STOCHASTIC MODEL OF
A PENSION PLAN

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by

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Abstract

Structuring a viable pension plan is a problem that arises in the study of financial contracts pricing and bears special importance these days. A recently proposed Ten Pillar Program [1] for a deterministic pension model is based on projections that are based on several assumptions concerning the "average" long-time behavior of the stock market. The aim of the present dissertation is to examine the proposed plan in a more realistic setting of a stochastic model. A widely held belief is that investment in the stock market is similar to gambling in a casino, while purchasing companies, after due diligence, is safer. The Ten Pillar Program subscribes to this belief and differentiates itself from most pension plans by acting as a holding company that wholly owns other companies, thus avoiding some of the stock market risks. In this dissertation, we show that the stock market index faithfully reflects its companies’ profits at the time of their publication. We compare the shifted historical dynamics of the S&P500’s aggregated financial earnings to its value, and find a high degree of correlation. We conclude that there is no benefits for a pension fund to wholly own a super trust. We verify, by examining historical data, that stock earnings follow an exponential (geometric) Brownian motion and estimate its parameters. This stochastic model is adopted for the Ten Pillar Pension model. The robustness of this model is examined by an estimate of a pensioner’s accumulated assets over a saving period. We also estimate the survival probability and mean survival time of the accumulated individual fund with pension consumption over the residual life of the pensioner.
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Introduction

What is a Pension?

A pension plan is a method for a prospective retiree to transfer part of his or her current income stream toward a retirement income. Pension plans are usually classified into two categories.

1. Defined-benefit plan - the pension fund, (e.g., employer) guarantees that the pensioner will receive a fixed, predefined, benefits upon retirement, regardless of the investment’s performance.

2. Defined-contribution plan - the pension fund makes predefined contributions, usually tax exempt, toward a pool of funds, set aside for the pension fund’s future benefit. The pool of funds is then invested on the retiree’s behalf allowing the her/him to receive benefits upon retirement. The final benefit received by the retiree depends on the investment’s performance.

The benefits are paid upon the pensioner’s retirement, usually in a lump sum. However, some countries, such as the UK, members are legally required to purchase an annuity, which then provides a regular income.

Pensions have a long history in Western civilization. The notion of pension dates back to the Roman Empire [2], where rulers and parliaments provided pensions for their workers, who helped perpetuate their regimes. More than two thousand years ago, the fall of the Roman republic and the rise of the empire were inextricably linked to the payment, or rather the nonpayment, of military pensions. The first private pension was established in 1875 by the American Express Company in the United States [3]. Prior to 1870, private-sector plans did not exist, primarily because most companies were small family-run enterprises.

Public and Private Pension Funds

A public pension fund is one that is regulated under public sector law, while a private pension fund is regulated under private sector law. In certain countries the distinction between public or government pension funds and private pension funds may be difficult to assess. In others,
the distinction is made sharply in law, with very specific requirements for administration and investment. For example, local governmental bodies in the United States are subject to laws passed by the states, in which those localities exist and these laws include provisions, such as defining classes of permitted investments and a minimum municipal obligation [4].

The Pension Crisis

The obligation of a fixed, predefined amount of benefits upon retirement exposes the insurer to a great risk. The calculation of the the benefits amount is based on financial assumptions that are hard to measure or predict. These assumptions include lifespan of employees, returns earned by pension investments, future taxes, and rare events, such as natural disasters.

On the other hand, defined-contribution plans transfer the risk to the insured, who is dependent on the pension fund performance upon his/her retirement date. An individual that retired in 2009 received significantly less than he/she would have in 2011. The following data paints a grim picture of UK and US pension plans.

In the US, there is a $1 trillion gap at the end of the fiscal year 2008 between the $2.35 trillion that US states had to set aside to pay for their employees’ retirement benefits and the $3.35 trillion price tag of those promises [5]. The present value of unfunded obligations under Social Security as of August 2010 was approximately $5.4 trillion [6]. Moreover, US state and local pension plans exhibit a structural shortfall, that will likely pose a long-enduring problem, according to the US Congressional Budge Office [7]. In the UK, many employees face retirement with an income well short of their expectations. Employees who pay into a Defined-contribution plan for 40 years, may get only half of the retirement income they could have expected. [8]. According to the International Monetary Fund [9], Western economies would have to set aside an additional 50% of their 2010 GDP to support the retirees. Several reforms have been suggested to amend the pension crisis.

Reforms

Reform proposals can be classified into three.

1. To meet pre-existing defined benefit obligations, the retirement age should be raised.

2. To mitigate risk and reduce obligations, there should be a shift from defined-benefit to defined-contribution pension plans.

3. To improve accumulated wealth, there should be an increase in resource allocation to fund pensions by increasing contribution rates and taxes.

The first reform does not exhibit any structural solution, but rather tries to put out a fire. The second reform still contains the risks of defined-contribution plans, and the third reform involves raising taxes, which potentially reduces the reward of work and therefore the incentive to work.
Chapter 1

The Ten Pillars Program

The Ten Pillars Program is a retirement saving program, proposed in [1]. A concise description of the core elements of the program are as follows.

1.1 Overview

1. Government Grant at Birth: The Special Levy -
   This pillar is an hypothecated taxation on the GDP, dedicated to fund the investments and operations of the Super Trusts (see 9 below). This tax will be collected once a year, in which a new Super Trust will be established and become the tax recipient.

2. Family and Friends Gifting at Birth -
   It is proposed that the birth of a child should become a rallying point, at which time family and friends can try to help by making a worthwhile financial contribution towards the build-up of the child’s eventual pension benefits.

3. Family and Friends Gifting throughout Life -
   This pillar promotes the idea of gifting money by way of Pension Vouchers. These vouchers will be invested in the individual’s retirement account

4. Compulsory Minimum Pension Contributions: The Insured -
   As in Defined-contribution plan, the insured will make predefined contributions, usually tax exempt, toward the retirement account.

5. Compulsory Minimum Pension Contributions: The Employer -
   As in Defined-contribution plan, the employer will make predefined contributions, usually tax exempt, toward the employee’s retirement account.

6. Government First Job Pension Subsidy -
   The government will boost the first job contributions of both employer and the employee. The rationale for this is to provide significant boost to the overall pension accumulation of each individual at the earliest possible age.
7. Windfalls: The Individual contributing throughout life -  
   A certain portion of income sources that stem from hobbies, unexpected gifts, substantial bonuses, cash prize winnings, inheritance, selling of family assets and tax refunds, should be contributed to the individual’s retirement account.

8. The Family Pension Trust: Sharing Prosperity -  
   Governments will allow tax-free flow of funds between retirement accounts of the same family. The rationale is that older family members, might have sufficient pension assets in their pension account, and are willing to contribute towards their loved one’s retirements.

9. The Super Trusts -  
   Government-owned investment funds (see the next section.)

10. MAXILIFE -  
   An internet software and concept that could help transform a nation into a learning society; that is a society which is made up of individuals who, by using this software, never stop learning and acquiring new and productive work and life-related skills. The rationale is that an attitude-shift regarding savings is required.

1.2 The Super Trusts

Every year a new trust is established and seeded with The Special Levy grant. The Super Trusts will employ low-risk investment strategies in the form of purchasing stakes in stable, substantial, and profitable companies. The continuous influx of net earnings will be reinvested by purchasing more companies. As a retirement saving fund, the Super Trusts will not be concerned about short-term liquidity provisions, and will focus on long-term investments, making it less vulnerable to market fluctuations. The Super Trusts will promote a mind set of long term investments over short term “fireworks”. Upon its members’ retirements, the fund will make monthly pension payments. Hopefully, the funds accumulated over the saving period will suffice to support all insured members.

1.2.1 The Super Trusts investment approach

The Super Trusts aim to achieve stable and steady long-term growth of its assets. Therefore, it facilitates corporate governance mechanism that promotes thriftiness and prudence. For example, to save billions of pension savings, the Super Trusts will hand moderate management remuneration packages. Furthermore, the Super Trusts believe that its investment policies achieve low-volatility, low-risk, steady growth. These policies include 100% stake purchases of companies that produce basic products or commodities, with an underlying economic substance, and a high, stable demand. The Super Trusts refrain from investments that carry no fundamental value. Investments that are considered risky by the Super Trusts are stocks, bonds, currencies, arbitrage-trading, futures, options, and all forms of derivatives.
The motivation for this approach, is based on a long list of historical financial crises. To name but a few:

1. Black Monday (1987) - Dow Jones Industrial Average dropped 22.61% in one day [10].


3. Russian Financial Crisis (1998) - Several factors, such as artificially high fixed exchange rate and chronic fiscal deficit led the Russian government to devalue the Ruble, default on domestic debt, and declare a moratorium on payment to foreign creditors [12]. As a result, inflation reached 84% that year. Banks closed down. Millions of people lost their life savings. As a direct consequence, US Hedge funds collapsed, including Long Term Capital Management (LTCM), which received a $3.6B bailout [13], under the supervision of the Federal Reserve.

4. The Dot-com bubble (2000) - The NASDAQ Composite lost 78% of its value. $5 trillion loss in the market value of companies.

5. The Subprime Mortgage Crisis (2008) - Americans lost more than a quarter of their net worth. Housing prices dropped, GDP began contracting, unemployment rate rose from 5% to 10%. S&P500 fell 57% from its October 2007 peak. US total national debt rose from 66% GDP pre-crisis to over 103% post-crisis [14].

In addition, the rise of algorithmic trading, systematic trading, high frequency trading, and hedge funds gave birth to new type of stock market crashes - computer code crashes. For example, the 2010 Flash Crash [15], and the Knight Capital Group crash [16] in 2012, are results of crashing of computers running complex algorithms.

These crises led the Super Trusts to seek growth in the net income instead of stock market returns.

1.2.2 Suggested investment strategy

The Ten Pillar Program does not provide a constructive investment methodology, or any rule-based portfolio structure and rather a general approach is depicted. In order to construct a mathematical model of the proposed pension program, we suggest a constructive investment strategy that aligns with the Super Trusts’ financial views and investment approach. A representative strategy suggested for the Super Trusts is the S&P500 index methodology [17]. The Super Trust will purchase a company if it meets the following S&P500 eligibility criteria:

1. Market value of more than $4.6B.
2. Annual dollar value traded is greater than its market value in the 6-months period prior to inclusion.

3. At least 250,000 of its shares are traded each month in the 6-months period prior to inclusion.

4. It is a US company.

5. At least 50% of the company’s shares are offered to the public

6. At least 4 consecutive quarters of positive earnings prior to inclusion.

7. Hasn’t been initially offered to the public (IPO) for the past 6-12 months.

A Super Trust will sell its entire stake in a company if one of the following holds.

1. It is involved in a Merger & Acquisition (M&A) that causes at least 1 violation of the above eligibility criteria.

2. The company is violating at least 1 of the above eligibility criteria on an ongoing basis.

This investment methodology is aligned with the investment approach of the Super Trusts, because companies that meet the above requirements are profitable by definition; They exhibit 4 consecutive quarters of positive earnings and moreover, do not exhibit negative earnings on an ongoing basis. Furthermore, these companies are highly liquid and are worth more than $4.6B. In addition, the historical fact that 97% of removals from S&P500 are due to M&As and that the average time a company stays in the index is 16.7 years, strengthens the notion that these companies are generally profitable and stable. Consequently, these characteristics qualify them as companies with substantial economic substance, suitable for investments by the Super Trusts.

