Strategic Behavior in a Queue
and
A Model for Network Evolution

Thesis submitted for the degree of
“Doctor of Philosophy”

by

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Submitted to the Senate of Tel-Aviv University
October 2011
Abstract

This thesis deals with two subjects: strategic behavior in a queue; evolution of complex networks:
In the context of **Strategic behavior in a queue**, we consider an observable M/M/1 queue and explore:

- the regulation of this queue with mixed population of cooperative and non-cooperative customers.
  We identify the optimal control of cooperative customers in order to maximize the social welfare, while non-cooperative-customers maximize their own welfare. The main results are non-intuitive: First, the optimal control of cooperative-customers is independent of their proportion; Second, the gain of having the ability to control cooperative-customers after they join the queue (instructing them to renege) is relatively small.

- inefficiency in this queue.
  We use the Price-of-Anarchy (PoA) measure, which is the ratio of the optimal social welfare and equilibrium social welfare. We find that the PoA has an odd behavior in two aspects: First, it sharply increases, from 1.5 to 2, as the arrival rate comes close to the service rate; Second, it becomes unbounded exactly when the arrival rate is greater than the service rate, which is odd because the system is always stable.

In the context of **evolution of complex networks**, we suggest a novel approach to modeling network evolution, based on the dynamics of independent Markov chains. The approach demonstrates that a network of complex topology can be composed of identical elements that have independent behavior. Moreover, the evolution is measured in continuous time units, as opposed to other models, where it is measured by a discrete counter of iterations. Our model produces characteristics in agreement with real world networks.
Acknowledgements

I would like express my deepest gratitude to my supervisor - Prof. Refael Hassin - for being an inspiring supervisor. He provided me with the freedom to pursue new ideas, yet at the same time, steered me in the right direction when needed. He offered me invaluable time and advices. He set me an example of brilliance, punctiliousness and kindness.

I recognize that this research would not have been possible without the scholarships of the School of Mathematical Sciences and the complementary financial support of my supervisor. I express my gratitude to those agencies.

I would also like to thank my fellow graduate-students for their friendship and stimulating conversations.

Finally, I thank my family and friends - in particular my parents, my husband and my son - for their love through the duration of my studies.
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Chapter 1

Introduction

1.1 Strategic behavior in a queue

Strategic behavior has been much studied in the context of observable queues. Customers joining behavior, which is based on the queue length, does not maximize the social welfare. However, the arrival process is often controlled by a central entity, that will be referred to as the manager, to optimize social welfare. The control has been studied in two forms. In the first form, there is a certain pricing policies [53, 54, 62, 86]. In the second, the control is exercised by the facility, i.e. customers selection [25, 52, 60, 77].

We study the inefficiency of a queuing system when there are customers who maximizes their own welfare. Specifically, we consider this issue in Naor’s model of an observable M/M/1 queue [62]. It is the most fundamental queueing system, where the actions of an individual affect the welfare of others.

Naor assumes a first-come first-served queue, whose length can be observed by the decision makers, with linear waiting costs and fixed rewards from obtaining service. Balking is associated with zero cost or reward. See [36] for a survey of related literature. Naor analyzes the behavior of homogeneous customers. The equilibrium solution in Naor’s model is a pure threshold strategy. Namely, for some integer $n$, customers join the queue if and only if the observed queue length upon arrival is strictly shorter than $n$. The socially-optimal solution in this model is also characterized by a threshold strategy. Naor observes that the individually optimal strategy is less selective than the socially optimal one. Typically, the equilibrium threshold is strictly higher. This result is robust in the sense that it also holds for more general queueing models like M/M/s [46], M/G/1 [27] and GI/M/1 [76]. Differently, under a last-come first-served policy, the behavior of individuals is
socially optimal [35].

Our study deals with two cases in the context of the inefficiency in Naor’s model of an observable M/M/1. In one case, all the customers are self-optimizers. In the other, only part of them are self-optimizers and the others obey the manager. These cases are detailed in the following sections.

1.1.1 Mixed population of cooperative and non-cooperative customers

A main convention of strategic behavior in Queueing Theory is that it is either cooperative or non-cooperative. Some models assume that all customers are cooperative, obeying a manager who controls them to maximize the social-welfare. Other models assume that all agents are non-cooperative, maximizing their own welfare. In the real world, customers diverse by their behavior. For example, legislation may force different disciplines for different parts of the population: Hall, Koppalle and Pyke [33] consider a single exponential server, which can use its excess capacity and admit occasional fill-in customers, as long as it keeps its commitment to supply service to its core customers within a given expected waiting time; Caldentey and Wein [21] consider a model for a market with two customers types: regular buyers who sign the long-term contract or speculators that wait to see the spot price at the time $t$ of their demand. For another example, people diverse by their willingness to cede priority on the waiting list for cardiac surgery [16].

From the manager’s point of view, it is important to compare the cost and the benefit of having a greater proportion of cooperative-customers. It is also important to compare different levels of control.

A model where a manager controls only a fraction of the flow in a network can be modeled as a Stackelberg game, in which there is a leader who aims to achieve social optimality. The leader moves first and all other agents (the followers) move second. For example, there are interesting results in association with network-routing as a Stackelberg game: In many cases of practical interest there is a strategy for the leader that only applies to a controlled fraction but induces an optimal overall behavior [48, 49]. There is a minimum value for the fraction of flow needed to be centrally controlled for there to be any improvement in the social cost [74]. Other papers investigate how to lead the controlled part of the flow in order to reduce the routing cost [42, 50, 51, 57, 69, 73]. Similar models consider Stackelberg atomic routing network games [24], in which players with considerable market power make their decisions before the others. In another routing model, users seek to optimize some combination between their own performance and that of the
other users [8].

In similarity with the Stackelberg games approach, we explore a queuing system where the manager controls only a fraction of the flow. In contrast to routing of flow in a network, the decisions of the manager (join, balk, renege) are made dynamically in response to the dynamic state of the system (queue length). Another difference is that the decisions of the manager in a queuing system do not affect the equilibrium strategy of those who do not obey him.

We describe our model as an extension of Naor’s model, by adding a parameter $\alpha$, which is the proportion of cooperative (social optimizers) customers. The complementary proportion of customers are non-cooperative (self-optimizers). We present the optimal control strategy, considering two levels of control: admission control: in which the manager instructs cooperative-customers to join or balk upon their arrivals; dynamic control: in which the manager also instructs cooperative-customers to renege or give up their place in the line. We investigate the influence of $\alpha$ on the social welfare in association with different values of the other parameters of the model. We quantify the additional benefit from having a dynamic control instead of admission control.

Similar approach has been taken by Blanc, Nain and Towsleyto [13] to obtain the optimal admission policy for an $M/M/c$ with one controlled and one uncontrolled arrival stream. In difference from our approach: the uncontrolled customers always join the system and there is a state dependent reward. It is shown that when the discount factor is small, there exist an optimal admission policy of a threshold type.

1.1.2 Price of Anarchy (PoA)

A central issue in the context of strategic behavior in a queuing system is the inefficiency of selfish behavior. Naor [62] was the first to demonstrate this phenomena assuming a simple $M/M/1$ observable queue with linear waiting costs and a fixed service value. During the last decades, much work has been performed for more general observable queueing systems like multi-server queues [53], multiple user classes [58] or a renewal process for the arrival stream $GI/M/1$ [75]. It is not enough to know that there is a lack of optimality in the queuing system, it is also interesting to ask how significant it is, and more particularly, in what cases it is worthy to invest in the regulation of the customers, and in what cases it is not?

The inefficiency of selfish behavior is often measured by the Price of Anarchy (PoA) [50, 57]. PoA bounds the ratio of the social welfare under optimum to the social welfare under equilibrium. Thus, PoA measures the extent to which non-cooperation approximates cooperation. The PoA has
been studied in various settings: congestion game [50], routing [34, 72, 73, 82],
toll competition in a parallel network [90], network-creation game [12, 26],
supply chains [20, 21], system resource allocation [42, 70], greedy auctions
[55], multiple-items auctions [82], network resource allocation games [41],
spectrum-sharing games [34], network-pricing games [2] and more.

Surprisingly, there has been little research on quantifying the inefficiency
of queuing systems. Haviv and Roughgarden [37] consider a multi-server
queueing system, in which the arrivals are routed to the servers, and the
routing decisions are not based on the queue lengths. The PoA in such
system is bounded from above by the number of servers. Anselmi and Gaujal
[7] considers a system of parallel unobservable queues, in which the router
has a memory of previous dispatching choices and the demands grow with
the network size. We explore the PoA of an observable $M/M/1$, which is
the most fundamental queueing system that involves customers’ decision. To
the best of our knowledge, it is the first study to deal with the PoA of an
observable queuing system.

We measure the inefficiency in the observable $M/M/1$ by the PoA. The
PoA is a function of the model’s parameter as well as of the the optimal
strategy $n^*$. There is no explicit term for $n^*$. Instead, it is an implicit
solution of an exponential function and it has to be computed according to a
numeric procedure, designed by Naor [62]. Accordingly, the investigation of
the PoA is a combination of analytical and numerical approaches is required.

We find that the PoA has an odd behavior as a function of the utilization
factor $\rho$. Results can be separated into three regions: For $\rho < 0.98$, the PoA
is lower than 1.5; For $0.98 < \rho < 1$, the PoA sharply increases but remains
bounded by 2; For $\rho > 1$, the PoA is unbounded. We emphasize that the
model doesn’t need to assume $\rho \leq 1$ for stability, and in fact this number is
of no significance in Naor’s results. It comes therefore as a surprise that the
PoA is bounded if and only if $\rho \leq 1$.

1.2 Evolution of complex networks

Networks are found everywhere: human bodies [39, 44, 79], technological
systems [3, 6, 28], nature [1, 19, 61], social relationships [4, 83], etc. It is a
challenge to understand the evolution of real world networks. For example,
a common modeling goal is to explain how a given network comes to have
its particular degree distribution or clustering at time $t$. Network evolution
models have been proposed over the past few decades, such as the Erdős and
Rényi’s random graph [14, 30, 40], the small-world [85] and the scale free
network model [9]. We briefly describe these models, providing the basis and
the inspiration for our model.

Erdős Renyi’s random graph can be used to model an evolution of network as a series of random graphs: starting with a graph of \(n\) nodes and no edges, iteratively add a random edge according to a uniform distribution on the missing edges. The evolution ends when the proportion of chosen edges is \(p\). This process demonstrates a formation of a giant component \([14, 30]\), in agreement with real networks like scientific collaboration \([63]\) and neural connection \([80]\). It also demonstrate a small average shortest path \([15]\) like the “six degree separation” concept \([47, 59]\).

In 1998 Watts and Strogats reported the small world phenomena: a coexistence of small diameter and high clustering \([85]\). They measure the tendency to cluster by the clustering coefficient, which is the fraction of triples that have their third edge filled in to complete the triangle \([64, 83, 85]\). Examples for small world are the World Wide Web \([5]\) and the co-occurrence of words \([31]\). Watts and Strogats introduce a network evolution model that captures the small world’s criteria: starting with a ring lattice with \(n\) nodes in which every node is connected to its first \(k\) neighbors \((k/2\) on either side), reassign each edge to a distant node with probability \(p\), such that self-connections and duplicate edges are excluded. This model has been further investigated by Barthélemy and Amaral \([11]\).

