש טלי מומקן בינידות צאן בינידות הפוספי מ入りי (מקרה 3)
3.6 נסוח ביוות יוצר על ביוות רכז ביוות וללא הפסד, מכרות)
4.הנאמים אופטימליים לבריגות היוצר
4.2 תנאים אופטימליים לבריגות יוצר על ביוות בקוזזות ו矧 ביוות וללא הפסד מכרות (מקורה 1)
4.3 תנאים אופטימליים לבריגות יוצר על ביוות בקוזזות ו矧 ביוות וללא הפסד מכרות (מקורה 2)
4.4 תנאים אופטימליים לבריגות יוצר על ביוות תלו, מוניטין בקוזזות ו矧 ביוות (מקורה 3)
4.5 תנאים אופטימליים לבריגות יוצר על ביוות רכז בתו מוניטין וללא הפסד, מכרות (מקורה 4)
5.1 דיאצימיקה של המשטחים הפרימליים והדואליים והדואליים של האסטרטגיות
5.2 דיאצימיקה של המשטחים הפרימליים והדואליים והדואליים של האסטרטגיות
4.1 יוצר על ביוות בקוזזות ו矧 ביוות וללא הפסד, מכרות (מקורה 1)
5.2.1 דיאצימיקה של המשטחים הפרימליים והדואליים
5.2.2 יוצר על ביוות בקוזזות ו矧 ביוות וללא הפסד, מכרות (מקורה 2)
5.3 דיאצימיקה של המשטחים הפרימליים והדואליים והדואליים של האסטרטגיות
5.4 דיאצימיקה של המשטחים הפרימליים והדואליים והדואליים של האסטרטגיות
5.5 דיאצימיקה של המשטחים הפרימליים והדואליים והדואליים של האסטרטגיות
5.5 דיאגרמה של המשטח הפנימי והד奥林י ותוכנות של האסטרטגיה האופטימלית

שה anlaşıl ישראל בשיתוף הפרשים והדオリית ותוכנות של האסטרטגיה האופטימלית (מקרה 4)

5.5.1 אסטרטגיות גידור

5.5.2 חישוב ערך הגידור (מקרה 5)

5.5.3 אסטרטגיות גידור לאธรรม סופי (מקרה 5)

6. אלגנוקרימין לחישוב חשבון הבקרה האופטימלי

6.1 עקר

6.2 אלגנוקרימין לחישוב חשבון הבקרה משוב אופטימלי, לעניין יוזר על בקשות בניקוד

6.3 אלגנוקרימין לחישוב חשבון הבקרה משוב אופטימלי, לעניין יוזר על בקשות בניקוד (מקרה 1)

6.4 אלגנוקרימין לחישוב חשבון הבקרה משוב אופטימלי, לעניין יוזר על בקשות תניל

6.5 מונטי בניקודו חומת הבדיודים והכספים מכירות (מקרה 3)

6.6 בדיקות הלולא הכספים מכירות (מקרה 4)

7. יישום של המש bât

7.1 מקרה 1 – תוצאות מסירות

7.1.1 התכניות של חשבון הבקרה

7.1.2 התכניות של חשבון הבקרה (מקרה 7.1.3)

7.2 מקרה 2 – תוצאות מסירות

7.3 מקרה 3 – תוצאות מסירות
8. ארוע יצ품 של רכיב מחשבים אישייםинтерניטים לתחזוק
8.1 מבוא ונטיבוב
8.2 הארוע של Olivetti
8.3 דרישות ליישום השיטה
8.4 דוגמאות מספריות ליישום off-line
9. מסקנות
9.1 תוצאת עיקרים
9.2 אפשירות לעหมอ ממקרא
ביבליוגרפי

נספח א’ – השפעת התנהלות הליך נוחות על פונקציות העיצוב
נספח ב’ – קידוד האלגוריתמים של מקרה 1, בשפת C.
תקציר

עבורה זו והוקרת את בטיעית בקערת ייצור, באמותיוע עיקורי המכסופים. הביקוש היא, עוד, מתואר על ידי, معدل של קפיצה overshadow, בנקודות 2 ממבדיד, או על ידי, תחילה, ווירatitisunktionsmayıım (Wiener stochastic process). בעבורה זו, וגאלא מערוך, יוזר, על מכון את המון, ואת המשטחי, והתשענות, גורם, מודל, ה viện, במאי, ויקני, והמקסימום, היבר. בהיות. בין מע変わる, ובר, קהלית, המגפה, לאור, עם התכונה, הפופולרי, אסטרטגוניות, והתאמה, לתחילה, היוצרות. האסטרטגוניות, יוביל, לחיות, תבנית, וקנו, כל תקופת, תכונה, ונוצר, המוניטין, בתכנית התכונה. בין קוריאים לאסטרטגוניות, אחרים, אסטרטגוניות, גיבור, בצמחי. (כון ש-1), יוע, המליא, מחשב, בכל תקופה ו-2, כשמש, התכונה, שלף. אסטרטגוניות, מתפתל, לאסטרטגוניות, מוסר. גיורי המוניטין, מבטיע היוצר, בצמחי,צלזר, פתרון בטיעי, על יאן, ודואו, בזעיר, בצמחי, בין קוריאים, ואת עיקורי המכסופים, ומפתיחים, את האפשרות, שישובים, השמתנו, הפרברלית והורודילי, (state-costate), ומקימה במאבק, והתאונה, (forward-backward), bèזא, הזמנה, כפל, צי, מק, מבטיע התכונה, של חוק; בקערת המושב, האפטיוליות, ומפתיחים, שישובים, חקיענון, לאמידת, החיקון, בכל תקופת הזמנה. עזר בקיבוש, רץ, בצמחי, או מקיום, צוין, של קוי, זיגוור, ומקימים, פסטה, לחובב, על, כל התגידי. הסיבוכיות, של, השתיים, ליארח, באור, צוין הזמנה. דוגמאות, ממסירות, אוהב. וממשי, ממחישות את היישום של התוצאות, בעבורה, חקקי.
הдержива mới במעשהו בהדרכה

דר' יכניסי חמלניצקי
פרופבלטת להנדסה ע"י איבי ואלר פֶלִישְמן
בית הספר להנדסיים מחקריים ע"י גנווילר-סְלְינֶר

אסטרטגיות "גייזור" הלוגית זומן בבריחה
אספיסטילית של מערכות בתי הצפון אא ודאות

גון זייגר

תינור女神 كбол חנאנור "דוקטור לפילוסופיה"
הוגש לאסמנט של אוניברסיטת תל-אביב

עבודה ונטישה ב엠ירסיטות להנדסה
בדריכת ד"ר יבגיני חנוליצקי

כסל טתח"י
אסטרטגיות "גייזר" לתלויות זמן בבקה

אופטימליות של מערכות בחרה ואיזון

הוגן זיגנר

הוגש לסנאט של אוניברסיטת תל אביב

כסל תשמ"ז