1.2.3 The investment performance

The Super Trusts are the beneficiaries of the companies’ net profit and it is up to the management’s discretion to determine the amount of the net profit retained by its underlying companies and the amount accumulated into the Super Trust’s pension fund. In order to be able to gauge the Super Trust performance, we assume that none of the portfolio’s net profit is retained by the underlying companies and that the Super Trusts accumulate the entire portfolio’s net earnings. Therefore, the growth on the Super Trust net income, a time $t$, relative to initial time $t_0$ i given by

$$ R(t) = \frac{\sum_{i=1}^{500} N I_i(t)}{\sum_{i=1}^{500} N I_i(t_{io})}, $$

where $NI_i(t)$ is the net income of the $i$-th company in S&P500 at time $t$ and $t_{io}$ is the time it was first included in the index.
1.2.4 A refinement of the investment strategy

The Efficient-Market Hypothesis (EMH) was introduced in [18] by Eugene Fama. The hypothesis states that it is impossible to "beat the market," because stock market efficiency causes existing share prices to always incorporate and reflect all relevant information. According to the EMH, stocks always trade at their fair value on stock exchanges, making it impossible for investors to either purchase undervalued stocks or sell stocks for inflated prices. Therefore, it is reasonable to assume that the collective price of a market as a whole, as represented by S&P, for example, should incorporate and reflect all information about its constituents, including their earnings performance at the time of their publication. We assert this argument by comparing two S&P indices historical returns against their shifted net income earnings. The financial statement’s publication dates, being hard to obtain, are approximated by adding 3 months to the quarter of which the financial statements refer to. For example, if a company reported net income for the 2nd quarter, we assign September 30th as the publication date. The period of 3 months shift was chosen, because US companies are required by law to publish their quarterly financial reports by the end of the subsequent quarter. We obtain financial statements and price data from the CRSP/COMPUSTAT merged database [19], and plot the historical monthly performance of their shifted CPI-adjusted net income growth, $R(t)$, from 1970 through 2011. We also plot a 0.6% window simple moving average to highlight their trend. To emphasize the high correlation of their dynamics, we plot CPI-adjusted returns performance against the shifted trend of the net earnings curve. The market representing indices chosen are S&P50 and S&P500. See figures 4.1,4.2 (pp.46). In addition, we calculate the Pearson correlation coefficient between the shifted earning (raw, not smoothed) and the return performance. The Pearson coefficient is given by

$$\rho(X,Y) = \frac{\text{Cov}(X,Y)}{\sigma_X \sigma_Y} = \frac{\mathbb{E}[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}.$$ 

The results are

<table>
<thead>
<tr>
<th>Index</th>
<th>$\rho$ coefficient</th>
</tr>
</thead>
<tbody>
<tr>
<td>S&amp;P50</td>
<td>0.89</td>
</tr>
<tr>
<td>S&amp;P500</td>
<td>0.92</td>
</tr>
</tbody>
</table>

Table 1.1: Correlations between the shifted CPI-adjusted earnings performance and CPI-adjusted price performance of market-representative US indices.

Therefore, in contrast to the widely held belief that investment in the stock market is similar to gambling in a casino, while purchasing companies is safe, we conclude that the stock market index faithfully reflects the companies’ profits at the time of their publication, thus strengthening the Efficient Market Hypothesis. Moreover, based on analysis of historical data, stock prices perform better, while being just as safe.

In view of the above, we refine the suggested investment strategy to purchase the shares of S&P500 companies instead of a 100% stake in them. For the purpose of mathematical analysis, pillars 1,2,3,6,7, and 8 can be amortized and incorporated into the individual’s initial salary, hence yielding the mathematically equivalent pension model: a defined-contribution
plan, together with an augmented initial influx, invested in the S&P500 stock market index.

1.3 Economic Considerations

In this chapter we discuss several economic aspects of the Ten Pillars Program. We begin by considering the establishment of the Super Trusts, their centralization impact, and their incentive mechanism. We continue to the discussion of the Special Levy and the difficulty of stimulating the economy with government tax-based spending.

1.3.1 The Super Trusts

The Super-Trusts are government-owned financial institutions, whose goal is to accumulate wealth by the process of acquiring private-sector companies. Therefore, these government-owned entities are subject to the criticism of a centrally planned economy, who one of its notable critics is Ludwig von Mises. von Mises introduced this theory in his 1920 paper, "Economic Calculation in the Socialist Commonwealth" [20]. His theory is based on the notion that the free market relies on the price mechanism, wherein people individually have the ability to decide how resources should be distributed based on their willingness to give money for specific goods or services. The price conveys embedded information about the abundance of resources as well as their desirability, which in turn allows, on the basis of individual consensual decisions, corrections that prevent shortages and surpluses. According to von Mises, in the case where government owns the means of production, no prices could be obtained for capital goods as they were merely internal transfers of goods in the same system, and not "objects of exchange", unlike final goods. The total value of the assets under the Super Trusts' management is economically significant, as it involves an entire country’s pension contributions, resulting in a control of a significant share of the private sector. This implies that as the Super Trusts grow big, it is likely that a portion of their sales will become a transfer of goods in the same internal system, thus distorting the price mechanism. Theoretically, the Super Trusts can grow to a size large enough to the point where it becomes cumbersome to continue planning, similar to the Soviet economy in the early 1990s. Being too large to manage by state planners, the Super Trusts might fail to respond in a timely manner to continuous changes in the economy.

We turn to considerations of the incentive mechanism in the Ten Pillar economy. A Super Trust employs a moderate remuneration package to its management, while expecting to recruit to its lines the best portfolio managers. It is questionable why talented professionals would choose to work for the Super Trusts, while they can get better compensation elsewhere. Furthermore, existing managements of bought companies will not receive significant bonuses and/or big paychecks anymore, because they are now subject to the Super Trust’s thrifty policy. This could lead to a reduced incentive on their part to outperform and grow.

In the presence of a public policy, where government money is periodically injected into the private sector, opportunists and exploiters might plague the economy. Private companies
are not required by law to publish financial reports. They can misrepresent and manipulate their performance in order to qualify as the next Super Trusts purchase, making a short-term profit without conveying any substantial economic value. Moreover, the Super Trusts management can become corrupt, or be replaced by corrupt people, therefore preventing the goal of the Ten Pillar Program from being realized. According to the historian and politician Lord Acton, "Power tends to corrupt, and absolute power corrupts absolutely. Great men are almost always bad men." Milton Friedman argued that reducing private economic activity could grant politicians coercive powers [21].

1.3.2 The Special Levy

The Special Levy is a form of government spending. Government spending can affect long-term economic growth, both up and down. Economic growth is based on the growth of labor productivity and labor supply, which can be affected by how governments directly and indirectly influence the use of an economy’s resources. However, increasing the economy’s productivity rate often requires the application of new technology and resources. The Super Trusts’ goal is to stimulate the economy and create new jobs, therefore it needs to improve labor productivity and labor supply. Funded by tax money, the Super Trusts investments consist in merely redistributing purchasing power between various groups of people. For example, many lawmakers claim that every $1 billion in highway stimulus can create 47,576 new construction jobs [22]. But the government must first tax or borrow $1 billion from the private economy, which will then lose at least as many jobs. Highway spending simply transfers jobs and income from one part of the economy to another. According to the Heritage Foundation economist Ronald Utt [23], the only way that $1 billion of new highway spending can create 47,576 new jobs is that $1 billion appears out of nowhere as if it were manna from heaven. It is not obvious that purchasing 100% stake in private companies can stimulate economic growth, as its effect on the development of new technological breakthroughs is unclear.

Furthermore, the transfer of resources from the more productive private sector to the less productive public sector can reduce long-term productivity. According to McKinsey [24], the public sector is the largest employer in all advanced economies, yet its slow productivity growth has long made it a drag on the economy.

However, investing in higher education can grow labor productivity and labor supply. A more educated work-force earns higher salaries, resulting in higher government tax revenue, make fewer social services demand, such as welfare, and most importantly, contribute to the development of new technological breakthroughs. A study of Imperial College Business School [25] states that in the UK, publicly funded research generates a return of 1185% annually. It also suggests that the benefits of research spending in higher education are greater than those from other areas of government-supported R&D.
Chapter 2

A Stochastic Model and its Analysis

In this chapter, we present a stochastic mathematical model for the Ten Pillars Program. We formulate probabilistic questions about the robustness and soundness of the plan, and answer by deriving the Fokker-Planck equation for the joint probability density function of the pension fund and salary growths, presenting numerically convergent schemes, and implementing them with computer simulations.

We obtain the pension model phase by phase. First, we present a stochastic model for the growth of a stock-market portfolio, such as the S&P500 index. Second, we present a stochastic model for the salaries growth, and finally, we combine the former to obtain a stochastic model for the growth of the pension fund, which is dependent on the initial conditions of the pensioner, namely, his initial salary. We are interested in the probabilities of the benefits payable to the individual. To this end, we write the corresponding Fokker-Planck equation, which we solve by a numerical scheme. We analyze its computational complexity, and perform an optimization by using the Generalized Minimal Residue Method (GMRES). We summarize the results in tables, based on the model’s dimensionless parameters. We present 3D plots of the computer program’s output, which is an approximation to the solution of the FPE for the transition probability density function of the pension’s growth process.

In addition, we define a stochastic consumption process, describing the individual’s rate of resource consumption. We assume a constant dollar amount rate of a pensioner’s yearly expense, and calculate the survival probability of the pension consumption process, i.e the probability that there will still be pension money left after a given number of years. We also calculate the mean first passage time (MFPT) of the consumption process to 0, which is the expected time that the pension money will run out. Finally, we assume that the pensioner’s life span is randomly distributed, according to a certain density function, and calculate the probability that the pension plan survives the pensioner, that is, the probability that the pensioner will die before consuming all his pension money.
2.1 The Diffusion Model

When modelling asset prices, factors such as the Efficient Market Hypothesis and randomness are incorporated in the form of Markov processes. Markov processes of significant importance are diffusion and jump diffusion processes, which are the common practice of asset-pricing in mathematical finance [26]. Therefore, we model the salaries and the market’s fluctuations with stochastic diffusion processes, whose coefficients will be corroborated against historical data. We use historical price data for the stock market index growth model and historical annual wage data for the salaries model. We use diffusion approximations to the discrete-time processes.

A diffusion process is a continuous-time Markov process $x(t)$ with almost surely continuous trajectories satisfies the following conditions,

$$
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E} \left\{ x(t + \Delta t) - x(t) \mid x(t) = x \right\} = a(x, t)
$$

$$
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E} \left\{ (x(t + \Delta t) - x(t))^2 \mid x(t) = x \right\} = b^2(x, t)
$$

$$
\lim_{\Delta t \to 0} \frac{1}{\Delta t} \mathbb{E} \left\{ (x(t + \Delta t) - x(t))^{2+\delta} \mid x(t) = x \right\} = 0, \quad \text{for some } \delta > 0.
$$

We consider diffusion models that are solutions of Itô stochastic differential equations (SDE) of the form

$$
dx(t) = a(x(t), t) \, dt + b(x(t), t) \, dw(t),
$$

where $w$ is standard Brownian motion. We approximate the continuous trajectories by solutions to the discrete Euler approximation scheme for (2.2) with drift coefficient $a(x, t)$ and diffusion coefficient $b(x, t)$. Historical data are used to estimate $a(x, t)$ and $b(x, t)$ from the empirical trajectories of the model.

2.1.1 Model simplifications

As mentioned above, the coefficients $a(x, t)$ and $b(x, t)$ are obtained from sample averaging of historical data in (2.1) and fitting interpolated functions. Instead of approximating the Itô coefficients by interpolation, we could have assumed that $a(x, t)$ and $b(x, t)$ are stochastic themselves, and are governed by the stochastic equations of their own,

$$
da(x, t) = a_1(x, t) \, dt + a_2(x, t) \, dw_1(t)
$$

$$
rb(x, t) = b_1(x, t) \, dt + b_2(x, t) \, dw_2(t),
$$

where $w_1$ and $w_2$ are standard Brownian motions, thereby increasing the problem’s number of dimensions. Such practices have been implemented in the field of stochastic volatility modelling like the Heston model, SABR model, and Chen model [27], [28], [29]. However, these models describe the dynamics of price movements in relatively short time periods, ranging from microseconds to a couple of weeks. For our purposes, when long-term dynamics are considered, a diffusion model with deterministic coefficients seems to be sufficient.
2.2 Stochastic Preliminaries

The following summary is taken from [30].