In 1999 Barabási and Albert \([9]\) explored the observed static degree distribution of networks. The degree \(k_i\) of a node \(i\) is the number of its neighbors. In many real networks, such as scientific papers citations \([68]\), the degree distribution is typically right-skewed with a “heavy” tail. Moreover, the degree distribution follows a power-law. That is, the fraction \(P(j)\) of nodes in the network having \(j\) connections to other nodes goes as \(cj^{-\gamma}\), where \(c\) is a normalization constant. Barabási and Albert introduced a network evolution model, producing a power-law degree distribution: starting with a small number \((m_0)\) of nodes, iteratively add a new node and link it to \(m \leq m_0\) different nodes already present in the system. The new node is connected to node \(i\) with probability \(\frac{d_i}{\sum_j d_j}\) (preferential attachment process).

One can identify two approaches for modeling network evolution. The traditional approach, which is inspired by the theory of random graphs, emphasizes the advantage of having a simple model, which is exactly solvable for many of its properties. The later approach, which is known as the new science of networks \([10, 84]\), emphasizes the topological structure of the network. Motivated by these approaches, we raise the following questions:

- Can identical and consistent behavior of elements produce network with complex topology?
How can we describe network evolution over continuous time?

To explore these questions, we suggest a novel approach to model network evolution. The evolution is represented by the dynamics of independent Markov chains, each represents an element of the network. Chains move among a common space of states. Sometimes chains intersect, being in the same state at the same time. These intersections relate the chains with each other and imply many interesting processes, including network evolution. As a first step, we consider a simple version of the model: Markov chains are identical; The time a chain spends at a given state is exponential; There are only two states, one of them is a meeting-state. Starting with a network of no edges, we add an edge between two nodes when their representative chains are both in the meeting-state for the first time.

We achieve a closed formula for the degree evolution. We explore the structural features of the evolving network, which are in agreement with the real world: the diameter is small, the clustering is high and the degree distribution is highly skewed.
Chapter 2

Mixed population of cooperative and non-cooperative customers in the observable M/M/1

Regulation of entry to a queue attracts considerable interest among economists and operations researchers, see surveys [78] and the references therein. We explore the regulation of non-homogenous population in an observable M/M/1.

Our model is an extension of Naor’s model, adding a parameter $\alpha$, which is the proportion of cooperative customers, who obey a social manager. The complementary proportion of customers are non-cooperative, who optimize their own welfare by following a threshold strategy $n_e$. For short, we denote cooperative or non-cooperative customer by c-customer or n-customer, respectively.

2.1 Background: Noar’s model for homogenous population

Naor’s model of an observable M/M/1 queue assumes homogenous customers, being either self-optimizers or social-optimizers. The model assumes a Poisson arrival of customers with rate $\lambda$ and exponential service rate $\mu$. There is a reward of value $R$ the customer obtains while completing service, and a cost of $C$ per unit of time spent waiting or in service. Following Naor (see also [36]), the model’s parameters can be normalized so that there are only two relevant parameters: $\rho = \frac{\lambda}{\mu}$, which is the system’s utilization factor,
and \( \nu_e = \frac{R \mu}{C} \), which is the value of service in terms of expected waiting cost during a service duration.\(^1\)

A customer enters the queue according to a pure threshold strategy which is denoted as \( n_e \). He joins the queue if he observes \( n_e - 1 \) or fewer customers and balks if he observes \( n_e \) or more customers. It is straightforward that

\[
n_e = \left\lfloor \frac{R \mu}{C} \right\rfloor = \lfloor \nu_e \rfloor.
\]

The social welfare is optimized when all the customers adopt the strategy \( n^* \). Define

\[
g(\nu) = \nu (1 - \rho) - \rho (1 - \rho^e) \frac{1}{(1 - \rho)^2},
\]

then Naor showed that \( n^* = \lfloor \nu^* \rfloor \), where \( \nu^* \) is the unique solution to

\[
g(\nu) = \nu.
\]

Naor observes that \( n^* \leq n_e \).

### 2.2 Dynamic control

Dynamic control utilizes the discipline of \( c \)-customers to the maximal extent. The manager instructs \( c \)-customers to join or balk upon their arrival. Additionally, he can instruct them to renege or give up their state in the line. In particular, when \( n \)-customer arrives and a \( c \)-customer is served, service is preempted: the \( c \)-customers move back to the line and the arriving \( n \)-customer enters the service. Without lost of optimality, an optimal strategy is well-defined by the maximum number of \( c \)-customers \( g(i) \) who stay in the line as a function of the number \( i \) of \( n \)-customers ahead.

We use the verb *expel* for an instruction to leave the queue, either by balking or reneging. We use the index \( d \), to indicate that a term refers to dynamic control,\( ,\)

#### 2.2.1 Naive strategy \( g \)

We consider the following strategy:

\[
g(i) = \max(0, n^* - i),
\]

\(^1\)To avoid triviality, \( \nu_e \geq 1 \). Otherwise, an arriving customers would balk even if the system is empty.
where $n^*$ is the optimal threshold for homogenous cooperative population. We say that $g$ is naive, in the sense that $c$-customers behave as if all other customers are also cooperative: being expelled when the length of the queue exceeds $n^*$.

### 2.2.2 $g$ is socially optimal

In this section we prove a theorem, showing that the naive strategy $g$ is socially optimal. We start with a lemma, showing that it is socially optimal to expel $c$-customers when there are $n^*$ other customers in the system. To prove the lemma, we examine a specific customer who is given the lowest priority to be served. Such customer no longer imposes external effects. Similar approach has been taken to obtain the optimal admission control policy for a first-come first-served M/M/m queueing system [91].

**Lemma 2.2.1.** It is optimal to expel a $c$-customer when there are $n^*$ other customers in the system.

*Proof.* Suppose otherwise. Suppose that there is an optimal strategy $f$, and a state $s$ such that the total number of customers in the system is $n^* + 1$ with at least one $c$-customer, and $f$ doesn’t expel any $c$-customer when the state is $s$. Let $t_0$ denote an instant when this happens.

Define the Mixed process $p_M$ and the Naor process $p_N$ that start at $t_0$ with the initial state $s$. Both processes are on the same probability space so that they see the same arrivals and services completions. In process $p_M$, $c$-customers follow strategy $f$, $n$-customers follow the equilibrium threshold strategy $n_e$, and $n$-customers are given a higher priority to be served. In process $p_N$, all customers follow the socially-optimal threshold strategy $n^*$ and are served according to the order of their arrival.

We tag a $c$-customer $c$, who stays in the system at $t_0$. In order to compute the expectation of the additional social welfare from keeping $c$ in the system at $t_0$, we make the following assumptions on his actions following $t_0$:

- In $p_M$, $c$ reneges when $f$ expels a $c$-customer. Specifically, $c$ reneges and one $c$-customer stays instead of balking or reneging.

- In $p_N$, $c$ reneges when he reneges in $p_M$.

- In both $p_M$ and $p_N$, $c$ has the lowest priority to be served.

We note that under these assumptions, in both systems defined by $P_M$ and $P_N$, $c$ has no influence on the decisions and the welfare of all other
customers. Therefore, the expected additional social welfare from keeping $c$ in the system at $t_0$ equals the expected utility of $c$ starting at $t_0$.

Let $t_1$ denote the first time following $t_0$ when $f$ expels a $c$-customer or the time when the service of $c$ is completed, whichever comes first.

During $[t_0, t_1)$, $P_M$ takes no actions of expelling $c$-customers. Therefore, the number of $c$-customers is not lower in $P_M$ than in $P_N$.

Also during $[t_0, t_1)$, $P_N$ takes at least the same number of $n$-customers’ balking actions like $P_M$. It is because of two reasons: First, in $P_N$ the threshold is lower than in $P_M$; Second, in $P_N$ an arriving $n$-customer is positioned after the $c$-customers who are already in the system, while in $P_M$ he is position before them. Therefore, the number of $n$-customers is also not lower in $P_M$ than in $P_N$.

In total: during $[t_0, t_1)$, the queue length maintained by $P_M$ is not lower than in $P_N$.

We now use these observations to conclude that under any sequence of events, the utility of $c$ under the system induced by $P_N$ is not worse than under $P_M$: (1) If $c$ completes service at $t_1$ under $P_M$, then it also does so at $t_1$ or earlier under $P_N$. (2) If $c$ is expelled at $t_1$ under $P_M$, then it has either completed his service earlier or it also reneges at $t_1$ under $P_N$.

By this conclusion and the fact that there are sequences of events under which the utility of $c$ is strictly better under $P_N$, we know that the expected additional social-welfare of $c$ in $P_M$ is strictly lower than in $P_N$.

By Naor’s result for the optimality of the threshold strategy $n^*$, keeping $c$ in the system when there are $n^*$ customers in the system would imply a non-positive expectation of additional social welfare. The expectation of additional social welfare which is implied by $p_M$ is even lower, so it is negative. Therefore, instructing $c$ to leave at $t_0$ would increase the expected social welfare. It contradicts the supposition that $f$ is optimal.

We continue the proof of the social optimality of $g$ by defining two system states according to $n_n$, which is the number of $n$-customers in the system:

- **State-1**: $0 \leq n_n \leq n^*$.
- **State-2**: $n^* + 1 \leq n_n \leq n_e$.

See Figure 2.1 to observe a simulation example of the alternating system-state. It is an evolution of the queue length when $n^* = 3$, $n_e = 9$ and the naive strategy is applied. It is shown as a function of a discrete counter of epochs (arrivals and service completions). The gray bars represent that the system is in State-1. The black bars represent that it is in State-2. One observes that the queue length tends to stay fixed when $n$ equals $n^*$ because
of balking of arriving c-customers and reneging of these customers when an n-customer arrive. The queue length tends to stay fixed also when \( n \) equals \( n_e \) because of balking of both customers types.

Figure 2.1: Queue length vs. discrete counter of epochs

Following Lemma 2.2.1 and the definition of State-1 and State-2, we show that \( g \) is optimal:

**Theorem 2.2.2.** The naive strategy \( g \) optimizes the expected social welfare.

**Proof.** Let \( S_1 \) and \( S_2 \) denote the expected rate of net gain during State-1 and State-2, respectively. The general expected rate of net gain \( S \) is a combination of \( S_1 \) and \( S_2 \):

\[
S = \kappa S_1 + (1 - \kappa) S_2,
\]

where \( \kappa \) is the proportion of time in State-1. \( \kappa \) is solely determined by the n-customers’ behavior and is independent of \( g \). Therefore, it is sufficient to prove that \( g \) maximizes \( S_1 \) and \( S_2 \), separately:

\( g \) maximizes \( S_1 \). By Lemma 2.2.1, the state of the system is the same when entering State-2 and when leaving it: there are \( n^* \) n-customers and no c-customer. Therefore, one can think of patching all time intervals when the
system is in State-1 to get a continuous process. We claim that during this process, the system behaves like an M/M/1/$n^*$ system with a homogenous population of c-customers:

- When $n_n < n^*$: an arriving customer joins the queue whether he is cooperative or not. This event is equivalent to joining of the arriving customer in M/M/1/$n^*$.

- Otherwise, when $n_n = n^*$: 1) an arriving c-customer balks; 2) an arriving n-customer joins to imply the reneging of a c-customer if there is such one in the line. These two events of expulsion are equivalent to balking of the arriving customer in M/M/1/$n^*$.

We do not consider the epoch of an arriving n-customer who joins a queue when there are $n^*$ n-customers and no c-customer, because this epoch has no influence on the patching process. The reason for it is that this epoch implies a transition from State-1 to State-2 and the following interval – until the system returns to System-1 when there are $n^*$ n-customers and no c-customer – is removed from the patching process.

In conclusion, $g$ maximizes $S_1$, because it implies an expected rate of net gain which is equal to the expected rate of net gain in M/M/1/$n^*$, and the latter is optimal by Naor.

$g$ maximizes $S_2$. By Lemma 2.2.1, any arriving c-customer who joins the system in State-2 will renge when $i_{nc}$ decreases to $n^*$, implying an additional negative utility as there is only waiting-cost and no reward. Therefore, $g$ maximizes $S_2$ by instructing c-customers to balk.