2.2.1 Itô’s formula

Consider the \( n \) Itô-differentiable processes

\[
dx_i = a_i \, dt + \sum_{j=1}^{m} b_{ij} \, dw_j, \quad i = 1, 2, \ldots, n
\]  

(2.3)

where \( a_i, b_{ij} \in H_2[0, T] \),

\[
H_2[0, T] = \left\{ f \in \mathcal{F}_t \left| \int_0^T \mathbb{E}f^2(s, w) \, ds < \infty \right. \right\},
\]

and \( \mathcal{F}_t \) is Brownian filtration. \( w_j \) are independent Brownian motions. Let \( f(x_1, x_2, \ldots, x_n, t) \) be a function that has continuous partial derivatives of second order in \( x_1, x_2, \ldots, x_n \) and a continuous partial derivative with respect to \( t \). Assume \( a_i(t) \) and \( b_{ij} \) are continuous functions and that \( f(x, t) \) is twice continuously differentiable function such that \( |f_{xx}(x, t)| \leq A(t) e^{\alpha(t)|x|} \) for some positive continuous function \( A(t) \) and \( \alpha(t) \). Then

\[
df(x(t), t) = \left[ \frac{\partial f(x(t), t)}{\partial t} + L^* f(x(t), t) \right] \, dt + \sum_{i=1}^{n} \sum_{j=1}^{m} b_{ij}(t) \frac{\partial f(x(t), t)}{\partial x_i} \, dw_j,
\]

where

\[
L^* f(x, t) = \sum_{i=1}^{n} \sum_{j=1}^{n} \sigma_{ij} \frac{\partial^2 f(x, t)}{\partial x_i \partial x_j} + \sum_{i=1}^{n} a_i(t) \frac{\partial f(x, t)}{\partial x_i}
\]

and

\[
\sigma_{ij} = \frac{1}{2} \sum_{k=1}^{m} b_{ik}(t) b_{kj}(t).
\]  

(2.4)

The \( n \times n \) matrix \( \{\sigma_{ij}(t)\} \) is called the diffusion matrix. In matrix notation,

\[
B(t) = \{b_{ij}(t)\}_{n \times m}
\]

is the noise matrix and the diffusion matrix \( \sigma(t) \) is given by

\[
\sigma(t) = \frac{1}{2} B(t) B^T(t).
\]
2.2.2 The lognormal SDE

The Itô equation of the exponential (geometric) Brownian motions has the form [30]

\[
dx(t) = a(t)x(t)\,dt + b(t)x(t)\,dw(t)
\]
\[x(0) = x_0. \tag{2.5}\]

It is solved by applying Itô’s formula to \(y(t) = \log(x(t))\), which gives

\[
dy(t) = \left[ a(t) - \frac{1}{2}b^2(t) \right] dt + b(t)\,dw(t),
\]

which has the integral form

\[
y(t) = y(0) + \int_0^t \left[ a(s) - \frac{1}{2}b^2(s) \right] \,ds + \int_0^t b(s)\,dw(s).
\]

Hence

\[
x(t) = x_0 \exp \left\{ \int_0^t \left[ a(s) - \frac{1}{2}b^2(s) \right] \,ds + \int_0^t b(s)\,dw(s) \right\}. \tag{2.6}
\]

2.2.3 Moments of lognormal diffusions

Using Itô’s formula, we compute the moments \(m_k(t) = \mathbb{E}x^k(t)\) for all \(k > 0\), where \(x(t)\) is the stochastic process governed by (2.5),

\[
\mathbb{E}x^k(t) = x_0^k \exp \left\{ k \int_0^t \left[ a(s) - \frac{1}{2}b^2(s) \right] \,ds \right\} \mathbb{E} \exp \left\{ k \int_0^t b(s)\,dw(s) \right\}. \tag{2.7}
\]

We begin by applying Itô’s formula to \(f(y(t)) = e^{y(t)}\), where \(y(t)\) is the solution of the SDE

\[
dy(t) = kb(t)\,dw(t), \quad y(0) = 0, \tag{2.8}
\]

to obtain

\[
df(y(t)) = \frac{1}{2}k^2b^2(t)f(y(t))\,dt + kb(t)f(y(t))\,dw(t)
\]
\[f(y(0)) = 1. \tag{2.9}\]

By taking expectation on both sides of the equation, we get the ODE

\[
d\mathbb{E}f(y(t)) = \frac{1}{2}k^2b^2(t)\mathbb{E}f(y(t))\,dt, \quad \mathbb{E}f(y(0)) = 1, \tag{2.10}
\]
whose solution is given by
\[
Ef(y(t)) = \mathbb{E}\exp \left\{ \int_0^t kb(s) dw_i(s) \right\} = \exp \left\{ \frac{k^2}{2} \int_0^t b^2(s) ds \right\}. \tag{2.11}
\]

Therefore
\[
m_k(t) = x_0^k \exp \left\{ k \int_0^t \left[ a(s) - \frac{1}{2} b^2(s) \right] ds \right\} \mathbb{E}f(y(t))
\]
\[
= x_0^k \exp \left\{ k \int_0^t a(s) ds + \left( \frac{k^2 - k}{2} \right) \int_0^t b^2(s) ds \right\}. \tag{2.12}
\]

In particular,
\[
\mathbb{E}x(t) = x_0 \exp \left\{ \int_0^t a(s) ds \right\}
\]
\[
\text{Var}[x(t)] = x_0^2 \exp \left\{ 2 \int_0^t a(s) ds + \int_0^t b^2(s) ds \right\} - x_0^2 \exp \left\{ 2 \int_0^t a(s) ds \right\}
\]
\[
= x_0^2 \exp \left\{ 2 \int_0^t a(s) ds \right\} \left( \exp \left\{ \int_0^t b^2(s) ds \right\} - 1 \right). \tag{2.13}
\]

Alternatively, we compute the moments of \(x(t)\) by observing that \(x(t)\) has the lognormal distribution; \(x(t) \sim LN(\mu, \sigma^2)\), where
\[
\mu = \log(x_0) + \int_0^t \left[ a(s) - \frac{1}{2} b^2(s) \right] ds, \quad \sigma^2 = \int_0^t b^2(s) ds
\]
and therefore has the moments
\[
\mathbb{E}[x(t)] = e^{\mu + \frac{1}{2} \sigma^2}
\]
\[
= x_0 \exp \left\{ \int_0^t a(s) ds + \frac{1}{2} \int_0^t b^2(s) ds \right\} = x_0 \exp \left\{ \int_0^t a(s) ds \right\}
\]
\[
\text{Var}[x(t)] = \left( e^{\sigma^2} - 1 \right) e^{2\mu + \sigma^2}
\]
\[
= x_0^2 \exp \left\{ 2 \int_0^t a(s) ds \right\} \left( \exp \left\{ \int_0^t b^2(s) ds \right\} - 1 \right). \tag{2.14}
\]
2.2.4 Nonhomogeneous linear SDE

Consider the nonhomogeneous linear SDE
\[
\frac{dX(t)}{dt} = [a_1(t)X(t) + a_2(t)]\, dt + [b_1(t)X(t) + b_2(t)]\, dW(t)
\]
\[X(t_0) = X_0,
\]
and the homogeneous linear SDE
\[
\frac{dH(t)}{dt} = a_1(t)H(t)\, dt + b_1(t)H(t)\, dW(t)
\]
\[H(t_0) = X_0.
\]

The solution to the homogeneous SDE (2.16), is given by (2.6)
\[
H(t) = X_0 \exp \left\{ \int_{t_0}^{t} \left[ a_1(s) - \frac{1}{2} b_1^2(s) \right] \, ds + \int_{t_0}^{t} b_1(s)\, dW(s) \right\}.
\]

To obtain the general solution, we apply Itô’s lemma to \( f(X(t), H(t), t) = X(t)/H(t) \), where \( X(t), H(t) \) satisfy the SDEs (2.15). The derivatives of \( f \) are given by
\[
\frac{\partial f}{\partial t} = 0
\]
\[
\frac{\partial f}{\partial X} = \frac{1}{H}; \quad \frac{\partial f}{\partial H} = -\frac{X}{H^2}
\]
\[
\frac{\partial^2 f}{\partial X^2} = 0 ; \quad \frac{\partial^2 f}{\partial H\partial X} = \frac{\partial^2 f}{\partial X\partial H} = -\frac{1}{H^2}; \quad \frac{\partial^2 f}{\partial H^2} = \frac{2X}{H^3}
\]
and
\[
\sigma^{12} = \sigma^{21} = \frac{1}{2} b_1 H(b_1 X + b_2) ; \quad \sigma^{22} = \frac{1}{2} b_1^2 H^2.
\]

Therefore we get from (2.4)
\[
df (X(t), H(t), t) = \left[ -\frac{1}{2} b_1 H (b_1 X + b_2) \frac{2}{H^2} + \frac{1}{2} b_1^2 H^2 \frac{2X}{H^3} + (a_1 X + a_2) \frac{1}{H} - a_1 H \frac{X}{H^2} \right] \, dt
\]
\[
+ \left[ (b_1 X + b_2) \frac{1}{H} - (b_1 H) \frac{X}{H^2} \right] \, dW(t).
\]

Thus,
\[
df (X(t), H(t), t) = \left( \frac{a_2(t) - b_1(t)b_2(t)}{H(t)} \right) \, dt + \frac{b_2(t)}{H(t)} \, dW(t),
\]
and written in the integral form
\[
\frac{X(t)}{H(t)} = \frac{X(t_0)}{H(t_0)} + \int_{t_0}^{t} \left( \frac{a_2(s) - b_1(s)b_2(s)}{H(s)} \right) \, ds + \int_{t_0}^{t} \frac{b_2(s)}{H(s)} \, dW(s),
\]
we finally get
\[
X(t) = H(t) \left[ 1 + \int_{t_0}^{t} \left( \frac{a_2(s) - b_1(s)b_2(s)}{H(s)} \right) \, ds + \int_{t_0}^{t} \frac{b_2(s)}{H(s)} \, dW(s) \right].
\]
2.3 Stochastic Model for Stock Returns

The S&P500 index returns has only one trajectory, so in order to construct a long-term diffusion model for the S&P500 index returns, we represent them as a weighted average of the underlying stocks returns. Therefore, we begin with modeling the dynamics and fluctuations of the CPI-adjusted returns of the S&P500 constituent stocks. We assume that the S&P500 stocks’ returns $x_i(t)$ are the outputs of the SDE

$$dx(t) = a(x(t), t) \, dt + b(x(t), t) \, dw(t), \quad x(t_0) = 1.$$  

Equivalently, $x_i(t)$ can be considered outputs of the identical and independent SDEs

$$dx_i(t) = a(x_i(t), t) \, dt + b(x_i(t), t) \, dw_i(t), \quad x_i(t_0) = 1 \text{ for } i = 1, 2, \ldots, (2.19)$$

where $w_i(t)$ are independent standard Brownian motions.

2.3.1 Discrete approximation scheme for the diffusion coefficients

We denote by $S(\tau, x)$ the set of all stock returns in the composition of S&P500 at the end of month $\tau$, whose price had multiplied $x$ times relative to their index inclusion price. If a stock was included in S&P500 prior to $\tau$ more than once, the last inclusion date is taken. The trajectory of the stock return process attributable to the $j$-th stock is denoted by $x_j(t)$.

The continuous trajectories of (2.19), discounted by the CPI index, are considered to be approximations to the discrete monthly CPI-adjusted return vectors. Thus the drift and diffusion coefficients of the stock returns are approximated with

$$a(x, \tau) = \frac{1}{|S(\tau, x)|} \sum_{s \in S(\tau, x)} [x_s(\tau + 1) - x_s(\tau)]$$

$$b^2(x, \tau) = \frac{1}{|S(\tau, x)|} \sum_{s \in S(\tau, x)} [x_s(\tau + 1) - x_s(\tau)]^2. (2.20)$$

2.3.2 Drift and diffusion surface interpolation

We use the CSRP/COMPSTAT® merged database for historical monthly stock prices and historical S&P500 compositions. We use the US Bureau of Labor Statistics for the historical Consumer Price Index (CPI) values. We compute (2.20) for $\tau$ between January 1970 and December 2011, to obtain the drifts and volatility surfaces $G_a, G_b : \mathbb{R}^2 \to \mathbb{R}$, where

$$G_a = \{(x, t, a(x, t))\}$$

$$G_b = \{(x, t, b^2(x, t))\}.$$
We interpolate these surfaces by projecting them onto the $t$-axis, to obtain $G_a[t], \ G_b[t] : \mathbb{R} \to \mathbb{R}$, where

$$G_a[t] = \{(x, a(x, t))\}, \ \text{for} \ 1970 \leq t \leq 2011,$$

$$G_b[t] = \{(x, b^2(x, t))\}, \ \text{for} \ 1970 \leq t \leq 2011.$$

We interpolate the 2-D curves $G_a[t]$, and $G_b[t]$ with a linear function $\tilde{G}_a[t]$ and a quadratic polynomial $\tilde{G}_b[t]$, for every $t$. Finally, we assemble back the 2-D interpolators to obtain the interpolated surfaces $\tilde{G}_a, \tilde{G}_b$, where

$$\tilde{G}_a = \left\{(x, t, \tilde{G}_a[t](x)) \in \mathbb{R}^3 \mid 1970 \leq t \leq 2011\right\},$$

$$\tilde{G}_b = \left\{(x, t, \tilde{G}_b[t](x)) \in \mathbb{R}^3 \mid 1970 \leq t \leq 2011\right\}.$$

### 2.3.3 Numerical results

We write the interpolating functions $\tilde{G}_a[t], \tilde{G}_b[t]$ in the form

$$a(x, t) = \tilde{G}_a[t](x) = q(t)x + q_2(t)$$

$$b^2(x, t) = \tilde{G}_b[t](x) = r(t)x^2 + r_2(t)x + r_3(t),$$

and look for $q(t), q_2(t), r(t), r_2(t), r_3(t) \in \mathbb{R}$ that minimize the residuals in the least square sense

$$\sum_x \left[\tilde{G}_a[t] - G_a[t]\right]^2, \ \sum_x \left[\tilde{G}_b[t] - G_b[t]\right]^2, \ \text{for every} \ 1970 \leq t \leq 2011.$$