\square

### 2.3 Admission control

In different with the previous section, we now assume that the manager cannot control customers after they join the queue. He only controls their admission: instructing them to join or balk. As in Naor’s model, decisions are made upon customers’ arrival. Therefore, the socially-optimal-strategy is a pure threshold strategy $n_a^*$.

As a first step, we analyze a general model of a mixed population with two thresholds. Following the solution of the social welfare in the general model, we associate the higher threshold with $n_e$ and compute the value of $n_a^*$ that maximizes the social welfare.

We use the index $a$, to indicate that a term refers to admission control.
2.3.1 General solution for mixed population

Consider a mixed population of arriving customers, with two thresholds:

- Customers with a threshold strategy \( n_1 \) arrive as a Poisson stream with parameter \( \alpha \lambda \).
- Customers with a threshold strategy \( n_2 > n_1 \) arrive as a Poisson stream with parameter \( (1 - \alpha) \lambda \).

The steady state equation is

\[
p_{i+1} = \begin{cases} 
\rho p_i & (0 \leq i < n_1), \\
(1 - \alpha) \rho p_i & (n_1 \leq i < n_2).
\end{cases}
\] (2.2)

The solution of (2.2) is:

\[
p_i = \begin{cases} 
\frac{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)(1 - \rho(1 - \alpha))^{n_2-n_1}}{1 - \rho(1 - \alpha)} & (0 \leq i \leq n_1) \\
\frac{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)(1 - \rho(1 - \alpha))^{n_2-n_1}}{1 - \rho(1 - \alpha)} & (n_1 < i \leq n_2).
\end{cases}
\] (2.3)

It is the same as equations (7) and (8) in Para-Frutos [66].

The generating function is

\[
g(z) = \sum_{i=0}^{n_2} p_i z^i = \frac{1}{1 - \rho z^{n_1+1}} \frac{1 - \rho(1 - \alpha) z^{n_2-n_1}}{1 - \rho(1 - \alpha)} \times \left[ \frac{1 - (\rho z)^{n_1+1}}{1 - \rho z} + (1 - \alpha) (\rho z)^{n_1+1} \frac{1 - (\rho(1 - \alpha) z)^{n_2-n_1}}{1 - \rho(1 - \alpha) z} \right].
\] (2.4)

The expected queue size \( L \) (including the single customer who might be in service), is given by the derivative of (2.4) with respect to \( z \) at \( z = 1 \):

\[
L = \frac{\rho}{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)(1 - \rho(1 - \alpha))^{n_2-n_1}} \times \left[ \frac{(1 + n_1) \rho^{n_1}}{\rho - 1} + \frac{1 - \rho^{n_1+1}}{(\rho - 1)^2} \right] + \frac{(1 - \alpha)^2 \rho((1 - \alpha) \rho)^{n_1} - ((1 - \alpha) \rho)^{n_2}}{((1 - \alpha) \rho - 1)^2} + \frac{(1 - \alpha)^{1-n_1}(1 - (1 - \alpha) \rho)^{n_2} - (1 - \alpha) \rho^{n_1}}{(1 - \alpha) \rho - 1} + \frac{(1 - \alpha)^{1-n_1}(n_2((1 - \alpha) \rho)^{n_2} - n_1((1 - \alpha) \rho)^{n_1})}{(1 - \alpha) \rho - 1}.
\] (2.5)
The expected number of customers diverted from the service in unit time, is given by

\[ \zeta = \lambda [\alpha(p_{n_1} + \cdots + p_{n_2}) + (1 - \alpha)p_{n_2}] \]
\[ = \lambda [\alpha(p_{n_1} + \cdots + p_{n_2-1}) + p_{n_2}]. \]

The expected number of customers joining the queue in a unit of time is:

\[ \lambda - \zeta = \lambda [1 - p_{n_2} - \alpha(p_{n_1} + \cdots + p_{n_2-1})]. \tag{2.6} \]

The expected rate of net gain is:

\[ S = R(\lambda - \zeta) - CL, \tag{2.7} \]

Substituting (2.5) and (2.6) in (2.7):

\[ S = R\lambda [1 - p_{n_2} - \alpha(p_{n_1} + \cdots + p_{n_2-1})] - \]
\[ \frac{C}{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)\frac{1 - (\rho(1 - \alpha))^{n_2-n_1}}{1 - \rho(1 - \alpha)}} \times \left[ \frac{\rho[1 - (n_1 + 1)\rho^{n_1} + n_1\rho^{n_1+1}]}{(1 - \rho)^2} + \right. \]
\[ \left. \frac{(1 - \alpha)\rho^{n_1+1}[(n_1 + 1) - n_1\rho(1 - \alpha) - (n_2 + 1)(\rho(1 - \alpha))^{n_2-n_1} + n_2(\rho(1 - \alpha))^{n_2-n_1+1}]}{(1 - \rho(1 - \alpha))^2} \right]. \]

We replace \( R\lambda \) with \( \nu_e\rho C \) and divide both sides by \( C \), to compute the benefit by units of the cost for waiting a single time unit:

\[ \frac{S}{C} = \nu_e\rho [1 - p_{n_2} - \alpha(p_{n_1} + \cdots + p_{n_2-1})] - \]
\[ \frac{\rho}{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)\frac{1 - (\rho(1 - \alpha))^{n_2-n_1}}{1 - \rho(1 - \alpha)}} \times \left[ \frac{(1 + n_1)\rho^{n_1}}{\rho - 1} + \frac{1 - \rho^{n_1+1}}{(\rho - 1)^2} \right. \]
\[ + \left. \frac{(1 - \alpha)^2-n_1\rho((1 - \alpha)\rho^{n_1} - ((1 - \alpha)\rho)^{n_2})}{((1 - \alpha)\rho - 1)^2} + \frac{(1 - \alpha)^{1-n_1}(((1 - \alpha)\rho)^{n_2} - ((1 - \alpha)\rho)^{n_1})}{(1 - \alpha)\rho - 1} \right. \]
\[ + \left. \frac{(1 - \alpha)^{1-n_1}(n_2((1 - \alpha)\rho)^{n_2} - n_1((1 - \alpha)\rho)^{n_1})}{(1 - \alpha)\rho - 1} \right]. \tag{2.8} \]

Substituting \( p_i \) for \( i = n_1, \ldots, n_2 \) according to (2.3), \( \frac{S}{C} \) is as function of \( n_1, n_2, \nu_e, \rho \) and \( \alpha \). It is the general solution for a mixed population with two thresholds.
2.3.2 Socially optimal strategy $n_a^*$

Following the solution in (2.8), we associate $n_2$ with the equilibrium threshold $n_e$, to achieve the expected social welfare in our mode:

$$\frac{S}{C} = \nu_e \rho \left[ 1 - p_{n_e} - \alpha (p_{n_1} + \ldots + p_{n_e-1}) \right]$$

$$- \frac{\rho}{1 - \rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)^{1 - (1 - \alpha)\rho^{n_e - n_1}}} \times \left[ \frac{(1 + n_1)\rho^{n_1}}{\rho - 1} + \frac{1 - \rho^{n_1+1}}{(\rho - 1)^2} \right]$$

$$+ \frac{(1 - \alpha)^{2-n_1}\rho \left( ((1 - \alpha)\rho)^{n_1} - ((1 - \alpha)\rho)^{n_e} \right)}{(1 - \alpha)^2(\rho - 1)^2} + \frac{(1 - \alpha)^{1-n_1}((1 - \alpha)^{n_e} - (1 - \alpha)^{n_1})}{(1 - \alpha)\rho - 1}$$

$$+ \frac{(1 - \alpha)^{1-n_1}(n_e((1 - \alpha)\rho)^{n_e} - n_1((1 - \alpha)\rho)^{n_1})}{(1 - \alpha)\rho - 1}$$

where

$$p_i = \begin{cases} \frac{\rho^i}{\rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)^{1 - (1 - \alpha)\rho^{n_e - n_1}}} & (0 \leq i \leq n_1) \\ \frac{\rho^i(1 - \alpha)^{-n_1}}{\rho^{n_1+1} + \rho^{n_1+1}(1 - \alpha)^{1 - (1 - \alpha)\rho^{n_e - n_1}}} & (n_1 < i \leq n_e) \end{cases}$$

We also associate $n_1$ with the threshold of c-customers’ strategy, and search for the value of $n_1$ that optimizes the social welfare. Specifically, for a fixed $\nu_e$ and $\rho$, we assume that $n_2 = \lfloor \nu_e \rfloor$ and detect $n_1$ that maximize (2.9) as a function of $\alpha$. For examples, Figure 2.2 demonstrates $n_a^*(\alpha)$ for $\nu_e = 10$ and different $\rho$ values, and Figure 2.3 demonstrates $n_a^*(\alpha)$ for $\rho = 0.6$ and different $\nu_e$ values.
Figure 2.2: The optimal admission threshold vs. proportion of c-customers

Figure 2.3: The optimal admission threshold vs. proportion of c-customers

Our numerical study gives the following observations on the behavior of
$n^*_a$: 1) When all customers are customers ($\alpha = 0$): $n^*_a = n^*$, which is in agreement with the fact that $n^*$ is the optimal threshold for homogenous cooperative population; 2) When the proportion of c-customers is higher ($\alpha$ increases): $n^*_a$ is also higher. The more there are n-customers (low $\alpha$), the higher are the expectations of the queue lengths in future times. As a consequence, c-customers compensate for the long-waiting time and decrease their strategic threshold in order to optimize the overall gain; 3) For any fixed $\alpha$, $n^*_a$ is non-increasing with $\rho$. A higher $\rho$ increases the the expectations of the queue lengths in future times, which can be compensated by decreasing the strategic threshold of c-customers; 4) For any fixed $\alpha$, $n^*_a(\alpha)$ is non-decreasing with $\nu_e$.

We examine the limit $\alpha \to 0$, for which $n^*_a(\alpha)$ represents the optimal strategy for an infinitely low proportion of c-customers. Figure 2.4 demonstrates the optimal admission control as a function of $\rho$ when $\alpha = 0.001$ and $\nu_e = (2, 8, 12)$. It demonstrates that when $\rho$ is low, the minority of c-customers follow the equilibrium strategy as the server is almost surely idle. When $\rho$ increases, the expectations of the queue lengths in future times and the threshold of the optimal strategy for c-customers decreases.