We obtain an approximation for the minimizing coefficients by applying the MATLAB "polyfit()" function for every $t$. The interpolating functions, $\tilde{G}_a[t], \tilde{G}_b[t]$, plotted against the projections $G_a[t], \ G_b[t]$ are given in figures 4.3-4.8 (p.47-52). In figure 4.9, $q(t)$ is plotted for $1970 \leq t \leq 2011$. For simplicity (see 2.1.1), we approximate the function $q(t)$ with a simple moving average filter, i.e every point equals the average of the $N$ preceding points. The resulting approximation is the constant

$$q(t) \equiv 0.002742,$$

$q_2(t)$ is plotted for $1970 \leq t \leq 2011$. $q_2(t)$ under a simple moving average filter results in the constant

$$q_2(t) \equiv 0.$$

In figure 4.10, $r(t)$ is plotted for $1970 \leq t \leq 2011$. $r(t)$ under a simple moving average filter results in the constant

$$r(t) \equiv 0.01,$$

$r_2(t)$ is plotted for $1970 \leq t \leq 2011$. $r_2(t)$ under a simple moving average filter results in the constant

$$r_2(t) \equiv 0,$$
$r_3(t)$ is plotted for $1970 \leq t \leq 2011$. $r_3(t)$ under a simple moving average filter results in the constant

$$r_3(t) \equiv 0.$$  

Substituting $q, q_2, r, r_2, r_3$ into (2.19), we get

$$dx_i(t) = qx_i(t)\, dt + rx_i(t)\, dw_i(t)$$

$$x_i(t_0) = x_0.$$  

(2.22)

### 2.3.4 Change of time scale

The stochastic process, $x_i(t)$, has time units of months. We wish to change to time scale to years, to match the time scale of the yearly income data. The transformation

$$w_2(t) = cw(t/c^2)$$  

(2.23)

preserves the Brownian motion [30], where $c$ is any positive constant. Using (2.6), the solution of (2.22) is given by

$$x_i(t) = x_0 \exp \left\{ \left( q - \frac{1}{2} r^2 \right) t + r w_i(t) \right\}.$$  

(2.24)

together with (2.23) where $c = 1/\sqrt{12}$, we get $x_i(t)$ scaled to years

$$x_i(t) = x_0 \exp \left\{ 12 \left( q - \frac{1}{2} r^2 \right) t + r \sqrt{12} w_i(t) \right\}.$$  

(2.25)

and the SDE of $x_i(t)$, scaled to years, is given by

$$dx_i(t) = \psi x_i(t)\, dt + \phi x_i(t)\, dw_i(t)$$

$$x_i(t_0) = x_0,$$

(2.26)

where

$$\psi = 12q = 0.0329, \quad \phi = \sqrt{12} r = 0.3464.$$  

### 2.4 Stochastic Model for Stock Market Index Returns

We next seek to identify the stochastic dynamics of the stock market index returns. The index return is determined by the weighted average of its constituents’ returns. We show that for a large number of summands, the behavior of a weighted average, under certain assumptions, coincide with the arithmetic mean.
2.4.1 A weak law of large numbers for weighted averages

We consider a sequence of i.i.d. random variables $x_i$ with finite first moment $\mu$ and variance $\sigma^2$. For an increasing double sequence of weights $\lambda_{i,n}$, $i = 1, 2, \ldots, n$ and $n = 1, 2, \ldots$ such that $\sum_{i=1}^n \lambda_{i,n} = 1$ and $\sum_{i=1}^n \lambda_{i,n}^2 = O(n^{-1})$. The first two moments of the weighted average $X_n(t) = \sum_{i=1}^n \lambda_{i,n} x_i$ are given by

$$E[X_n] = \sum_{i=1}^n \lambda_{i,n} E[x_i] = \mu,$$

$$\text{Var}[X_n] = E[X_n^2] - E[X_n]^2 = E\left[\left(\sum_{i=1}^n \lambda_{i,n} x_i\right)^2\right] - \mu^2 =$$

$$= \sum_{i \neq j} \lambda_{i,n} \lambda_{j,n} E[x_i] E[x_j] + \sum_{i=1}^n \lambda_{i,n}^2 E[x_i^2] - \mu^2$$

$$= \mu^2 \sum_{i \neq j} \lambda_{i,n} \lambda_{j,n} + (\sigma^2 + \mu^2) \left(\sum_{i=1}^n \lambda_{i,n}^2\right) - \mu^2$$

$$= \mu^2 \left(\sum_{i=1}^n \lambda_{i,n}\right)^2 + \sigma^2 \left(\sum_{i=1}^n \lambda_{i,n}^2\right) - \mu^2$$

$$= \sigma^2 \sum_{i=1}^n \lambda_{i,n}^2$$

$$= \sigma^2 O(n^{-1}).$$

Tchebychev’s inequality gives

$$\Pr\{|X_n - \mu| > \epsilon\} \leq \frac{\text{Var}[X_n]}{\epsilon^2} = \frac{\sigma^2 O(n^{-1})}{\epsilon^2} = \sigma^2 O(n^{-1}),$$

hence

$$\lim_{n \to \infty} \Pr\{|X_n - \mu| > \epsilon\} = \lim_{n \to \infty} \sigma^2 O(n^{-1}) = 0.$$

As a corollary,

$$\lim_{n \to \infty} \Pr\left\{\left|\sum_{i=1}^n \lambda_{i,n} x_i - \frac{1}{n} \sum_{i=1}^n x_i\right| > \epsilon\right\} = 0. \quad (2.27)$$

2.4.2 Equal weight index model

The S&P500 end of year weights, from 2001 to 2011, are approximated with $\lambda_{i,n} = i^\alpha / \sum_{i=1}^n i^\alpha$, for $\alpha = 18$. See figures 4.11,4.12(pp.54–55). Such weights satisfy the weak law of large numbers for a weighted average. Indeed,

$$\lambda_{i,n} = \frac{i^\alpha}{\sum_{i=1}^n i^\alpha} = \frac{1}{n} \left(\frac{i}{n}\right)^\alpha \approx \frac{1}{n} \int_0^1 x^\alpha dx = \frac{(\alpha + 1)}{n} \left(\frac{i}{n}\right)^\alpha$$

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and
\[\sum_{i=1}^{n} \lambda_{i,n}^2 \approx \frac{(\alpha + 1)^2}{n} \sum_{i=1}^{n} \left( \frac{i}{n} \right)^{2\alpha} \frac{1}{n} \approx \frac{(\alpha + 1)^2}{n} \int_0^1 x^{2\alpha} dx = n^{-1} \frac{(\alpha + 1)^2}{2\alpha + 1} = O(n^{-1}).\]

Therefore, by corollary (2.27), we henceforth assume an equal-weight index
\[X_n(t) = \frac{1}{n} \sum_{i=1}^{n} x_i(t).\] (2.28)

The drift of the sum of lognormal stochastic processes is linear and therefore equal to the average of the underlying drifts. However, the diffusion coefficient is obtained by assembling \(n\) independent Brownian motions into one. Therefore, the SDE of \(X_n(t)\), is given by
\[dX_n(t) = \psi(t) \frac{1}{n} \sum_{i=1}^{n} x_i(t) dt + \phi(t) \frac{1}{n} \sum_{i=1}^{n} x_i(t) dw_i(t)\]
\[= \psi(t) X_n(t) dt + \frac{1}{n} \phi(t) \left( \sqrt{\sum_{i=1}^{n} x_i^2(t)} \right) dW(t).\] (2.29)

Much research has been done on the subject of identifying the distribution of the average of lognormals. Large Deviation Theory and Central Limit Theorem methods tend to fail, because a moment generating function does not exist for lognormals. Several numerical methods have been suggested for the approximation of the sum of lognormals. In [31], the steepest descent technique is used to numerically evaluate the sum of lognormals cdf integral, using the Lambert-W function. This method works well for few summands with relatively low variance. In our case, where long-term investment is considered, the variance becomes large. In the Fenton-Wilkinson (F-W) method, [32] [33], the sum is approximated with another lognormal whose first two moments are matched to the sum. Numerical simulations show that the (F-W) method is a good approximation of the average process, for long time periods.

### 2.4.3 Approximating the sum of lognormals with a lognormal

We employ the F-W moment matching technique to construct a stochastic process, \(Z_n(t)\), whose distribution is known, that approximates the distribution of \(X_n(t)\). We begin by assuming that \(Z_n(t)\) is a lognormal diffusion process, whose first and second moments concur with \(X_n(t)\)’s.

1. \(Z_n(0) = X_n(0)\)
2. \(\mathbb{E}[Z_n(t)] = \mathbb{E}[X_n(t)], \text{ for every } t \geq 0\)
3. \(\text{Var}[Z_n(t)] = \text{Var}[X_n(t)], \text{ for every } t \geq 0\)

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These constraints uniquely induce a SDE for \( Z_n(t) \). Indeed, being a diffusion process, \( Z_n(t) \) must have the same drift as \( X_n(t) \), for condition (2) to hold. Moreover, being a lognormal, the diffusion coefficient must be of the form \( \Phi(t)Z_n(t) \), for some function \( \Phi(t) \). Therefore, we write

\[
dZ_n(t) = \psi(t)Z_n(t)\,dt + \Phi(t)Z_n(t)\,dW(t). \tag{2.30}
\]

Next, we choose \( \Phi(t) \) to satisfy condition (3). Employing the formula for lognormal moments (2.12), we obtain

\[
\text{Var } [Z_n(t)] = Z_n^2(0) \exp \left\{ 2 \int_0^t \psi(s) \,ds \right\} \left( \exp \left\{ \int_0^t \Phi^2(s) \,ds \right\} - 1 \right). \tag{2.31}
\]

Calculating the variance of \( X_n(t) \) directly, we get

\[
\text{Var } [X_n(t)] = \frac{1}{n^2} \text{Var } \left[ \sum_{i=1}^n x_i(t) \right] = \frac{1}{n^2} \sum_{i=1}^n \text{Var } [x_i(t)] = \frac{1}{n} x_0^2 \exp \left\{ 2 \int_0^t \psi(s) \,ds \right\} \left( \exp \left\{ \int_0^t \phi^2(s) \,ds \right\} - 1 \right). \tag{2.32}
\]

Equating (2.31) and (2.32), together with \( Z_n(0) = X_n(0) = x_i(0) \), we get

\[
\exp \left\{ \int_0^t \Phi^2(s) \,ds \right\} = \frac{1}{n} \left( \exp \left\{ \int_0^t \phi^2(s) \,ds \right\} + n - 1 \right). \tag{2.33}
\]

Taking log from each side of the equation, we get

\[
\int_0^t \Phi^2(s) \,ds = \log \left( \exp \left\{ \int_0^t \phi^2(s) \,ds \right\} + n - 1 \right) - \log(n). \tag{2.34}
\]

Differentiating each side of the equation, we get

\[
\Phi^2(t) = \frac{\phi^2(t)e^{\int_0^t \phi^2(s) \,ds}}{e^{\int_0^t \phi^2(s) \,ds} + n - 1}. \tag{2.35}
\]

Therefore, the diffusion process \( Z_n(t) \), governed by the SDE

\[
dZ_n(t) = \psi(t)Z_n(t)\,dt + \left( \frac{\phi^2(t)e^{\int_0^t \phi^2(s) \,ds}}{e^{\int_0^t \phi^2(s) \,ds} + n - 1} \right) Z_n(t)dW(t), \tag{2.36}
\]
satisfies conditions (1)-(3). We note that \( \Phi^2(t) \to \phi^2(t) \) as \( t \to \infty \), meaning that the asymptotic behaviour of \( Z_n(t) \) aligns with the underlying stocks of \( X_n(t) \). The solution of (2.30) is given by

\[
Z_n(t) = x_0 \exp \left\{ \int_0^t \left[ \psi(s) - \frac{1}{2} \phi^2(s) \right] ds + \int_0^t \Phi(s) dW(s) \right\},
\]

(2.37)

and in terms of the underlying stocks,

\[
Z_n(t) = x_0 \exp \left\{ \int_0^t \left[ \psi(s) - \frac{1}{2} \phi^2(s) e^{\phi^2 s + n - 1} \right] ds + \int_0^t \left( \frac{\phi^2(s)}{e^{\phi^2 s + n - 1}} \right)^{\frac{1}{2}} dW(s) \right\}.
\]

Finally, we incorporate the estimated coefficients, \( \psi(s) = \psi, \phi(t) = \phi \), to get

\[
Z_n(t) = x_0 \exp \left\{ \int_0^t \left[ \psi - \frac{1}{2} \frac{\phi^2 e^{\phi^2 s}}{e^{\phi^2 s + n - 1}} \right] ds + \int_0^t \left( \frac{\phi^2 e^{\phi^2 s}}{e^{\phi^2 s + n - 1}} \right)^{\frac{1}{2}} dW(s) \right\}
\]

\[
= x_0 \exp \left\{ \psi t - \frac{1}{2} \log \left( e^{\phi^2 t + n - 1} \right) + \frac{1}{2} \log(n) + \int_0^t \left( \frac{\phi^2 e^{\phi^2 s}}{e^{\phi^2 s + n - 1}} \right)^{\frac{1}{2}} dW(s) \right\}
\]

\[
= x_0 \left( \frac{e^{\phi^2 t + n - 1}}{n} \right)^{-\frac{1}{2}} \exp \left\{ \psi t + \int_0^t \left( \frac{\phi^2 e^{\phi^2 s}}{e^{\phi^2 s + n - 1}} \right)^{\frac{1}{2}} dW(s) \right\}.
\]

(2.38)

### 2.4.4 Numerical simulations of \( X_n(t) \) - the Euler scheme

The Wiener interpretation of stochastic differential equations is useful for both the conceptual understanding of SDEs and for deriving differential equations that govern the evolution of the pdf’s of their solutions [30]. Itô’s definition of the stochastic integral on the lattice \( t_k = t_0 + k\Delta t \), with \( \Delta t = T/N \) and \( \Delta w(t) = \Delta w(t + \Delta t) - w(t) \), defines the solution of the SDE

\[
dx = a(x, t) dt + b(x, t) dw, \quad x(0) = x_0,
\]

or equivalently, of the Itô integral equation

\[
x(t) = x_0 + \int_0^t a(x(s), s) ds + \int_0^t b(x(s), s) dw(s),
\]

(2.39)
as the limit of the solution of the Euler scheme

\[ x_N(t + \Delta t) = x_N(t) + a(x_N(t), t)\Delta t + b(x_N(t), t)\Delta w(t) \]

\[ x_N(0) = x_0 \]  \hspace{1cm} (2.40)

as \( \Delta t \to 0 \). The increments \( \Delta w(t) \) are independent random variables that can be constructed by Levy’s method [30], as \( \Delta w(t) = n(t)\sqrt{\Delta t} \), where the random variables \( n(t) \), for each \( t \) on the numerical mesh, are independent standard Gaussians variables \( \mathcal{N}(0, 1) \). according to the recursive scheme (2.40), at any time \( t \) on the numerical mesh, the process \( x_N(t) \) depends on the sampled trajectory \( w(s) \) for \( s \leq t \), so it is \( \mathcal{F}_t \)-adapted. The limit \( x(t) = \lim_{N \to \infty} x_N(t) \) existence is guaranteed by the following theorem

**Theorem (Skorokohd [34]).** If \( a(x, t) \) and \( b(x, t) \) are uniformly Lipschitz continuous functions in \( x \in \mathbb{R} \), \( t \in [t_0, T] \), then the limit \( x(t) \overset{\text{Pr}}{=} \lim_{N \to \infty} x_N(t) \) (convergence in probability) exists and is the solution of (2.39).

The convergence of the pdf is guaranteed by

**Theorem ([30]).** the pdf \( p_N(x, t \mid x_0) \) of the solution \( x_N(t, \omega) \) of (2.40) converge to the solution \( p(x, t \mid x_0) \) of the FPE

\[ \frac{\partial p(y, t \mid x, s)}{\partial t} = \frac{1}{2} \frac{\partial^2 [b^2(y, t)p(y, t \mid x, s)]}{\partial y^2} - \frac{\partial [a(y, t)p(y, t \mid x, s)]}{\partial y} \]

with the initial condition \( \lim_{t \downarrow s} p(y, t \mid x, s) = \delta(y - x) \), as \( N \to \infty \).

We construct 10,000 trajectories of \( X_n(t) \), by averaging its 500 underlying trajectories \( x_i(t) \), for \( 0 \leq t \leq 600 \) months. The \( x_i(t) \)'s are constructed with

\[ x_i(t) = x_i(t - 1) + \phi x_i(t - 1) \cdot \mathcal{N}(0, 1), \quad 1 \leq t \leq 600, \]

\[ x_i(0) = 1 \].

We plot the lognormal pdf of \( Z_n(t) \) against the histogram of the euler scheme for every \( t \), together its least square fit. See figures 4.13,4.14(pp.56-57).

### 2.5 Stochastic Model for Salaries

We assume that the salaries growth of member \( i \) after time \( t \), denoted \( s_i(t) \), is governed by the diffusion SDE

\[ ds_i(t) = a(t, s_i) \, dt + b(t, s_i) \, dw_i(t) \]

\[ s_i(t_0) = 1; \]  \hspace{1cm} (2.41)

where \( w_i \) is Brownian motion.
2.5.1 Discrete approximation scheme for the diffusion coefficients

We approximate continuous trajectories with discrete yearly CPI-adjusted wage vectors. We denote by \( S(\tau, x) \) the set of all individuals whose salary, at time \( \tau \), had multiplied \( x \) times relative to their initial salary. The trajectory of the salary growth process, attributable to the \( j \)-th individual is denoted by \( x_j(t) \). The coefficients of (2.41) are approximated with

\[
a(x, \tau) = \frac{1}{|S(\tau, x)|} \sum_{s \in S(\tau, x)} [x_s(\tau + 1) - x_s(\tau)]
\]
\[
b^2(x, \tau) = \frac{1}{|S(\tau, x)|} \sum_{s \in S(\tau, x)} [x_s(\tau + 1) - x_s(\tau)]^2.
\] (2.42)

2.5.2 Data preparation

We use the Panel Study of Income Dynamics® database [35]. PSID is a longitudinal survey of a representative sample of US individuals and families, which has been ongoing since 1968. Information on individuals and their descendants has been collected continuously, including data covering employment, income, wealth, expenditures, health, marriage, child-bearing, child development, philanthropy, education, and numerous other topics. For pension purposes, we are interested in the individual’s pension plan contributions. Unfortunately, the PSID database does not offer a full, cross-year individual time series of pension contributions, so we use the total wage earned from labor instead, and assume contributed ratio. Furthermore, income attributable to bonuses, independent businesses, secondary professional practices, comission, tips and other sources were not incorporated, due to the discontinuous and sparse nature of data. The trajectories constructed span from 1970 through 1992. Overall, 58,807 individuals were considered, out of which only 3,669 began working in 1970. The amount of individuals starting to work each year is given in figure 4.15 (p.58).

All of the wages data have been adjusted to the cost of living, using the Bureau of Labor Statistics’s historical CPI values. For highly-noisy volatility estimation, 3% of the largest volatility values were omitted as outliers, and 5% of the highest salary growths were discarded as well (25,000% growth rates etc).

2.5.3 Drift and diffusion surface interpolation

We repeat the process of section 2.3.2 for data between January 1970 and December 1992, to obtain the drifts and volatility surfaces \( G_a, G_b : \mathbb{R}^2 \to \mathbb{R} \).

Similarly, we repeat the process of 2.3.3 with \( q(t), q_2(t), r(t), r_2(t), r_3(t) \) replaced by \( \xi(t), \xi_2(t), \eta(t), \eta_2(t), \eta_3(t) \), respectively, and obtain \( \xi(t) = -0.0328, \xi_2(t) = 0, \eta(t) = \sqrt{\frac{1}{6}}, \eta_2(t) = \eta_3(t) = 0 \).

The interpolating functions, \( \tilde{G}_a(t), \tilde{G}_b(t) \), plotted against the projections \( G_a[t], G_b[t] \) are given
in figures 4.16-4.18 (p.59-60). In figure 4.19, $\xi(t)$ and $\xi_2(t)$ are plotted for $1970 \leq t \leq 1992$. In figure 4.20, $\eta(t)$ and $\eta_2(t)$ are plotted for $1970 \leq t \leq 1992$. Substituting $\xi, \xi_2, \eta, \eta_2, \eta_3$ into (2.41), we get the lognormal SDE

$$ds_i(t) = \xi s_i(t) \, dt + \eta s_i(t) \, dw_i(t)$$

$$s_i(t_0) = 1,$$  \hspace{1cm} (2.43)

whose solution is given by

$$s_i(t) = s_0 \exp \left\{ \left( \xi - \frac{1}{2} \eta^2 \right) t + \eta w_i(t) \right\}.$$ \hspace{1cm} (2.44)

### 2.6 Stochastic Model for the Pension Fund

The funds of the Super Trusts will be invested in a stock-market index, like S&P500. The value of the index is the market-capitalization weighted average of its components’ stock prices.

#### 2.6.1 Model derivation

We denote by $v_i(t)$ the growth in the amount payable by the fund to member $i$ after time $t$. Let $T_i = \{t_{i_0} < t_{i_1} < \ldots < t_{i_n} = t\}$ be the equispaced time periods that member $i$ contributed to the pension fund. We also assume that the total contribution (both employer and employee) is a constant fraction of the salary, and we denote it by $c_i(t) = \Lambda s_i(t)$.  \hspace{1cm} (2.45)

In practice, $\Lambda$ is around 10%. In addition, we denote member’s $i$ first salary, in dollar amount, by $\alpha_i$. We incorporate the model of the approximated portfolio returns to derive the equation for $v_i(t)$ by making the following observation. For every $0 \leq j \leq n$, the contributed dollar amount at time $t_{i_j}$ is given by $\alpha_i c_i(t_{i_j})$, where the interest on this amount compounded from $t_{i_j}$ through $t_{i_n}$ and is given by $Z_n(t_{i_n})/Z_n(t_{i_j})$. Therefore, the portion of the pension’s total amount, attributable to $j$-th contribution is given by

$$\alpha_i c_i(t_{i_j}) \frac{Z_n(t)}{Z_n(t_{i_j})}$$ \hspace{1cm} (2.46)

and the portion of the pension’s total growth, attributable to the $j$-th contribution is obtained from (2.46), divided by $\alpha_i$. Therefore, the total growth of the pension fund, from time $t_{i_0}$ to time $t$, is given by

$$v_i(t) = \sum_{\tau \in T_i} c_i(\tau) \frac{Z_n(t)}{Z_n(\tau)} = Z_n(t) \sum_{\tau \in T_i} \frac{c_i(\tau)}{Z_n(\tau)}.$$ \hspace{1cm} (2.47)

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The continuous model for \( v_i(t) \) is obtained by representing (2.47) as the Riemann sum

\[
v_i(t) = \frac{Z_n(t)}{\Delta t} \sum_{j=1}^{n} c_i(j \Delta t) \Delta t
\]

(2.48)

where \( \Delta t = (t_i - t_{i-1}) \) is the constant time elapsed between consecutive salaries. Based on the PSID database, \( \Delta t = 1 \) year. We write \( v_i(t) \) in the integral form

\[
v_i(t) = Z_n(t) \left( \int_{v_0}^{t} \frac{c_i(u)}{Z_n(u)} \, du \right),
\]

(2.49)

and the absolute amount in dollars, payable to member \( i \) can now be expressed as

\[
V_i(t) = \alpha v_i(t)
\]

(2.50)

We obtain the SDE for \( v_i(t) \) by differentiating (2.49), and applying the chain rule

\[
dv_i(t) = dZ_n(t) \left( \int_{v_0}^{t} \frac{c_i(u)}{Z_n(u)} \, du \right) + Z_n(t) \left( \frac{c_i(t)}{Z_n(t)} \right) dt.
\]

(2.51)

Substituting (2.30), (2.45) and (2.49) into (2.52), we obtain the SDE for \( v_i(t) \)

\[
dv_i(t) = \left[ \psi(t)v_i(t) + \Lambda s_i(t) \right] dt + \Phi(t)v_i(t) \, dW(t).
\]

(2.52)

The Fokker-Planck equation for the joint probability density function \( p(v,s,t) \) of the solution \((s_i(t), v_i(t))\) of the system (2.43) and (2.52) is given by

\[
\frac{\partial p}{\partial t} = -\frac{\partial}{\partial v} \left[ (\psi v + \Lambda s) p \right] - \frac{\partial}{\partial s} (\xi s p) + \frac{1}{2} \frac{\partial^2}{\partial s^2} (\eta^2 s^2 p) + \frac{1}{2} \frac{\partial^2}{\partial v^2} \left[ \Phi^2(t)v^2 p \right].
\]

(2.53)

### 2.7 Probabilities for the Pension Fund

The probability distribution function of \( V_i(t) \) is the probability that the pension payable to individual \( i \) at time \( t \) exceeds \( y \) dollars. To compute,

\[
\Pr (V_i(t) > y) = \Pr \left\{ v_i(t) > \frac{y}{\alpha t_o} \right\} = 1 - \Pr \left\{ v_i(t) \leq \frac{y}{\alpha t_o} \right\},
\]

(2.54)

we compute the joint transition probability density function \( p(v,s,t \mid v_0, s_0, t_{i_0}) \) of the process \( v_i(t) \) and \( s_i(t) \) from the FPE (2.53) with the initial condition

\[
p(v,s,t_{i_0} \mid v_0, s_0, t_{i_0}) = \delta(v - v_0, s - s_0),
\]

(2.55)

where \( v_0 = s_0 = 1 \). We give a numerical solution by applying the finite difference method (FDM). Being a numerical method, the approximated solution has several limitations.