![Figure 2.4: The optimal admission threshold vs. proportion of c-customers](image-url)

Figure 2.4: The optimal admission threshold vs. proportion of c-customers
2.4 The social welfare

2.4.1 Optimal social welfare $S^*_d$ - dynamic control

Let $S^*_d$ denote the expected social-welfare obtained when the manager implements the naive strategy $g$, which is optimal. $S^*_d$ is achieved by the following steps:

1. Computing the steady state distribution. More details are given below.

2. Computing the expected queue length ($L$) and the expected proportion of time when the server is idle ($\beta$).

3. Computing the expected social welfare in terms of $C$:
   \[
   \frac{S^*_d}{C} = \nu_c (1 - \beta) - L.
   \]

Steady state distribution of a dynamic control

Given the optimal strategy $n^*$, we define:

\[
\delta_1(n) = \begin{cases} 
1 & \text{if } n \leq n^* \\
0 & \text{otherwise ,}
\end{cases}
\]

\[
\delta_2(n) = \begin{cases} 
1 & \text{if } n \leq n_e \\
0 & \text{otherwise ,}
\end{cases}
\]

and

\[
\delta_3(n) = \begin{cases} 
0 & \text{if } n = 0 \\
1 & \text{otherwise .}
\end{cases}
\]

Let $\mathbf{l} = (l_c, l_n)$ denote a system state, where and $l_c$ and $l_n$ denote the number of c-customers and n-customers in the system, respectively. We compute the corresponding steady state distribution $p(\mathbf{l})$ as the solution of
the following equations:

\[ p(l_c, l_n)[\lambda \alpha \delta_1(l_c + l_n + 1) + \lambda (1 - \alpha)\delta_2(l_n + l_c + 1) + \mu (\delta_3(l_n) + \delta_3(l_c)(1 - \delta_3(l_n)))] \]

\[ = (1 - \alpha)\lambda [\delta_3(l_c)\delta_1(l_c + l_n) + (1 - \delta_3(l_c))\delta_2(l_n)]\delta_3(l_n)p(l_c, l_n - 1) + (1 - \alpha)\lambda [1 - \delta_1(l_c + l_n + 1)]\delta_3(l_n)p(l_c + 1, l_n - 1) + \alpha \lambda \delta_1(l_c + l_n)\delta_3(l_c)p(l_c - 1, l_n) + \mu (1 - \delta_3(l_n))\delta_1(l_c + l_n + 1)p(l_c + 1, l_n) + \mu \delta_1(l_c + l_n + 1)p(l_c, l_n + 1), \]

where, by the definition of strategy \( g \), \( 0 \leq l_n \leq n_e \) and \( 0 \leq l_c \leq (n^* - l_n) \). One can divide the equations by \( \mu \), converting the system to be a function of \( \rho \).

### 2.4.2 Optimal social welfare \( S^*_a \) - admission control

Let \( S^*_a(\alpha) \) denotes the expected optimal-social-welfare, which is obtained when the manager implements the optimal admission control. It is computed by substituting \( \nu_e, \rho, \alpha, n_2 = \lfloor \nu_e \rfloor \) and \( n_1 = n^*_a(\alpha) \) in (2.9).

### 2.4.3 \( S^*_a \) and \( S^*_d \) as a function of \( \alpha \)

For \( \alpha = 1 \) : there are only c-customers. In such a case, they are never instructed to renege, even under dynamic control. Therefore, the optimal behavior is the same in both models and \( S_a(1) = S_d(1) \). For \( \alpha = 0 \) : there are no c-customers so the type of control is not relevant. Therefore, \( S_a(0) = S_d(0) \). The ratios of social welfare, \( \frac{S_a(1)}{S_a(0)} \) and \( \frac{S_d(1)}{S_d(0)} \), are the price of anarchy, which is studied by Gilboa-Freedman and Hassin [32]. For \( 0 \leq \alpha \leq 1 \) : it is trivial that \( S_a \) and \( S_d \) are increasing with \( \alpha \). However, we are also interested in the shape of \( S_a \) and \( S_d \) as a function of \( \alpha \). The shape is demonstrated in Figures 2.5 and 2.6.

In Figure 2.5 \( \nu_e \) is fixed: equals 2, 4, or 6, in the upper middle or lower graph, respectively. \( S_a \) or \( S_d \) are demonstrated for various values of \( \rho \) in solid or dashed lines, respectively.

The shape of the curves depends on \( \rho \). For low values of \( \rho \), \( S_d \) is concave, which means that the influence of increasing the proportion of c-customers is higher when there are few of them. For higher values of \( \rho \), both \( S_a \) and \( S_d \) are sigmoid, which means that the influence is highest for intermediate proportion of c-customers.
Plots intersect. It is due to the fact that the social welfare is monotonous increasing with $\rho$ when $\alpha = 1$ and non monotonous when $\alpha = 0$. The reason for monotonicity at $\alpha = 1$ is that having a higher arrival rate of cooperative customers is preferable in terms of social welfare.

Figure 2.5: Fixed $\nu_e$: social welfare vs. proportion of c-customers

In Figure 2.6 $\rho$ is fixed: equals 0.6, 0.9, or 2, in the upper middle or lower graph, respectively. The value of $\nu_e$ is varied. Unlike Figure 2.5, plots don’t intersect. The graphs demonstrate that for fixed $\rho$ and $\alpha$, both $S_a$ and $S_d$ increase with $\nu_e$. 
2.4.4 Gain of dynamic control (GoD)

We quantify the additional gain from having dynamic control instead of admission control by the ratio of $S_d$ to $S_a$. We find interest in high values of this ratio in dependency with the parameters of the model. Specifically we suggest the Gain of Dynamic-control (GoD) measure. For each choice of $\nu_e$ and $\rho$, GoD is the maximal ratio of $S_d$ to $S_a$ as a function of $\alpha$:

$$GoD(\nu_e, \rho) = \max_{\alpha} \frac{S_d(\nu_e, \rho, \alpha)}{S_a(\nu_e, \rho, \alpha)}.$$ 

We saw in the previous section (see Figures 2.5 and 2.6), that the ratio of $S_d$ and $S_a$ is usually quite small. In our numerical study we find that GoD is the highest for $\nu_e = 2$. Figure 2.7 demonstrates GoD (upper graph) and the value of $\alpha$ which induces it (lower graph) as a function of $\rho$, for $\nu_e = 2$. 

Figure 2.6: Fixed $\rho$: Social welfare vs. proportion of c-customers

![Graph showing social welfare vs. proportion of c-customers for different values of $\rho$.]
Figure 2.7: GoD and the proportion of c-customers which induces GoD vs. $\rho$

2.5 Summery and conclusions

This study explores optimal control, when only part of the population is controllable.

We consider two levels of control in association with when does the social manager expel a cooperative customer: 1) only at the time of customer’s arrival (admission control); 2) any time after his arrival until his service is completed (dynamic control).

We analyze the optimal policy for each level of control. The optimal dynamic control is to expel cooperative customers when the length of the queue exceeds a threshold, which is the same for any proportion of cooperative customers. We prove the optimality of this strategy. The optimal admission control is a threshold strategy, increasing with the proportion of cooperative customers.

We give numerical observations on the social welfare when the control is partial. The qualitative behavior of the social welfare depends on $\rho$: When $\rho$ is low, $S_d$ is concave. It means that the influence of increasing the proportion of c-customers is higher when there are few of them; For higher values of $\rho$, both $S_a$ and $S_d$ are sigmoid. It means that the influence is highest for
intermediate proportion of c-customers.

Another numerical observation is that the gain from having dynamic control instead of admission control is usually quite small. One can conclude that in most real situations it is not worth to control the customers who have already joined the queue.

A further study could explore the regulation of other more general queuing systems with a mixed population of cooperative and non-cooperative customers.
Chapter 3

Price of anarchy (PoA) in the observable M/M/1 queue

The interest for the PoA in the context of queuing system is currently growing because of the large spectrum of applications [65]. We quantify the inefficiency in the observable M/M/1 queue by the Price of Anarchy measure. It is the ratio of the optimal and the Nash equilibrium social welfare. The optimal and Nash equilibrium strategies (and their associated social welfare) have been studied by Naor (see [62] and section 2.1).

3.1 PoA general behavior

The social welfare associated with a threshold $n$ is

$$S_n = R\lambda \frac{1 - \rho^n}{1 - \rho^{n+1}} - C \left[ \frac{\rho}{1 - \rho} - \frac{(n + 1)\rho^{n+1}}{1 - \rho^{n+1}} \right].$$

PoA is defined as the ratio of the expected optimal net gain per time unit $S_{n^*}$, and the expected equilibrium one $S_{\nu_e}$:

$$\text{PoA}(\rho, \nu_e) = \frac{\frac{1-\rho^{n^*}}{1-\rho^{n^*+1}} - \frac{1}{\nu_e} \left[ \frac{1}{1-\rho} - \frac{(n^*+1)\rho^{n^*+1}}{1 - \rho^{n^*+1}} \right]}{\frac{1-\rho^{\nu_e}}{1-\rho^{\nu_e+1}} - \frac{1}{\nu_e} \left[ \frac{1}{1-\rho} - \frac{((\nu_e+1)\rho^{\nu_e})}{1 - \rho^{\nu_e+1}} \right]}$$

$$= \frac{p_{\text{join}}(n^*) - \frac{1}{\nu_e} q(n^*)}{p_{\text{join}}(n_e) - \frac{1}{\nu_e} q(n_e)},$$

where $q(n)$ is the expected queue length under a threshold $n$, and $p_{\text{join}}(n)$ is the probability that an arriving customer joins the queue. PoA$(\rho, \nu_e)$ is shown in Figures 3.1 and 3.2.
Figure 3.1: PoA Vs. $\rho$ and $\nu_e$

Figure 3.2: PoA Vs. $\rho$ and $\nu_e$ - zoom in
In the following subsections, we isolate the influence of $\nu_e$ and $\rho$ on the PoA.

### 3.2 PoA as a function of $\nu_e$

The PoA as a function of $\nu_e$ is demonstrated in Figure 3.3 for three different values of $\rho$. For $\rho \geq 1$ the PoA is monotone increasing when $\nu_e \to \infty$. The asymptotic behavior is described by the following lemmas:

**Lemma 3.2.1.** $\lim_{\nu_e \to \infty} \text{PoA}(1, \nu_e) = 2$.

**Proof.** Consider the function

$$h(x) = \frac{x}{x+1} - \frac{x}{2\nu_e}.$$ 

$h(n)$ is the social welfare when customers follow a threshold strategy $n$ and $\rho = 1$. In such case, the expected length of the queue is $= n/2$ and $p_{\text{join}} = n/(n+1)$, because the queue length distribution is uniform on $0, 1, \ldots, n$. In particular - $\text{PoA}(1, \nu_e) = \frac{h(n^*)}{h(n_e)}$, where $h(n^*)$ and $h(n_e)$ are the optimal and equilibrium social welfare, respectively.
Replacing $h(n*)$ with $h(\sqrt{2\nu_e} - 1)$ increases the value of the denominator of $PoA(1, \nu_e)$. A continuous (with respect to $x$) analysis of $h(x)$ yields that it is maximized by $x = \sqrt{2\nu_e} - 1$.

Also, replacing $h(n_e)$ with $h(nu_e)$ decreases the value of the nominator of $PoA(1, \nu_e)$. By concavity of $h(x)$, $h(x_2) \leq h(x_1)$ for any $x_1$ and $x_2$ such that $\sqrt{2\nu_e} - 1 \leq x_1 \leq x_2$. Specifically, $h(n_e) \leq h(nu_e)$ since $\sqrt{2\nu_e} - 1 \leq n_e = \lfloor \nu_e \rfloor \leq \nu_e$. The reason why $\sqrt{2\nu_e} - 1 \leq n_e$ lies at the fact that $n^* \leq n_e$ and if $\sqrt{2\nu_e} - 1$ is greater than $n_e$ than both $n^*$ and $n_e$ are lower than the $\sqrt{2\nu_e} - 1$. This implies (by the concavity of $h(x)$) that $h(n^*) \leq h(n_e)$ and it contradicts the optimality of $n^*$.

Therefore:

$$PoA(1, \nu_e) \leq \frac{\sqrt{2\nu_e} - 1 - \sqrt{2\nu_e}}{\frac{\nu_e}{\nu_e + 1} - \frac{1}{2}},$$

which tends to 2 when $\nu_e \to \infty$.

\[\square\]

**Lemma 3.2.2.** $\forall \rho > 1, \lim_{\nu_e \to \infty} PoA(\rho, \nu_e) \to \infty$.

**Proof.** When $\rho > 1$ and when $\nu_e \to \infty$, by (2.1)

$$n^* = \log_\rho \nu_e + o(\log_\rho \nu_e).$$

By (3.1)

$$PoA(\rho, \nu_e) \sim \frac{1 - \rho^{\log_\rho \nu_e}}{1 - \rho^{\log_\rho \nu_e + 1}} = \frac{1}{\nu_e} \left[ \frac{1 - \rho^{\log_\rho \nu_e}}{1 - \rho^{\log_\rho \nu_e + 1}} \right] - \frac{\frac{1}{1 - \rho} \left( \log_\rho \nu_e + 1 \right) \rho^{\log_\rho \nu_e}}{1 - \rho^{\log_\rho \nu_e + 1}}.$$

Since

$$\frac{1 - \rho^{\log_\rho \nu_e}}{1 - \rho^{\log_\rho \nu_e + 1}} = \frac{1}{\nu_e} \left[ \frac{1}{1 - \rho} - \frac{\log_\rho \nu_e + 1}{1 - \rho^{\log_\rho \nu_e + 1}} \right] = \frac{1 - \nu_e}{1 - \rho \nu_e} - \frac{1}{\nu_e} \left[ \frac{1}{1 - \rho} - \frac{\log_\rho \nu_e + 1}{1 - \rho \nu_e} \right] \sim \frac{1 - \nu_e}{1 - \rho \nu_e}$$

and

$$\frac{\frac{1}{1 - \rho} \left( \log_\rho \nu_e + 1 \right) \rho^{\log_\rho \nu_e}}{1 - \rho^{\log_\rho \nu_e + 1}} = \frac{\nu_e (1 - \rho) - (1 - \rho) \log_\rho \nu_e}{\nu_e (1 - \rho) \rho} = \frac{\nu_e (1 - \rho) - (1 - \rho) \log_\rho \nu_e}{\nu_e (1 - \rho) \rho} \leq \frac{1 - 2 \rho}{\nu_e (1 - \rho) \rho} \nu_e \to 0,$$

we conclude that

$$\lim_{\nu_e \to \infty} PoA(\rho, \nu_e) = \infty$$

\[\square\]
In contrast, \( \forall \rho < 1 \), the PoA is monotone decreasing to 1 when \( \nu_e \to \infty \):

**Lemma 3.2.3.** \( \forall \rho < 1 \), \( \lim_{\nu_e \to \infty} \operatorname{PoA}(\rho, \nu_e) = 1 \).

**Proof.** When \( \rho \leq 1 \), the solution of Equation (2.1) satisfies

\[
\lim_{\nu_e \to \infty} \frac{n^*}{\nu_e} = 1 - \rho.
\]

Substituting this relation into (3.1) gives:

\[
\lim_{\nu_e \to \infty} \operatorname{PoA}(\rho, \nu_e) = \frac{1 - \rho \nu_e (1 - \rho)}{1 - \rho \nu_e (1 - \rho) + 1} - \frac{1}{\nu_e} \left[ \frac{1}{1 - \rho} - \frac{(\nu_e(1 - \rho) + 1) \rho \nu_e (1 - \rho)}{1 - \rho \nu_e (1 - \rho) + 1} \right] \quad \rightarrow 1
\]

By Lemma 3.2.2, the PoA is unbounded when \( \rho > 1 \). By Lemma 3.2.3 and Lemma 3.2.1, the bound of the PoA cannot be achieved asymptotically \( \rho \leq 1 \) as the limit is finite. In the next section, we study this the bound of the PoA in this range (\( \rho \leq 1 \)). We will show that, except from a small range of \( \rho \) values near but below 1, the global maximum is achieved when \( \nu_e = 2 \).

### 3.3 PoA as a function of \( \rho \)

In this section we consider the upper envelope of the function \( \operatorname{PoA}(\rho, \nu_e) \) for \( \nu_e \geq 1 \):

\[
\operatorname{PoA}(\rho) = \sup_{\nu_e \geq 1} \operatorname{PoA}(\rho, \nu_e).
\]

**Lemma 3.3.1.** For any \( \rho > 1 \), \( \operatorname{PoA}(\rho) \) is unbounded.

**Proof.** By Lemma 3.2.2. \( \square \)

**Remark.** Under the limit case, when \( \rho \to \infty \), \( n_e \) remain the same and \( n^* = 1 \). An explanation to the latter is that social-optimizers don’t need to wait in the line in order to prevent idleness of the server in the future, because idleness never occurs. Also, when \( \rho \to \infty \), given any threshold \( n \), the number of customers in the system is almost always \( n \). This is because at any departure instant, an arrival occurs. Therefore, the PoA is

\[
\frac{R \mu - C}{R \mu - C n_e} = \frac{1 - \frac{1}{\nu_e}}{1 - \frac{[\nu_e]}{\nu_e}} = \frac{\nu_e - 1}{\nu_e - [\nu_e]} \quad \nu_e \to \infty \to \infty
\]

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Lemma 3.3.2. PoA(1) = 2.

Proof. By (3.3)

\[
PoA(1, \nu_e) \leq \frac{\sqrt{2\nu_e - 1} - \sqrt{2\nu_e - 1}}{\nu_e + 1} \leq \frac{(\sqrt{2\nu_e - 1})^2(\nu_e + 1)}{\nu_e(\nu_e - 1)}, \tag{3.3}
\]

For \( \nu_e = 1, 2, 3 \), the righthand side is less than 1.5, and for \( \nu_e > 2 + \sqrt{3} \) its derivative is positive. Therefore, \( PoA(1, \nu_e) \) is bounded by the limit when \( \nu_e \) goes to infinity, which is 2.

The following lemmas will be used in the proof of Theorem 3.3.5.

Lemma 3.3.3. For any \( \rho < 1 \), if \( PoA(\rho, x) \) is maximized at \( x = \nu_e \) then \( \nu_e \) is an integer.

Proof. We show that \( PoA \) is monotone decreasing with \( \nu_e \) in the range where \( n_e \) is fixed, i.e. \( \nu_e \in [n, n + 1) \). This range is divided to a finite number of intervals, such that in each interval \( n^* \) is also fixed. The \( PoA \) is continuous where \( n^* \) changes because at these values \( S_{n^*} = S_{n^*+1} \). Therefore it is sufficient to show that \( PoA \) is monotone decreasing with \( \nu_e \) in each of the intervals, where both \( n^* \) and \( n_e \) are fixed.

Consider two values \( \nu_e^1 < \nu_e^2 \) in such interval. The derivative of (3.2) with respect to \( \nu_e \) is

\[
PoA'(\rho, \nu_e) = \frac{q(n^*)p_{\text{join}}(n_e) - q(n_e)p_{\text{join}}(n^*)}{\nu_e^2 \left[ p_{\text{join}}(n_e) - \frac{1}{\nu_e} q(n_e) \right]^2}.
\]

which is negative when

\[
q(n^*)p_{\text{join}}(n_e) < q(n_e)p_{\text{join}}(n^*)
\]

or

\[
\left( \frac{1}{1 - \rho} - \frac{(n^* + 1)\rho^{n^*}}{1 - \rho^{n^*+1}} \right) \left( \frac{1 - \rho^{n_e}}{1 - \rho^{n_e+1}} \right) < \left( \frac{1}{1 - \rho} - \frac{(n_e + 1)\rho^{n_e}}{1 - \rho^{n_e+1}} \right) \left( \frac{1 - \rho^{n^*}}{1 - \rho^{n^*+1}} \right).
\]

This inequality holds when

\[
(1 - \rho^{n_e})(1 - \rho^{n^*+1}) - (n^* + 1)\rho^{n^*}(1 - \rho^{n_e})(1 - \rho) -
(1 - \rho^{n^*})(1 - \rho^{n_e+1}) + (n_e + 1)\rho_e^{n_e}(1 - \rho^{n^*})(1 - \rho) < 0.
\]

Simplifying the last expression, it is left to show that

\[
n_e\rho_e^{n_e}(1 - \rho^{n^*}) - n^*\rho^{n^*}(1 - \rho_e^{n_e}) < 0
\]
\[
\frac{\rho^{-n^*}(1 - \rho^{n^*})}{n^*} - \frac{\rho^{-n^e}(1 - \rho^{n^e})}{n^e} < 0,
\]
which is true since \(n^* < n^e\) and \(\rho^{-n}(1-\rho^n)\) is increasing with \(n\) when \(\rho < 1\). We conclude that \(\text{PoA}(\rho, \nu_e)\) is decreasing with \(\nu_e \in [n, n+1]\). In particular this means that the maximum value of \(\text{PoA}\) in this range is obtained at \(\nu_e = n\).

\[\square\]

**Lemma 3.3.4.** For \(\rho < 1\), \(\frac{1+(n+1)\rho^n-\rho^{n+1}(n+2)}{1-\rho^{n+1}}\) is a monotone decreasing function of \(n\).

**Proof.** We have

\[
\frac{1+(n+1)\rho^n-\rho^{n+1}(n+2)}{1-\rho^{n+1}} = 1 + \frac{(n+1)(1 - \rho)\rho^n}{1 - \rho^{n+1}} = 1 + \frac{n + 1}{\sum_{i=0}^{n} \rho^{-i}}.
\]

Thus, we want to show that \(\frac{n+1}{\sum_{i=0}^{n} \rho^{-i}}\) is decreasing, or alternatively \(\frac{\sum_{i=0}^{n} \rho^{-i}}{n+1}\) is increasing. As the sequence \(\rho^{-n}\) is increasing (because \(\rho < 1\)), so is the sequence of averages. \(\square\)

**Theorem 3.3.5.** \(\forall \rho < 1, \text{PoA}(\rho) < 2\).

**Proof.** By (3.1), we need to prove that \(\forall \rho < 1\) and \(\nu_e \geq 1\):

\[
\frac{1 - \rho^{n^*}}{1 - \rho^{n^*+1}} - \frac{1}{\nu_e} \left( \frac{1}{1 - \rho} - (n^* + 1) \frac{\rho^{n^*}}{1 - \rho^{n^*+1}} \right) 
\leq 2 \left[ \frac{1 - \rho^{[\nu_e]}}{1 - \rho^{[\nu_e]+1}} - \frac{1}{\nu_e} \left( \frac{1}{1 - \rho} - ([\nu_e] + 1) \frac{\rho^{[\nu_e]}}{1 - \rho^{[\nu_e]+1}} \right) \right].
\]

By Lemma 3.3.3 it is sufficient to consider \(\nu_e = [\nu_e]\). Since \(\nu_e\) is an integer and \(n^* < \nu_e\), it would be sufficient to prove that for any two integers \(n < m\):

\[
\frac{1 - \rho^{n}}{1 - \rho^{n+1}} - \frac{1}{m} \left( \frac{1}{1 - \rho} - (n + 1) \frac{\rho^{n}}{1 - \rho^{n+1}} \right) 
\leq 2 \left[ \frac{1 - \rho^{m}}{1 - \rho^{m+1}} - \frac{1}{m} \left( \frac{1}{1 - \rho} - (m + 1) \frac{\rho^{m}}{1 - \rho^{m+1}} \right) \right].
\]

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Since \( \frac{1-\rho^n}{1-\rho^{n+1}} \) is monotone increasing in \( n \), it is sufficient to prove

\[
\frac{1}{m} \frac{1}{1-\rho} + \frac{1}{m(n+1)} \frac{\rho^n}{1-\rho^{n+1}} \leq \frac{1-\rho^m}{1-\rho^{m+1}} + \frac{2}{m} \frac{(m+1)}{1-\rho^{m+1}}.
\]

Multiplying by \( m(1-\rho) \), it is sufficient to prove

\[
\frac{1+(n+1)\rho^n - \rho^{n+1}(n+2)}{1-\rho^{n+1}} \leq \frac{m + \rho^m(m+2)(1-\rho)}{1-\rho^{m+1}}.
\]

By Lemma 3.3.4, the left-hand side is monotone decreasing with \( n \). Therefore it is sufficient to show that the inequality holds for \( n = 1 \):

\[
\frac{1 + 2\rho - 3\rho^2}{1-\rho^2} < \frac{m + \rho^m(m+2)(1-\rho)}{1-\rho^{m+1}},
\]

or

\[
0 < (m-1) - 2\rho - (m-3)\rho^2 + (m+2)\rho^m - (m+1)\rho^{m+1} - m\rho^{m+2} + (m-1)\rho^{m+3}
\]

\[
= \left[ 0 < 2 + (m-3)(1+\rho) + (m+1)\rho^m + \rho^m(1+\rho) - (m-1)\rho^{m+2} \right] (1-\rho).
\]

Since \( \rho < 1 \), it is left to show that

\[
0 < 2 + (m-3)(1+\rho) + (m+1)\rho^m + \rho^m(1+\rho) - (m-1)\rho^{m+2}
\]

which is true by \( m > n \geq 2 \) and

\[
(m+1)\rho^m > (m-1)\rho^{m+2}.
\]

The behavior of \( \text{PoA}(\rho) \) in the range \( \rho < 1 \) is shown in Figure 3.4. One sees that except from a small range near \( \rho = 1 \), \( \text{PoA}(\rho) \) is bounded by 1.5. This boundary is much smaller than the boundary proved in Theorem 3.3.5. However, when \( \rho \) becomes close to 1 the boundary sharply increases to 2. To understand this oddness, we examine the behavior of the PoA for various values of \( \nu_e \).