1. Finite difference methods may become computationally expensive, as the grid-spacing gets finer.
2. The domain for the process \((v_i, s_i)\) and its pdf is finite (see explanation in section 2.7.2)
2.7.1 Finite difference scheme for (2.53)

We use the stable implicit BTCS (First Order Backward Time Central Space) [36] method to approximate \( p(v, s, t) \).

2.7.2 Boundary Conditions

The stochastic process \( s_i(t) \) is always positive, because it is a lognormal. Moreover, the stochastic process \( v_i(t) \) is always positive, because it is a sum of a product of a lognormal and a ratio of two lognormal processes. Therefore the joint density cannot contain any mass on the boundary and thus

\[
p(v, 0, t) = p(0, s, t) = 0. \tag{2.56}
\]

Because the boundaries \( v = 0 \) and \( s = 0 \) are unattainable by the stochastic processes, the conditions (2.56) are set numerically.

The distant boundaries of the grid describe the possibility of the market/salaries to get within given time to unheard of levels. Because there is no data at such levels, a zero condition for the FPE at distant boundaries becomes part of our model and concurs with the data. Consequently, we set

\[
p(v, s, t) \bigg|_{\partial G} = 0. \tag{2.57}
\]

Assigning zero conditions sufficiently far affects the solution in a negligible way inside the finite domain, see section 2.7.3

2.7.3 The initial condition

An infinite value is unattainable by a computer program of finite memory, therefore we approximate \( \delta(v - v_0, s - s_0) \) numerically by a multivariate normal distribution with a small standard deviation and we set \( p_0^{ij} \) to be the multivariate Gaussian, with standard deviation \( \Sigma = \begin{pmatrix} \sigma_v^2 & 0 \\ 0 & \sigma_s^2 \end{pmatrix} \), where \( \sigma_v, \sigma_s << 1 \), and mean \( \mu = (v_0, s_0) = (1, 1) \). In [37], it is shown that the Gaussian is equivalent to the Dirac distribution in the limit \( \sigma_v, \sigma_s \rightarrow 0 \). For every \( v = (v, s) \in \mathbb{R}^2 \)

\[
p_{0,ij} = \frac{1}{2\pi \sqrt{\det(\Sigma)}} \exp \left\{ -\frac{1}{2} (v - \mu)^T \Sigma^{-1} (v - \mu) \right\} = \frac{1}{2\pi \sigma_v \sigma_s} \exp \left\{ -\frac{(v - 1)^2}{2\sigma_v^2} - \frac{(s - 1)^2}{2\sigma_s^2} \right\}. \tag{2.58}
\]

We discretize \( \mathbb{R}^3 \) on a \((t, v, s)\) grid with steps \((\Delta k, \Delta h, \Delta m)\) and abbreviate

\[
p(v_j, s_l, t_n) = p_{0,j,l}^{n}, \text{ for } j, l, n \geq 0, \tag{2.59}
\]
where \( v_j = j \Delta h \), \( s_l = l \Delta m \) and \( t_n = n \Delta k \). The initial density is normalized by \( p_{j,l}^0 = p_{j,l}^0 / \int_{\mathbb{R}^2} p_{j,l}^0 \) to insure \( \int_0^{\infty} \int_0^{\infty} p(v, s, 0) dsdv = 1. \)

The location of the zero conditions has negligible effect on the solution in the domain where it does not vanish. Extending the grid boundaries from \( (N_v, N_s) \) to \( (N_v', N_s') \) where \( N_v' > N_v \), \( N_s' > N_s \) introduces a change in the linear equations of the finite difference scheme that corresponds to the boundaries. Under an implicit method scheme (see below), the error infiltrates the interior of the domain by a coefficient that depend on \( \Delta m^2 \), \( \Delta h^2 \) and \( \Delta k \). Therefore, if we show that

\[
\int_0^{N_v} \int_0^{N_s} p_{j,l}^0 dsdv - \int_0^{N_v'} \int_0^{N_s'} p_{j,l}^0 dsdv = (2.60)
\]

is negligible, then by the above argument we conclude that the change in the solution is negligible. There are several numerical methods for approximating the normal CDF [38],[39],[40].

We compute (2.60) numerically for the grid size we will use and a grid double its size. i.e., \( 18 \times 5 \) grid extended to \( 36 \times 10 \). The results are summarized in the table below.

<table>
<thead>
<tr>
<th>( \Delta h )</th>
<th>( \Delta m )</th>
<th>( N_v )</th>
<th>( N_s )</th>
<th>( N_v' )</th>
<th>( N_s' )</th>
<th>( N_v \Delta h )</th>
<th>( N_s \Delta m )</th>
<th>(2.60)</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.025</td>
<td>0.2</td>
<td>720</td>
<td>25</td>
<td>1440</td>
<td>50</td>
<td>18</td>
<td>5</td>
<td>1.0805 \cdot 10^{-17}</td>
</tr>
</tbody>
</table>

We denote the numerical solution at time \( t_n \) by

\[
G[t_n, N_v, N_s] = \{ (v_j, s_l, p_{j,l}^n) = (j \Delta h, l \Delta m, p_{j,l}^n) \in \mathbb{R}^3 \mid 0 \leq j \leq N_v, 0 \leq l \leq N_s \}. \quad (2.61)
\]

Next, we derive \( G[t_n, N_v, N_s] \) from \( G[t_{n-1}, N_v, N_s] \) by discretizing the derivatives of \( p \) in the FPE, thus obtaining an implicit relation between the numerical solutions at two consecutive times. The discretizations of the derivatives in the FPE are given by

\[
\frac{\partial p}{\partial t} \bigg|_{(v_j, s_l, t_n)} = \frac{p_{j,l}^n - p_{j,l}^{n-1}}{\Delta k}
\]

\[
\frac{\partial}{\partial s} (\xi sp) \bigg|_{(v_j, s_l, t_n)} = \xi \left[ \frac{(l + 1) \Delta m p_{j,l+1}^n - (l - 1) \Delta m p_{j,l-1}^n}{2\Delta m} \right]
\]

\[
\frac{\partial}{\partial v} \left[ (\psi v + \Lambda s) p \right] \bigg|_{(v_j, s_l, t_n)} = \left[ \frac{(\psi (j + 1) \Delta h + \Lambda l \Delta m) p_{j+1,l}^n - (\psi (j - 1) \Delta h + \Lambda l \Delta m) p_{j-1,l}^n}{2\Delta h} \right]
\]

\[
\frac{\partial^2}{\partial s^2} \left( \frac{1}{2} \eta^2 s^2 p \right) \bigg|_{(v_j, s_l, t_n)} = \frac{\eta^2}{2} \left[ \frac{(l + 1) \Delta m^2 p_{j,l+1}^n - 2(l \Delta m) \Delta m p_{j,l}^n + ((l - 1) \Delta m)^2 p_{j,l-1}^n}{\Delta m^2} \right]
\]

\[
\frac{\partial^2}{\partial v^2} \left( \frac{1}{2} \Phi^2(t) v^2 p \right) \bigg|_{(v_j, s_l, t_n)} = \frac{\Phi^2(t_n)}{2} \left[ \frac{((j + 1) \Delta h)^2 p_{j+1,l}^n - 2(j \Delta h)^2 p_{j,l}^n + ((j - 1) \Delta h)^2 p_{j-1,l}^n}{\Delta h^2} \right]
\]
with the initial condition, \( p_{j,l}^0 = \delta(v - v_0, s - s_0) \) for \( j, l > 0 \) and the boundary condition \( p_{0,l}^0 = p_{n,l}^0 = 0 \), for \( j, l \geq 0 \). We substitute the scheme (2.62) into (2.53), to get
\[
\frac{p_{j,l}^n - p_{j,l}^{n-1}}{\Delta k} = - \left[ \left( \psi(j+1)\Delta h + \Lambda l\Delta m \right) p_{j+1,l}^n - \left( \psi(j-1)\Delta h + \Lambda l\Delta m \right) p_{j-1,l}^n \right] \\
\frac{1}{2\Delta h} \xi \left[ \left( l + 1 \right)\Delta m p_{j,l+1}^n - \left( l - 1 \right)\Delta m p_{j,l-1}^n \right] \\
+ \frac{1}{2} \eta^2 \left[ \left( l + 1 \right)\Delta m^2 p_{j,l+1}^n - 2 \left( l\Delta m \right)^2 p_{j,l}^n + \left( l - 1 \right)\Delta m^2 p_{j,l-1}^n \right] \\
+ \frac{1}{2} \Phi^2(t_n) \left[ \left( j + 1 \right)\Delta h^2 p_{j+1,l}^n - 2 \left( j\Delta h \right)^2 p_{j,l}^n + \left( j - 1 \right)\Delta h^2 p_{j-1,l}^n \right]
\]
which reduces to
\[
\frac{p_{j,l}^n - p_{j,l}^{n-1}}{\Delta k} = - \left[ \left( \frac{1}{2} \psi(j+1) + \frac{\Lambda l\Delta m}{2\Delta h} \right) p_{j+1,l}^n - \left( \frac{1}{2} \psi(j-1) + \frac{\Lambda l\Delta m}{2\Delta h} \right) p_{j-1,l}^n \right] \\
- \frac{1}{2} \xi \left[ \left( l + 1 \right) p_{j,l+1}^n - \left( l - 1 \right) p_{j,l-1}^n \right] \\
+ \frac{1}{2} \eta^2 \left[ \left( l + 1 \right)^2 p_{j,l+1}^n - 2l^2 p_{j,l}^n + \left( l - 1 \right)^2 p_{j,l-1}^n \right] \\
+ \frac{1}{2} \Phi^2(t_n) \left[ \left( j + 1 \right)^2 p_{j+1,l}^n - 2j^2 p_{j,l}^n + \left( j - 1 \right)^2 p_{j-1,l}^n \right]
\]
Finally, we implicitly express \( p_{j,l}^n, p_{j,l+1}^n, p_{j,l-1}^n, p_{j+1,l}^n, p_{j-1,l}^n \) in terms of \( p_{j,l}^{n-1} \),
\[
p_{j,l}^{n-1} = \lambda_1(j, l, n)p_{j,l}^n + \lambda_2(j, l, n)p_{j,l+1}^n + \lambda_3(j, l, n)p_{j,l-1}^n + \lambda_4(j, l, n)p_{j,l+1}^n + \lambda_5(j, l, n)p_{j,l-1}^n,
\]
where
\[
\lambda_1(j, l, n) = \Delta k \left[ \frac{1}{\Delta k} + l^2\eta^2 + j^2\Phi^2(t_n) \right] \\
\lambda_2(j, l, n) = \Delta k \left[ \left( \frac{1}{2} \psi(j+1) + \frac{\Lambda l\Delta m}{2\Delta h} \right) - \frac{(j + 1)^2\Phi^2(t_n)}{2} \right] \\
\lambda_3(j, l, n) = \Delta k \left[ - \left( \frac{1}{2} \psi(j-1) + \frac{\Lambda l\Delta m}{2\Delta h} \right) - \frac{(j - 1)^2\Phi^2(t_n)}{2} \right] \\
\lambda_4(j, l, n) = \Delta k \left[ \frac{1}{2} \xi \left( l + 1 \right) - \frac{1}{2} \eta^2 \left( l + 1 \right)^2 \right] \\
\lambda_5(j, l, n) = \Delta k \left[ - \frac{1}{2} \xi \left( l - 1 \right) - \frac{1}{2} \eta^2 \left( l - 1 \right)^2 \right]
\]
and
\[
\Phi^2(t_n) = \frac{\phi^2 e^{\phi^2 n\Delta k}}{e^{\phi^2 n\Delta k} + N - 1}.
\]
2.7.4 Solution of the linear system

The numerical solution \( G[t_n, N_v, N_s] \) is recursively derived from the previous numerical solution \( G[t_{n-1}, N_v, N_s] \) by solving a system of linear equations. Because the density vanishes on the grid’s boundaries, we solve (2.65) for

\[
\text{int}(G) = G \setminus \partial G = \left\{ (j \Delta h, l \Delta m, p_{j,l}^n) \in \mathbb{R}^3 \mid 1 \leq j \leq N_v - 1, 1 \leq l \leq N_s - 1 \right\}, \tag{2.68}
\]

written in matrix form as

\[
A(n) \vec{p}_n = \vec{p}_{n-1}, \tag{2.69}
\]

where \( \vec{p}_n \) is a 1-D compression of \( \text{int}(G) \), and \( A(n) \) is the coefficient matrix obtained from the \( Q = (N_v - 1)(N_s - 1) \) linear equations in (2.65). The compression is achieved by mapping every point \( (j \Delta h, l \Delta m, p_{j,l}^n) \in \text{int}(G) \) to the \( k \)-th coordinate of \( \vec{p}_n \), where \( k = (j - 1)(N_s - 1) + l \).

The construction of \( A(n) \) is as follows: for every \( 1 \leq j \leq N_v - 1, 0 \leq l \leq N_s - 1 \), the coefficients of \( p_{j+1,l}^n, p_{j+1,l}^n, p_{j-1,l}^n, p_{j+1,l-1}^n, p_{j+1,l+1}^n, p_{j-1,l}^n, p_{j-1,l-1}^n, p_{j-1,l+1}^n \), in the linear system (2.65), are located respectively in the \( k \)-th column of \( A(n) \). Consequently, \( A(n) \) is a block diagonal matrix, with tridiagonal matrices as the main diagonal blocks with two additional diagonals, spaced \( N_s - 1 \) entries from the main diagonal, and has the form

\[
\begin{pmatrix}
\vdots & & & & & & & \\
0 \ldots 0 & \cdots & \cdots & \cdots & \cdots & 0 \ldots 0 & \\
0 \ldots 0 & p_{k,k-N_s+1}^n & \cdots & \cdots & \cdots & 0 \ldots 0 & \\
\vdots & \cdots & \cdots & \cdots & \cdots & \cdots & \vdots
\end{pmatrix}_{Q \times Q}
\]

Indeed, for every \( 1 \leq k \leq Q \), we write

\[
k = (j - 1)(N_s - 1) + l,
\]

where

\[
j = \left\lfloor \frac{k}{(N_s - 1)} \right\rfloor + 1
\]

\[
l = [k \mod (N_s - 1)] + 1
\]

and the \((k, r)\) entry of \( A(n) \) is

\[
A_{kr} = \begin{cases} 
\lambda_1(j, l, n) & r = k \\
\lambda_2(j, l, n) & (1 \leq j \leq N_v - 2) \land (r = k + N_s - 1) \\
\lambda_3(j, l, n) & (2 \leq j \leq N_v - 1) \land (r = k - N_s + 1) \\
\lambda_4(j, l, n) & (1 \leq l \leq N_s - 2) \land (r = k + 1) \\
\lambda_5(j, l, n) & (2 \leq l \leq N_s - 1) \land (r = k - 1) \\
0 & \text{else}
\end{cases} \quad \tag{2.70}
\]
The upper secondary diagonal is composed of $\lambda_2$-s and the lower secondary diagonal is composed of $\lambda_3$-s. The main diagonal is composed of a tridiagonal matrix, with $\lambda_1$ on the main diagonal, $\lambda_4$ above, and $\lambda_5$ below it.

### 2.7.5 Analysis of the computation

The process of reconstructing $G[t_n, N_v, N_s]$ requires the computation of the inverse of a large sparse matrix $A(n)$. This procedure is costly, because $A^{-1}(n)$ is dense. To this end, we apply the Generalized Minimal Residual Method (GMRES) [41]. This method approximates the exact solution $\vec{p}_n$ by the vector $\vec{\xi} \in K_m[A(n), \vec{p}_{n-1}]$, where $K_m[A(n), \vec{p}_{n-1}]$ is the Krylov subspace

$$\text{span}\left\{ \vec{p}_{n-1}, A(n)\vec{p}_{n-1}, \ldots, A(n)^{m-1}\vec{p}_{n-1} \right\}$$

and $\xi$ satisfies

$$\xi = \min_{\nu \in \mathbb{R}^M} \|\nu - [A(n)\nu - \vec{p}_{n-1}]\|.$$  \hspace{1cm} (2.71)

We use the Arnoldi Iteration [42] eigenvalue method to find $\xi$. The Arnoldi iteration uses the stabilized Gram–Schmidt process [43] to produce a sequence of orthonormal vectors, $\xi_0, \xi_1, \xi_2, \ldots$, called the Arnoldi vectors, such that for every $n$, the vectors $\xi_1, \ldots, \xi_n$ span the Krylov subspace $K_n$. while $\xi_0$ is chosen at random, for every $j > 1$, $\xi_j = \phi_j A \xi_{j-1}$, where $\phi_j \in \mathbb{R}$ is a normalizing factor. For sparse matrices, the computational cost is reduced to $O(M)$, as opposed to the $O(M^3)$ matrix inversion cost. Applying the implicit finite difference method in a straightforward manner might be computationally inefficient. The complexity lies in advancing the numerical solutions in time, which requires solving a system of linear equations.

We employ the GMRES method by employing the ”gmres()” function in MATLAB. The Arnoldi method is employed with tolerance $10^{-4}$, i.e. it returns when $\|\xi - [A(n)\xi - \vec{p}_{n-1}]\| < 10^{-4}$. The unrestarted version of GMRES is used with $O(\max(N_v, N_s))$ total number of iterations.

### 2.7.6 Results

The probability that the pension fund will be of size $y$ equals the probability that the fund’s growth will equal the proportion of $y$ and the initial salary $\alpha$. This ratio is the dimensionless parameter of the problem. In the table below, we show probabilities for different ratios. We assume a 10% salary contribution. We say that a target pension $y$, with initial salary $\alpha$, and $t$ years of savings, has an implied annual return $r$, if $\sum_{i=1}^{t} \alpha (1 + r)^i = y$. 

33
<table>
<thead>
<tr>
<th>Target Pension Size</th>
<th>Saving Period</th>
<th>Implied Annual Return</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>3.11</td>
<td>25 Years</td>
<td>1.64%</td>
<td>65.40%</td>
</tr>
<tr>
<td>3.33</td>
<td>25 Years</td>
<td>2.15%</td>
<td>54.40%</td>
</tr>
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<td>3.55</td>
<td>25 Years</td>
<td>2.61%</td>
<td>45.17%</td>
</tr>
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<td>25 Years</td>
<td>3.60%</td>
<td>28.27%</td>
</tr>
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<td>4.44</td>
<td>25 Years</td>
<td>4.17%</td>
<td>16.16%</td>
</tr>
<tr>
<td>5.00</td>
<td>25 Years</td>
<td>4.98%</td>
<td>7.37%</td>
</tr>
<tr>
<td>5.83</td>
<td>25 Years</td>
<td>6.02%</td>
<td>1.72%</td>
</tr>
<tr>
<td>6.67</td>
<td>25 Years</td>
<td>6.90%</td>
<td>0.34%</td>
</tr>
</tbody>
</table>

Table 2.1: Pension size probabilities for 25 years of savings.

<table>
<thead>
<tr>
<th>Target Pension Size</th>
<th>Saving Period</th>
<th>Implied Annual Return</th>
<th>Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>5.00</td>
<td>40 Years</td>
<td>1.05%</td>
<td>59.38%</td>
</tr>
<tr>
<td>6.50</td>
<td>40 Years</td>
<td>2.23%</td>
<td>54.51%</td>
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<td>7.00</td>
<td>40 Years</td>
<td>2.55%</td>
<td>49.17%</td>
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<tr>
<td>7.50</td>
<td>40 Years</td>
<td>2.85%</td>
<td>41.77%</td>
</tr>
<tr>
<td>9.50</td>
<td>40 Years</td>
<td>3.83%</td>
<td>21.69%</td>
</tr>
<tr>
<td>11.00</td>
<td>40 Years</td>
<td>4.43%</td>
<td>14.86%</td>
</tr>
<tr>
<td>15.00</td>
<td>40 Years</td>
<td>5.65%</td>
<td>1.07%</td>
</tr>
</tbody>
</table>

Table 2.2: Pension size probabilities for 40 years of savings.

The 3-D graph of the joint density function, \( p(v, s, t) \) for \( t = 25 \) are given in figures 4.21 and 4.22 (pp.62).

## 2.8 Stochastic Model for Pension Consumption

We consider the pension consumption process, \( \tilde{V}_i(t) \), which is the remaining dollar amount individual \( i \) has after being retired for \( t \geq t_0 \) years, where \( t_0 \) is the time of retirement, and \( \beta_i \)
is a constant annual dollar consumption rate. Assume $T_i = \{t_{i0} < t_{i1} < \ldots < t_{in} = t\}$ are the equispaced years that member $i$ has been retired. The initial condition for $\tilde{V}_i(t)$ is given by $\tilde{V}_i(t_{i0}) = V_r$, where $V_0$ is the pension accumulated during the working life of individual $i$. Upon retirement the salary stops. We incorporate the model of the approximated portfolio returns to derive the equation for $\tilde{V}_i(t)$ by shifting the initial condition of $Z_n$, thereby obtaining a new process $\tilde{Z}_n(t)$,

$$d\tilde{Z}_n(t) = \psi(t)\tilde{Z}_n(t)\,dt + \Phi(t)\tilde{Z}_n(t)dW(t), \quad \text{for } t > t_{i0}$$

$$\tilde{Z}_n(t_{i0}) = Z_r$$

(2.72)

and making the following observation; for every $0 < j \leq n$, the value of $\tilde{V}_i(t_{ij})$ grew by $\tilde{Z}_n(t_{ij})/\tilde{Z}_n(t_{ij-1})$, relative to the previous year $\tilde{V}_i(t_{ij-1})$, while $\beta_i$ dollars were consumed. Therefore, the value of $\tilde{V}_i(t_{ij})$ is given by the recursion relation

$$\tilde{V}_i(t_{ij}) = \frac{\tilde{V}_i(t_{ij-1})\tilde{Z}_n(t_{ij})}{\tilde{Z}_n(t_{ij-1})} - \beta_i, \quad \text{for } 0 < j \leq n$$

$$\tilde{V}_i(t_{i0}) = V_r$$

and in its closed form,

$$\tilde{V}_i(t) = V_r \frac{\tilde{Z}_n(t)}{Z_r} - \beta \sum_{j=1}^{n} \frac{\tilde{Z}_n(t)}{\tilde{Z}_n(t_{ij})} = \tilde{Z}_n(t) \left( \frac{V_r}{Z_r} - \beta \sum_{j=1}^{n} \frac{1}{\tilde{Z}_n(t_{ij})} \right)$$

(2.73)

The continuous model for $\tilde{V}_i(t)$ is obtained by representing (2.73) as the Riemann sum

$$\tilde{V}_i(t) = \tilde{Z}_n(t) \left( \frac{V_r}{Z_r} - \beta \int_{t_{i0}}^{t} \frac{du}{\tilde{Z}_n(u)} \right)$$

(2.74)

where $\Delta t = (t_{ij} - t_{ij-1}) = 1$ is the constant time elapsed between consecutive time periods (time is measured in years). The process $\tilde{V}_i(t)$ is approximated by the integral

$$\tilde{V}_i(t) = \tilde{Z}_n(t) \left( \frac{V_r}{Z_r} - \beta \int_{t_{i0}}^{t} \frac{du}{\tilde{Z}_n(u)} \right)$$

(2.75)

We differentiate (2.75) to obtain the SDE

$$d\tilde{V}_i(t) = d\tilde{Z}_n(t) \left( \frac{V_r}{Z_r} - \beta \int_{t_{i0}}^{t} \frac{du}{\tilde{Z}_n(u)} \right) + \tilde{Z}_n(t) d \left( \int_{t_{i0}}^{t} \frac{du}{\tilde{Z}_n(u)} \right)$$

$$= d\tilde{Z}_n(t) \left( \frac{V_r}{Z_r} - \beta \int_{t_{i0}}^{t} \frac{du}{\tilde{Z}_n(u)} \right) + \tilde{Z}_n(t) \left( -\beta \int_{t_{i0}}^{t} \frac{dt}{\tilde{Z}_n(u)} \right)$$