Figure 3.5 demonstrates the behavior of the PoA as a function of \( n\nu_e \). When \( \rho < 0.98175 \), \( \text{PoA}(\rho) = \text{PoA}(\rho,2) \). In contrast, when \( \rho \in [0.98175,1] \), \( \text{PoA}(\rho) \) is not equal to a single function \( \text{PoA}(\rho,\nu_e) \). Instead, there is an infinite number of functions which define the upper envelope of the PoA in this range.

**Observation 3.3.6.** If \( \rho \in [0,0.98175] \), \( \text{PoA}(\rho,\nu_e) \leq \frac{1+\rho+\rho^2}{1+\rho} < 1.48635 \)

**Proof.** We observe that \( \text{PoA}(\rho) = \text{PoA}(\rho,2) \) when \( \rho < 0.98175 \). Substituting \( \nu_e = 2 \) in (3.1) we have

\[
\text{PoA}(\rho,\nu_e) \leq \frac{1+\rho+\rho^2}{1+\rho}.
\]

We also have that \( \text{PoA} \) is uniformly bounded by the maximum of \( \frac{1+\rho+\rho^2}{1+\rho} \) over \([0,0.98175]\) which is 1.48635. \( \square \)
3.4 Summery and conclusions

This study explores the PoA in the $M/M/1$ model:

- For $\rho \geq 1$, the PoA is monotone increasing with $\nu_e$. Accordingly, the upper bound of the PoA is achieved asymptotically when $\nu_e$ goes to $\infty$: For $\rho = 1$, the PoA increases to 2; $\forall \rho > 1$, the PoA increases to $\infty$.

- For $\rho < 1$, the PoA is monotone decreasing with $\nu_e$ and the upper bound for this range is not achieved asymptotically. The main result
of this paper is that when $\rho$ is in $[0, 0.98175]$, the bound is $\frac{1+\rho+\rho^2}{1+\rho}$, which is lower than 1.5. In most real situations $\rho$ falls into this range. One can conclude that the PoA in $M/M/1$ is often small comparing to other models discussed in the literature. When $\rho$ is in the small range between 0.98175 and 1, the PoA is bounded by 2. We prove that this bound is tight.

A further study could assess the PoA in other queuing systems, in which the self-optimization by individual customers does not optimize public good. For example, it would be interesting to explore the PoA in $GI/M/1$, where the arrival process is a general one, and in $GI/M/s$, where there are $s > 1$ parallel servers. Like the $M/M/1$, these systems also have a pure threshold strategy for the Nash equilibrium solution [87, 88].
Chapter 4

Continuous time model for complex network evolution

we suggest a novel approach to model network evolution. The evolution is represented by the dynamics of independent Markov chains, each represents an element of the network.

4.1 The model

We list the assumptions underlying the model:

- $N$ Markov chains are identical and independent.
- Each chain has two states: $M$ and $L$.
- The duration times in $M$ and $L$ are independent, identically and exponentially distributed with parameters $\mu$ and $\lambda$, respectively.

The evolution is a series of changing graphs, based on the dynamics of the chains. Evolution starts with $G = (V, E)$, where $V = 1, 2, \ldots, N$ and $E = \phi$. Each node is associated with a chain. Chains move between state $M$ and state $L$. $M$ is a meeting state: if at any given time two chains are both in $M$, we say that the chains meet. When two chains, say chain $i$ and chain $j$, meet for the first time, we add an undirected edge $(i, j)$ to $G$.

The model parameters can be normalized, so there are only two relevant parameters: $N$ which is the number of chains, and $\rho = \frac{\lambda}{\mu}$, which is the ratio of the expected times spent in $M$ and $L$. The model is an extension of the random graph model: when $\rho$ goes to zero, edges are randomly added one by one. On the other hand, when $\rho$ is high, there is a high probability that the corresponding evolving network is a series of cliques with growing sizes.
4.2 Analysis of degree distribution

We derive a closed formula for the expected time until the degree of an arbitrary node in the evolving network is $h$. One distinct chain plays the role of a leader. All other $N$ chains are non-leaders. The meetings of the leader represent the edges incident to a specific node in the corresponding evolving network. We analyze these meetings. Specifically, we derive the expected time until the leader has met $h$ non-leaders.

4.2.1 Recursion

Let $S_{i,m,l}$ denote a state of the system, where $i$ equals 1 or 0 to indicate whether the leader is in state $M$ or $L$, respectively; $m \in 0, \ldots, N - 1$ is the number of non-leaders in state $M$ which are not yet acquainted with the leader; $l \in 0, \ldots, N - m$ is the number of non-leaders in state $L$ which are not yet acquainted with the leader.

Let $M_l$ denote the expected time until the system goes from state $S_{1,0,l}$ to state $S_{1,0,0}$. Let $L_{m,l}$ denote the expected time until the system goes from state $S_{0,m,l}$ to state $S_{1,0,0}$. $M_l$ and $L_{m,l}$ are the expected times until the leader has been in state $M$ with all of the non-leaders (at least once). We compute $M_N$ and $L_{0,N}$ as a function of $\mu$, $\lambda$ and $N$.

When $l = 0$, $M_l = 0$. For any $l > 0$, the first transition takes the system from $S_{1,0,l}$ to $S_{1,0,l-1}$ or $S_{0,0,l}$, depending on whether it is a transition of the leader or one of the non-leaders, respectively. Hence, for $l = 1, \ldots, N$:

$$M_l = \frac{1}{\mu + l \lambda} + \frac{l \lambda}{\mu + l \lambda} M_{l-1} + \frac{\mu}{\mu + l \lambda} L_{0,l}. \quad (4.1)$$

When $l = 0$, $L_{m,l} = 0$, for all $m$ values. For any $l > 0$, the first transition takes the system from $S_{0,m,l}$ to $S_{0,m+1,l-1}$, $S_{0,m-1,l+1}$ or $S_{1,l}$, depending on the type of the transiting chain and the direction of this transition. Hence, for $m = 0, \ldots, N - l$ and $l = 1, \ldots, N - m$:

$$L_{m,l} = \frac{1}{m \mu + (l + 1) \lambda} + \frac{l \lambda}{m \mu + (l + 1) \lambda} L_{m+1,l-1} + \frac{m \mu}{m \mu + (l + 1) \lambda} L_{m-1,l+1} + \frac{\lambda}{m \mu + (l + 1) \lambda} M_l. \quad (4.2)$$

$L_{-1,l+1}$ is not defined. However, its coefficient is 0.

4.2.2 Recursion for an embedded process

Solving the recursion system 4.1 and 4.2 is not straightforward, since $L_{m,l}$ is not induced by lower values of $m$ and $l$. To derive a closed solution, we
embed the system in states $S_{1,0,l}$ and $S_{0,0,l}$, where the non-leaders who are not acquainted with the leader are in state $L$. Let $L_l = L_{0,l}$. When the system is in state $S_{0,0,l}$ and $l > 0$, the first transition of the leader takes the system to $S_{1,0,r}$, for some $0 \leq r \leq l$, depending on the number of non-leaders, that have not yet met the leader and are in state $M$ at the time of this transition. Let $T$ denote the time of the first transition of the leader. For $l = 1, \ldots, N$, (4.2) is replaced by:

$$L_l = \int_{T=0}^{\infty} \left[ T + \sum_{r=0}^{l} [P_{l,r}(T)M_r] \right] \lambda e^{-\lambda T} dT$$

$$= \frac{1}{\chi} + \int_{T=0}^{\infty} \left[ \sum_{r=1}^{l} [P_{l,r}(T)M_r] \right] \lambda e^{-\lambda T} dT;$$

where $P_{l,r}(T)$ denotes the probability that $X = r$, where $X$ is the number of chains in state $L$ at time $T$, if initially there are $l$ chains and all of them are in state $L$. $X$ has been studied by Enns [29]. Enns’ model is equivalent to the generalization of Uppuluri [81] for the Ehrenfest model [43]. As a conclusion from Enns study, $X$ is distributed according to binomial distribution $B(l, u)$, where $u$, the probability that a chain is in state $L$ at time $T$, given that it was in state $L$ at time 0, satisfies:

$$u = \frac{\mu + \lambda e^{-(\lambda+\mu)T}}{\lambda + \mu}.$$ 

Therefore,

$$L_l = \frac{1}{\chi} + \int_{T=0}^{\infty} \left[ \sum_{r=1}^{l} P\left( B\left( l, \frac{\mu + \lambda e^{-(\lambda+\mu)T}}{\lambda + \mu} \right) = r \right) M_r \right] \lambda e^{-\lambda T} dT$$

$$= \frac{1}{\chi} + \sum_{r=1}^{l} \left[ M_r \int_{T=0}^{\infty} P\left( B\left( l, \frac{\mu + \lambda e^{-(\lambda+\mu)T}}{\lambda + \mu} \right) = r \right) \lambda e^{-\lambda T} dT \right]$$

$$= \frac{1}{\chi} + \sum_{r=1}^{l} M_r \left( \frac{l}{r} \right) \left( \frac{1}{1 + \rho} \right)^l f(r,l), \quad (4.3)$$
where

\[
 f(r, l) = \frac{1}{\mu^l} \int_{T=0}^{\infty} (\mu + \lambda e^{-(\lambda+\mu)T})^r (\lambda(1 - e^{-(\lambda+\mu)T}))^{l-r} \lambda e^{-\lambda T} dT
\]

\[
 = \rho^{l-r} G\left[-r, \frac{\lambda}{\mu + \lambda}; 1 - r + l + \frac{\lambda}{\lambda + \mu}; \frac{-\lambda}{\mu}\right]
\]

\[
 \times G\left[\frac{\lambda}{\lambda + \mu}, r - l; \frac{2\lambda + \mu}{\lambda + \mu}; 1\right],
\]

\[
 = \rho^{l-r} G\left[-r, \frac{\rho}{1 + \rho}; 1 - r + l + \frac{\rho}{\rho + 1}; -\rho\right]
\]

\[
 \times G\left[\frac{\rho}{\rho + 1}, r - l; \frac{2\rho + 1}{\rho + 1}; 1\right],
\]

(4.4)

where \( G \) is the Gauss’s hypergeometric function:

\[
 G[a, b; c, z] = \sum_{l=0}^{\infty} \frac{(a)_l(b)_l z^l}{(c)_l l!},
\]

and \((a)_k\) is the Pochhammer symbol of the rising factorial:

\[
 (a)_k = \begin{cases} 
 a(a+1) \cdots (a+k-1) & k \geq 1 \\
 1 & k = 0.
\end{cases}
\]

Substituting (4.3) in (4.1), \( M_l \) can be computed recursively, for \( l = 1, \ldots, N \) as a function of all \( M_a, a = 0, \ldots, l - 1 \). Substituting \( M_0 = 0 \), we get:

\[
 M_l = \chi(l) + \sum_{a=1}^{l-1} \xi(r, l)M_a,
\]

(4.5)

where

\[
 \lambda \chi(l) = \frac{1 + \rho}{1 + l \rho - \left(\frac{1}{1 + \rho}\right)^l f(a, l)},
\]

\[
 \xi(a, l) = \frac{\binom{l}{a} \left(\frac{1}{1 + \rho}\right)^l f(a, l) + \delta(a, l) l \rho}{1 + l \rho - \left(\frac{1}{1 + \rho}\right)^l f(l, l)},
\]

and \( \delta(a, l) \) is the Kronecker function:

\[
 \delta(a, l) = \begin{cases} 
 1 & a = l - 1 \\
 0 & \text{otherwise.}
\end{cases}
\]
4.2.3 Expected time until a node is connected with all others

Consider a directed acyclic weighted graph $H = (V, E)$, with $V = \{1, \ldots, N\}$ and $E = \{(i, j)|i < j\}$. Denote the weight of edge $(i, j)$ by $w(i, j)$, and the weight of vertex $i$ by $v(i)$.

For $1 \leq k \leq n \leq N$, let $P_{k,n}$ denote the set of directed pathes from $k$ to $n$ in $H$. Define

$$\Omega(k, n) = \left\{ \sum_{p \in P_{k,n}} \prod_{(i,j) \in p} w(i, j) \right\}_{1 \leq k < n, k = n}. \tag{4.6}$$

Notice that the last edge in a path from $k$ to $n$ is $(q, n)$ for some $k \leq q < n$. Therefore, for any $k$ and $n$,

$$\Omega(k, n) = \sum_{p \in P_{k,n}} \prod_{(i,j) \in p} w(i, j) = \sum_{q=k}^{n-1} w(q, n) \Omega(k, q). \tag{4.7}$$

Consider the function $q(n)$, defined recursively by $q(1) = v(1)$ and for $n = 2, \ldots, N$

$$q(n) = v(n) + \sum_{s=1}^{n-1} w(s, n) q(s). \tag{4.8}$$

**Lemma 4.2.1.** $q(n) = \sum_{k=1}^{n} v(k) \Omega(k, n)$, for $n = 1, 2, \ldots, N$.

**Proof.** We verify that the claim is true for $n = 1$. By the definitions of $q(n)$ and $\Omega(k, n)$, $q(1) = v(1)$ and $\Omega(1, 1) = 1$. Therefore,

$$q(1) = v(1) = v(1) \Omega(1, 1) = \sum_{k=1}^{1} v(k) \Omega(k, 1).$$

We assume that the claim is true for all $n < n_1$, for some $n_1 \leq N$. We prove

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it for \( n_1 \). From (4.7) and (4.8):

\[
q(n_1) = v(n_1) + \sum_{s=1}^{n_1-1} w(s, n_1)q(s)
\]

\[
= v(n_1) + \sum_{s=1}^{n_1-1} w(s, n_1) \sum_{k=1}^{s} v(k)\Omega(k, s)
\]

\[
= v(n_1) + \sum_{k=1}^{n_1-1} v(k) \left[ \sum_{s=k}^{n_1-1} w(s, n_1)\Omega(k, s) \right]
\]

\[
= v(n_1) + \sum_{k=1}^{n_1-1} v(k)\Omega(k, n_1)
\]

\[
= v(n_1)\Omega(n_1, n_1) + \sum_{k=1}^{n_1-1} v(k)\Omega(k, n_1)
\]

\[
= \sum_{k=1}^{n_1} v(k)\Omega(k, n_1).
\]

\[\square\]

We return to our problem and calculate \( M_N \) and \( \lambda L_N \) as functions of \( \rho \) and \( N \). We consider an acyclic weighted graph \( H = (V, E) \), in which the weights of the edges are \( \xi(i, j) \) and the weights of the vertices are \( \chi(i) \). Let \( \tilde{\Omega} \) be defined by substituting \( w(i, j) = \xi(i, j) \) in (4.6):

\[
\tilde{\Omega}(k, n) = \left\{ \sum_{p \in P_{k,n}} \prod_{(i,j) \in p} \xi(i, j) \right\}
\]

\[1 \leq k < n, \quad 1 \quad k = n.\]

**Theorem 4.2.2.** \( M_N = \sum_{k=1}^{N} \chi(k)\tilde{\Omega}(k, N) \).

**Proof.** By (4.5), (4.7) and Lemma 4.2.1. \[\square\]

**Theorem 4.2.3.** \( \lambda L_N = 1+\sum_{r=0}^{N} \left[ \left( \begin{array}{c} N \\ r \end{array} \right) \left( \frac{1}{\rho+1} \right)^N f(r, N) \sum_{k=1}^{r} \chi(k)\tilde{\Omega}(k, r) \right] \).

**Proof.** By (4.3):

\[
\lambda L_N = 1+\sum_{r=0}^{N} \left( \begin{array}{c} N \\ r \end{array} \right) \left( \frac{1}{\rho+1} \right)^N f(r, N)M_r,
\]

and using Theorem 4.2.2,

\[
\lambda L_N = 1+\sum_{r=0}^{N} \left[ \left( \begin{array}{c} N \\ r \end{array} \right) \left( \frac{1}{\rho+1} \right)^N f(r, N) \sum_{k=1}^{r} \chi(k)\tilde{\Omega}(k, r) \right],
\]
where $f(r, N)$ is given in (4.4).

Figure 1 shows $\lambda L_N$ as a function of $N$ for various $\mu$ values. Functions are monotonously increasing as it takes more time to become acquainted with a larger population. The functions are also concave. It is a result of the fact that, adding a fixed number of non-leaders $\Delta N$ has a stronger effect when the original number of chains is small. Intuitively, $L_N$ increases as a result of adding chains, only because the leader may meet some of the new $\Delta N$ chains after he meets all the original $N$ chains. When $N$ is higher, the period to infect $N$ chains becomes longer and less of the new $\Delta N$ chains are left to the end of the process. The inset shows the same plots using a horizontal log scale. The affine relation of $L_N$ and $\log(N)$, graphically shows that $L_N$ can be well-approximated by a linear function of $\log(l)$.

Figure 4.1: Expected time to meet all chains. The solution for $\lambda L_N$ is shown as a function of the size of the network $N$, for different $\rho$ values. The inset shows the same plot, using a horizontal log scale.
4.2.4 Expected time until a node has a certain degree

We extend the formula for $M_N$ and $L_N$, to compute the expected time until the leader meets a subset of non-leaders. These subsets represent the neighbors of the node corresponding to the leader. The growing size of these subsets represents the degree evolution of the same node.

Let $M^h_l$ denote the expected time until the system goes from state $S_{1,0,l}$ to any state $S_{1,0,z}$ where $z \leq N - h$. Let $L^h_{m,l}$ denote the expected time until the system goes to the same states from $S_{0,m,l}$. $M^h_l$ and $L^h_{m,l}$ are the expected time until the leader is acquainted with at least $h$ out of the $N$ non-leaders. We compute $M^h_l$ and $L^h_{m,l}$ as a function of $\lambda$, $\mu$, $N$ and $h$.

By definition, $M^h_l$ equals 0 for all $l \leq N - h$. The development of $M^h_l$ is similar to the development $M_l$. The only difference lies at the range of index $r$ in (4.3), which now starts at $N - h + 1$ to compute $L^h_l$. Substituting $L^h_l$ to (4.1), we get the following modification of (4.5):

$$M^h_l = \chi(l) + \sum_{a=N-h+1}^{l-1} \xi(r,l)M^h_a.$$  

A corresponding modification of index $a$ in Theorem 4.2.2 gives:

$$M^h_l = \sum_{k=N-h+1}^{l} [\chi(k)\Omega(k,l)].$$  

The solution for $M^h_l$ is a generalization of the solution for $M_l$, as $M^N_l = M_l$.

Let $L^h_l$ denote $L^N_{0,l}$. $L^N_l = 0$ for $m + l \leq N - h$, otherwise it is achieved by substituting (4.4) and (4.10) in (4.3):

$$L^h_l = \frac{1}{\lambda} + \sum_{r=0}^{l} \left( \binom{N}{r} \left( \frac{1}{\lambda + \mu} \right)^N f(r,N) \sum_{k=N-h+1}^{l} [\chi(k)\Omega(k,l)] \right).$$

4.3 Network features during the evolution

We use numerical simulations to investigate the structural features of the evolving network. We start a simulation from a state where all the Markov chains are in $L$. We continue with an iterative process of acquiring the next transition of any chain. We update the states of the system and at the same time we update the link propagation in the evolving network. Specifically, in every epoch when chain $i$ transits into state $M$, we examine the chains in $M$. Some of the them, say chains $j_1, j_2, \ldots$, represent nodes which are not
yet connected to \( i \) in the evolving network. We connect these nodes to \( i \) by edges \((i, j_1), (i, j_2)\,...\). The process terminates when the network becomes a clique with \( N \) nodes.

We investigate the network evolution in dependency with the parameters of the model. For each choice of \( \rho \) and \( N \), the results are averaged over ten runs of simulations. We recognize interesting features in our model: small average shortest path (section 4.1), high clustering (section 4.2) and highly skewed degree distribution (section 4.3).

Following the format of previous studies, we demonstrate the evolution of the main structures versus \( p \) - the proportion of edges. Furthermore, we demonstrate the same structures versus \( \lambda t \) - the time, normalized by the expected time a chain spends in state \( L \). Each feature is demonstrated in a graph, for a specific case when \( N = 200 \), for different values of \( \rho \). We find special interest when \( \rho \) goes to zero, or when \( \rho \) is high enough to imply a series of growing cliques.

### 4.3.1 Small average shortest path

Two nodes are connected if the network contains a path between them. For any couple of connected nodes, the shortest path is the minimum number of edges to traverse from one node to another. Let \( l \) denote the average of all the shortest pathes in a graph. For a connected graph, \( l \) characterizes the spread of the graph. For a disconnected graph, \( l \) is defined as the average of the average lengths of shortest pathes of its components.

Figure 2 demonstrates the evolution of \( l \) in our model.
Both Random Graphs [15, 22, 23, 17, 18, 45, 56] and our model demonstrate low values of \( l \), provided \( p \) is not too small. It is in agreement with real world networks, like the World Wide Web [5] and the co-occurrences of words [31].

Moreover, when \( p \) is small (see zoom-in figure), the average value of \( l \) in our model is bounded from above by \( \ln N \), which is the average shortest path of a random graph with the same size (dashed line). The reason for this is that when \( p \) is small, \( l \) decreases with \( \rho \) as a chain meets smaller sets of chains when it moves to \( M \). In the limit case, edges are added one by one, like in the evolution of Random Graphs.

### 4.3.2 High clustering

Inspired by a concept in sociology, named “fraction of transitive triples” [83], Watts and Strogats quantified the tendency to cluster by the *clustering coefficient* [85]. The clustering coefficient of a node quantifies how close the node and its neighbors are to being a clique. For each selected node \( i \), the nodes which are connected to it are examined. These are called the *neighbors*
of node $i$. Having $k_i$ neighbors, let $e_i$ be the number of edges that exist between pairs of neighbors of node $i$. The clustering coefficient of node $i$ is defined as $Cl_i = \frac{2e_i}{k_i(k_i-1)}$. In words, $Cl_i$ gives the proportion of ‘triangles’ that go through node $i$, whereas $\frac{k_i(k_i-1)}{2}$ is the total number of triangles that could pass through node $i$, should all of its neighbors be connected to each other. The clustering coefficient of a network is the average of its local coefficients: $Cl = \sum_{i=1}^{n} \frac{Cl_i}{n}$. Figure 3 demonstrates $Cl$ in our model.

![Figure 4.3: Clustering coefficient during the evolution. (A) The evolution of $Cl$ versus $p$, for different $\rho$ values. (B) The evolution of $Cl$ versus continuous normalized time, for the same $\rho$ values.](image)

In general, $Cl$ increases from 0 to 1 as the network evolves to a clique. Let as look at Figure 3(A). When $\rho \to 0$, $Cl$ goes to a limit function which is $Cl = p$ as for a random graph with the same size (shown as a dashed line). Otherwise, $Cl$ is higher than $p$. Figure 3(B) shows the evolution $Cl$ versus time. The clustering in most, if not all, real networks is much higher than the clustering coefficient of a random graph. The clustering coefficient of the
internet, for example, ranges between 0.18 and 0.3 [89], in comparison with \( \sim 0.001 \) for random networks with similar parameters. Together with the result in the previous section, our model gives a satisfactory result: exhibition of a short average shortest path along with a clustering coefficient which is higher than that of a random graph. Our model represents the small world criteria better than a random graph.

### 4.3.3 Highly skewed degree distribution

A degree of a node is the number of its neighbors. Many real networks, the degree distribution is typically right-skewed with a “heavy” tail, meaning that a small fraction of the nodes are many times better connected than the average. These nodes are called *hubs*.

Figure 6 demonstrates some samples of degree distributions in our model for different \( \rho \) value for the same \( p \).
Figure 4.4: Histograms of degree distribution, for $p = 0.12$, for different $\rho$ values, in comparison with the distribution of a random graph (red dashed line). (A) when $\rho$ is low, the distribution is well approximated by a Poisson distribution like in random graphs. (B) for intermediate value of $\rho$, the distribution is highly skewed. (C) when $\rho$ is high, the evolution is a series of cliques and there is a single degree value.

When $\rho$ is low (upper graph), the distribution is well approximated by a Poisson distribution with the same mean. When $\rho$ is high (lower graph), the network is a clique and the connected nodes have the same degree.

In random graphs, the number of nodes with degree $k$ follows a Poisson distribution, of which the variance is small (dashed lines in all graph). The high skewness is not captured by random graphs. In contrast, it is captured in our model, for intermediate values of $\rho$ (middle graph).
4.4 Another application of the model - spread of a disease

Let $S_{i,m,h}$ denote the states of the system, where $i$ equals 1 or 0 to indicate whether there is at least one sick chain in state $M$; $m = 0, \ldots, N$ is the total number of chains in state $M$; and $h$ is the number of healthy chains in state $L$.

4.4.1 Expected time until $N$ healthy elements become sick

We compute $|T|$, the expected time until all $N$ non-leaders are acquainted with the leader.

Recursion

Let $C_{m,h}$ denote the expected time until the system goes from state $S_{1,m,h}$ to any state $S_{1,m',0}$, for some $m' > 0$. Let $B_{m,h}$ denote the expected time until the system goes from state $S_{0,m,h}$ to any state $S_{1,m',0}$, for some $m' > 0$.

By definition, $C_{m,0} = B_{m,0} = 0$.

For $m > 0, h > 0$:

$$C_{m,h} = \frac{1}{m\mu + (N - m)\lambda} + \frac{m\mu}{m\mu + (N - m)\lambda}[(1 - \hat{\delta}(m))C_{m-1,h} + \hat{\delta}(m)B_{0,h}]$$

$$+ \frac{(N - m - h)\lambda}{m\mu + (N - m)\lambda}C_{m+1,h} + \frac{h\lambda}{m\mu + (N - m)\lambda}C_{m+1,h-1},$$

where $\hat{\delta}(x)$ is the Kronecker function:

$$\hat{\delta}(x) = \begin{cases} 1 & x = 1 \\ 0 & \text{otherwise} \end{cases}$$

For $m \geq 0, h > 0$:

$$B_{m,h} = \frac{1}{m\mu + (N - m)\lambda} + \frac{m\mu}{m\mu + (N - m)\lambda}B_{m-1,h+1}$$

$$+ \frac{(N - m - h)\lambda}{m\mu + (N - m)\lambda}C_{m+1,h} + \frac{h\lambda}{m\mu + (N - m)\lambda}B_{m+1,h-1}.$$
Recursion for the embedded process

We embed the system in states $S_{0,1,0}$, $S_{1,m,0}$, and such that there is no more than one healthy chain in state $M$. Let $C_{m,h}$ denote the expected time until the system goes from state $S_{1,m,h}$ to any state $S_{1,m',0}$, for some $m' > 0$. Let $B_h$ denote the expected time until the system goes from state $S_{0,1,h}$ to any state $S_{1,m',0}$, for some $m' > 0$.

For $h > 0$ and $m \geq 1$:

$$C_{m,h} = \int_{T=0}^{\infty} \left[ T + P_{N-m,h,m,0}(T)B_{h-1} + \sum_{m'=1}^{N-m-h} [P_{N-m,h,m,m'}(T)C_{m',h-1}] \right] h\lambda e^{-h\lambda T} dT$$

(4.10)

where $P_{n,m,m'}(T)$ is the probability that $m'$ chains are in state $M$ at time $T$, if there are $n$ chains in total and initially $m$ of them were in state $M$.

Again we follow Enns [29]. Enns shows that

$$P_{n,m,m'}(T) = a^{m'} b^{n-m'-m} d^n \sum_{j=0}^{m} \binom{n-m}{m-j} \binom{m}{j} (bc)^{n-j}$$

(4.11)

where

$$a = \frac{\lambda(1 - e^{-(\lambda+\mu)T})}{\lambda + \mu}$$
$$b = \frac{\mu + \lambda e^{-(\lambda+\mu)T}}{\lambda + \mu}$$
$$c = \frac{\lambda + \mu e^{-(\lambda+\mu)T}}{\lambda + \mu}$$
$$d = \frac{\mu(1 - e^{-(\lambda+\mu)T})}{\lambda + \mu}.$$  (4.12)

When the system is in state $S_{0,1,h}$ and $h > 0$, the first transition of a sick chain from $L$ to $M$ takes the system to $S_{1,h-h',2,h'}$. Hence, for $h > 0$:

$$B_h = \int_{T=0}^{\infty} \left[ T + \sum_{h'=1}^{h+1} [P_{h+1,h',h,h'}(T)C_{h-h',2,h'}] \right] (N - h - 1)\lambda e^{-(N-h-1)\lambda T} dT,$$

where $P_{h+1,h',h}$ is computed according to (4.11) and (4.11). Substituting
(4.11) to (4.10), we get:

\[
C_{m,h} = \frac{1}{\lambda h} + \frac{N-m-h}{N-h} \left[ C_{m',h-1} \int_{T=0}^{\infty} P_{N,m,m'}(T) h \lambda e^{-h\lambda T} dT \right]
\]

\[+ \frac{1}{N-h} \int_{T=0}^{\infty} P_{N,m,0}(T) h \lambda e^{-h\lambda T} dT
\]

\[+ \sum_{h'=0}^{h} \left[ h(N-h) \lambda^2 \int_{T=0}^{\infty} P_{N,m,0}(T) e^{-h\lambda T} dT \int_{\tilde{T}=0}^{\infty} P_{h,h-1,h'}(\tilde{T}) e^{-(N-h)\lambda \tilde{T}} d\tilde{T}
\]

\[C_{h-h'+1,h'} \right]

One can notice that, \( C_{m,h} \) is a function of \( C_{1,h} \) and \( C_{m,h'} \), for \( h' < h \):

\[C_{m,h} = \epsilon(m, h) + \zeta(m, h) C_{1,h} + \sum_{m'=1}^{N-h} \eta(m,m',h) C_{m',h-1} + \sum_{h'=0}^{h-2} \theta(m,h) C_{h-h'+1,h'}, \]

where

\[\epsilon(m,h) = \frac{1}{\lambda h} + \frac{1}{N-h} \int_{T=0}^{\infty} P_{N,m,0}(T) h \lambda e^{-h\lambda T} dT, \]

\[\zeta(m,h) = h(N-h) \lambda^2 \int_{T=0}^{\infty} P_{N,m,0}(T) e^{-h\lambda T} dT \int_{\tilde{T}=0}^{\infty} P_{h,h-1,h'}(\tilde{T}) e^{-(N-h)\lambda \tilde{T}} d\tilde{T}, \]

\[\eta(m,m',h) = \int_{T=0}^{\infty} P_{N,m,m'}(T) h \lambda e^{-h\lambda T} dT + \tilde{\delta}(m') \zeta(m,h), \]

where \( \tilde{\delta}(m) \) is the Kronecker function:

\[\tilde{\delta}(m) = \begin{cases} 
1 & m = 2 \\
0 & \text{otherwise}
\end{cases} \]

and

\[\theta(m,h,h') = h(N-h) \lambda^2 \int_{T=0}^{\infty} P_{N,m,0}(T) e^{-h\lambda T} dT \int_{\tilde{T}=0}^{\infty} P_{h,h-1,h'}(\tilde{T}) e^{-(N-h)\lambda \tilde{T}} d\tilde{T}. \]

Therefore, \( C_{m,h} \) can be achieved numerically by computing \( C_{1,1} \), then \( C_{1,1} \) for all \( M > 1 \), then \( C_{1,2} \) and \( C_{i,2} \) for all \( M > 1 \), and so forth. For any \( N > 0 \), \( B_N \) is achieved by 4.13 as a function of \( C_{m,h} \) values.
4.5 Summery and conclusions

We suggest modeling network evolution according to the dynamics of Markov chains. We explore a simple version of such modeling. The model assumes a finite number of independent Markov chains with two states, one of which is a meeting state and a meeting there imply an addition of corresponding edge to the evolving network. Some properties of the model can be analyzed by the theory of stochastic process. For example, we formulate a closed formula for the expected time until a node has a certain degree.

Recall the questions in the introduction section:

- Can **identical** and **consistent** behavior of elements produce a network with complex topology?
  Yes. Our model demonstrates that complex features in real world networks are not necessarily produced by diverse behavior of the elements. Instead, interactions of identical elements, whose behavioral rules are not influenced by the dynamic structure of the network, may also imply familiar features like: high clustering, small diameter and a skewed degree distribution.

- **What is the right way to describe a network evolution over continuous time?**
  Our model gives an example for mechanism that relates the the duration of processes in a network with the underlying processes of its elements. The underlying processes in our model are the transitions of the Markov chains. These processes can be measured in real time units because they are associated with the changes of an element in any real network. For example, people move among feasible states: home, work, pub, etc, making new friends in these places. For another example, neurons may be spiking or non spiking, while getting connected through synapses, following a principle which is often summarized as “fire together, wire together” [38]. The spiking state of the neuron resembles the meeting state in our model.

As for further study, the model could be examined under more general assumptions: having any number of states for each Markov chain (instead of two), assuming any Poisson rate of meetings in each state (instead of either 0 or $\infty$ ) and considering a diversity of transition rates (instead of identical rates for all the chains).

Also, there is room for natural extension of the model: to allow an elimination of the edges; to distinguish between the chain which enters the meeting state and the chain which is already there (directed graph); or to allow multiplication of edges (weighted graph).
Beside network evolution, the model is a breeding ground for many interesting processes, like: A spread of a rumor (or a disease); Equilibrium behavior, when a target function is defined for the elements; Competitions between leaders.
Bibliography


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אוקטובר 2011