(2.76)
Now, substituting (2.72) and (2.75), we have
\[
d\tilde{V}_i(t) = \left[\psi(t)\tilde{V}_i(t) - \beta_i\right] dt + \Phi(t)\tilde{V}_i(t) dW(t)
\]
\[
\tilde{V}_i(t_{i_0}) = V_r,
\]
(2.77)
which is a nonhomogeneous linear SDE. The solution of a nonhomogeneous linear SDE is obtained in 2.2.4, which is the case of (2.18), the stochastic solution of (2.77) and is given by
\[
\tilde{V}_i(t) = H(t) \left[1 - \beta_i \int_{t_{i_0}}^{t} \frac{ds}{H(s)}\right], \quad \text{for } t > t_{i_0}
\]
(2.78)
where
\[
H(t) = V_r \exp \left\{ \int_{t_{i_0}}^{t} \left[\psi(s) - \frac{1}{2}\Phi^2(s)\right] ds + \int_{t_{i_0}}^{t} \Phi(s)dW(s) \right\}.
\]
We conclude from (2.78) that \(\tilde{V}_i(t) = 0\), for \(t > t_{i_0}\), such that
\[
\int_{t_{i_0}}^{t} \frac{ds}{H(s)} = \frac{1}{\beta_i}.
\]
The Fokker-Planck equation of (2.77) is given by
\[
\frac{\partial \tilde{p}}{\partial t} = -\frac{\partial}{\partial \tilde{v}} \left[\left(\psi\tilde{v} - \beta_i\right) \tilde{p}\right] + \frac{1}{2} \Phi^2(t) \frac{\partial^2}{\partial \tilde{v}^2} \left(\tilde{v}^2 \tilde{p}\right)
\]
(2.79)
for \(t > t_{i_0}, \tilde{v} > 0\), where \(\tilde{p} = \tilde{p}(\tilde{v}, t \mid V_r, t_{i_0})\) is the transition probability density function of \(\tilde{V}_i(t)\), with the initial condition
\[
\lim_{t \to t_{i_0}} \tilde{p}(\tilde{v}, t \mid V_r, t_{i_0}) = \delta (\tilde{v} - V_r).
\]
(2.80)

### 2.8.1 Numerical scheme for the Fokker-Planck equation of the consumption process \(\tilde{V}_i(t)\)

We solve (2.79) in a similar manner to (2.53). As above, we choose the implicit BTCS method to approximate \(\tilde{p}(v, t)\). We discretize \(\mathbb{R}^2\) with \(\Delta k\) jumps in the \(t\)-axis, \(\Delta h\) jumps in the \(v\)-axis. We write
\[
p(v_j, t_n) = p^n_j, \quad \text{for } j, n \geq 0,
\]
(2.81)
where \( v_j = j \Delta h \), and \( t_n = n \Delta k \). We denote the numerical solution at time \( t_n \) by

\[
G[t_n, N_v, \Delta h] = \left\{ (j \Delta h, p^n_j) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 0 \leq j \leq N_v \right\},
\]

or simply \( G \) when the context of \( N_v \) and \( \Delta h \) is clear. We derive \( G[t_n, N_v, \Delta h] \) from \( G[t_{n-1}, N_v, \Delta h] \) by approximating the the derivatives of \( \tilde{p} \), and replacing these estimations in the FPE, which will impose an implicit relation between the 2 consecutive numerical solutions in time. The derivatives approximations for the Fokker-Planck equation of the consumption process are given by

\[
\frac{\partial \tilde{p}}{\partial t} \bigg|_{(v_j, t_n)} = \frac{\tilde{p}^n_j - \tilde{p}^{n-1}_j}{\Delta k}
\]

\[
\frac{\partial}{\partial \tilde{v}} \left[ - (\psi \tilde{v} - \beta_i) \tilde{p} \right] \bigg|_{(v_j, t_n)} = - \left[ \frac{(\psi(j + 1) \Delta h - \beta_i) \tilde{p}^{n+1}_j - (\psi(j - 1) \Delta h - \beta_i) \tilde{p}^{n-1}_j}{2 \Delta h} \right]
\]

\[
\frac{\partial^2}{\partial \tilde{v}^2} \left[ \frac{1}{2} \Phi^2(t) \tilde{v}^2 \tilde{p} \right] \bigg|_{(v_j, t_n)} = \frac{1}{2} \Phi^2(t_n) \left[ \frac{(j + 1) \Delta h)^2 \tilde{p}^{n+1}_j - 2(j \Delta h)^2 \tilde{p}^n_j + ((j - 1) \Delta h)^2 \tilde{p}^{n-1}_j}{\Delta h^2} \right]
\]

(2.83)

with the initial condition, \( \tilde{p}^0_j = \delta(\tilde{v} - V_j) \), for \( j > 0 \) and the boundary condition \( \tilde{p}^n_0 = 0 \). We substitute the scheme (2.83) into (2.79), to get

\[
\frac{\tilde{p}^n_j - \tilde{p}^{n-1}_j}{\Delta k} = - \left[ \frac{(\psi(j + 1) \Delta h - \beta_i) \tilde{p}^{n+1}_j - (\psi(j - 1) \Delta h - \beta_i) \tilde{p}^{n-1}_j}{2 \Delta h} \right]
\]

\[
+ \frac{1}{2} \Phi^2(t_n) \left[ \frac{(j + 1) \Delta h)^2 \tilde{p}^{n+1}_j - 2(j \Delta h)^2 \tilde{p}^n_j + ((j - 1) \Delta h)^2 \tilde{p}^{n-1}_j}{\Delta h^2} \right]
\]

(2.84)

which reduces to

\[
\frac{\tilde{p}^n_j - \tilde{p}^{n-1}_j}{\Delta k} = - \left[ \frac{1}{2} \psi(j + 1) - \frac{\beta_i}{2 \Delta h} \right] \tilde{p}^{n+1}_j - \left( \frac{1}{2} \psi(j - 1) - \frac{\beta_i}{2 \Delta h} \right) \tilde{p}^{n-1}_j
\]

\[
+ \frac{1}{2} \Phi^2(t_n) \left[ (j + 1)^2 \tilde{p}^{n+1}_j - 2j^2 \tilde{p}^n_j + (j - 1)^2 \tilde{p}^{n-1}_j \right]
\]

(2.85)

Finally, we implicitly express \( \tilde{p}^n_j, \tilde{p}^{n+1}_j, \tilde{p}^{n-1}_j \) in terms of \( \tilde{p}^{n-1}_j \);

\[
\tilde{p}^{n-1}_j = \lambda_C^n(j) \tilde{p}^n_j + \lambda_R^n(j) \tilde{p}^{n+1}_j + \lambda_L^n(j) \tilde{p}^{n-1}_j,
\]

(2.86)
where
\[
\tilde{\lambda}^n_C(j) = \Delta k \left[ \frac{1}{\Delta k} + \Phi^2(t_n) j^2 \right]
\]
\[
\tilde{\lambda}^n_R(j) = \Delta k \left[ \frac{1}{2} \psi(j + 1) - \frac{\beta_i}{2\Delta h} - \frac{1}{2} \Phi^2(t_n) (j + 1)^2 \right]
\]
\[
\tilde{\lambda}^n_L(j) = \Delta k \left[ -\left( \frac{1}{2} \psi(j - 1) - \frac{\beta_i}{2\Delta h} \right) - \frac{1}{2} \Phi^2(t_n) (j - 1)^2 \right]
\]
(2.87)

and
\[
\Phi^2(t_n) = \frac{\sigma^2 e^{\sigma^2 n \Delta k}}{e^{\sigma^2 n \Delta k} + N - 1}.
\]
(2.88)

2.8.2 The initial condition

The initial condition (2.80), where \( V_r \) is the total pension accumulated upon retirement, is approximated by a normal density with small standard deviation. Thus, we choose \( \tilde{p}_0^j \) to be the Gaussian density with standard deviation \( \tilde{\sigma}_v << 1 \) and mean \( \tilde{\mu} = V_r \), that is, for every \( j \Delta h \)

\[
p_j^0 = \frac{1}{\sqrt{2\pi\tilde{\sigma}_v}} \exp \left\{ \frac{(j \Delta h - V_r)^2}{2\tilde{\sigma}_v^2} \right\}.
\]
(2.89)

The initial density must be normalized by

\[
\int_{G[0,Nv,\Delta h]} \tilde{p}(v,0) \, dv = 1,
\]

therefore we define the normalized initial condition as

\[
q_j^0 = \frac{p_j^0}{\int_{G[0,Nv,\Delta h]} \tilde{p}(v,0) \, dv}.
\]
(2.90)

2.8.3 Boundary conditions

1. Absorbing boundary condition

The stochastic process \( \tilde{V}_i(t) \) cannot be negative, as the pensioner cannot over consume his pension. Consequently, we impose the absorbing boundary condition

\[
\tilde{p}(0,t) = 0, \quad \text{for } t \geq 0.
\]
2. Artificial boundary conditions

The approximate solution of the FPE is defined on a finite grid with the number of points

\[ |G[t_n, N_v, \Delta h]| = N_v + 1 < \infty. \]

Because for all \( t \geq 0 \)

\[ \lim_{v \to \infty} \tilde{p}(v, t) = 0, \]

we set an artificial boundary at the edge of the grid,

\[ \tilde{p}(N_v \Delta h, t_n) = 0. \]

3. Boundary layer

The boundary condition forces \( \tilde{p}(v, t) \) to vanish at \( v = 0 \), yet, for sufficiently large \( t \) the solution \( \tilde{p}(v, t) \) peaks near \( v = 0 \), so the gradient at \( v = 0 \) becomes large, thus giving rise to a numerical boundary layer. This is expressed in a numerical instability near \( v = 0 \) for large \( t \). Even though the solution decays as \( t \to \infty \), there is a numerically unstable transition layer in the \((v, t)\) plane. To overcome the instability, we refine the grid near the boundary. The contribution of the boundary layer to the integral \( \int_G \tilde{p}(v, t) dv \) is an integral over the boundary layer, which is negligible \(< 10^{-6}\).

2.8.4 Solution of the linear system

The numerical solution \( G[t_n, N_v] \) is recursively derived from the previous numerical solution \( G[t_{n-1}, N_v] \) by solving a system of linear equations. Because the the density vanishes on the boundary of the grid, we solve (2.86) for

\[
\text{int}(G) = G \setminus \{(N_v \Delta h, 0), (0, 0)\} = \left\{ (j \Delta h, p^n_j) \in \mathbb{R}^+ \times \mathbb{R}^+ \mid 1 \leq j \leq N_v - 1 \right\}, \quad (2.91)
\]

written in the matrix form as

\[
\begin{pmatrix}
\lambda^n_C(1) & \lambda^n_R(1) \\
\lambda^n_L(k) & \lambda^n_C(k) & \lambda^n_R(k) \\
\vdots \\
\lambda^n_L(K) & \lambda^n_C(K)
\end{pmatrix}
\begin{pmatrix}
\tilde{p}^n_{\Delta h} \\
\tilde{p}^n_{2\Delta h} \\
\tilde{p}^n_{3\Delta h} \\
\vdots \\
\tilde{p}^n_{K \Delta h}
\end{pmatrix}
= 
\begin{pmatrix}
\tilde{p}^{n-1}_{\Delta h} \\
\tilde{p}^{n-1}_{2\Delta h} \\
\tilde{p}^{n-1}_{3\Delta h} \\
\vdots \\
\tilde{p}^{n-1}_{K \Delta h}
\end{pmatrix}
\]

where \( K = N_v - 1 \), or in the abbreviated form

\[ \tilde{A}(n)\tilde{p}_n = \tilde{p}_{n-1}, \]
where $\tilde{A}(n)$ is a $(N_v - 1) \times (N_v - 1)$ tridiagonal coefficient matrix, $\vec{\tilde{p}}_n, \vec{\tilde{p}}_{n-1}$ are vectors, whose $j$-th entry is the density of $v_j = j\Delta h$, at times $n$ and $n - 1$, respectively. The $(i, j)$-th entry of $\tilde{A}(n)$ is given by

$$
\tilde{A}_{ij} = \begin{cases} 
\lambda_L^n(i) & (2 \leq i \leq N_v - 1) \land (j = i - 1) \\
\lambda_C^n(i) & j = i \\
\lambda_R^n(i) & (1 \leq i \leq N_v - 2) \land (j = i + 1) \\
0 & \text{else}
\end{cases}
$$

(2.92)

The numerical solution $G[t_n, N_v]$ is obtained by computing the inverse the matrix $\tilde{A}(n)$, and multiplying it by the previous numerical solution, that is,

### 2.8.5 Survival probability for the consumption process

The first time of the consumption process $\tilde{V}_i(t)$ to zero is defined by

$$
\tau = \inf \left\{ t > t_0 \mid \tilde{V}_i(t) = 0 \right\}
$$

and the survival probability is defined as

$$
S(t \mid V_r, t_0) = \Pr (\tau > t \mid V_r, t_0) = \int_{\tilde{G} \setminus \tilde{G}} \tilde{p}(\tilde{v}, t \mid V_r, t_0) \, d\tilde{v},
$$

(2.93)

where the pdf of $\tilde{V}(t)$ is the solution for the FPE (2.79) for $\tilde{v} > 0$ with the initial condition and boundary conditions

$$
\tilde{p}(\tilde{v}, t_0 \mid V_r, t_0) = \delta(\tilde{v} - V_r), \quad \tilde{p}(0, t \mid V_r, t_0) \quad \text{for } t > t_0.
$$

Changing variables to $\tilde{v} = V_r x$ to (2.92), we obtain

$$
\frac{\partial q}{\partial \tilde{v}} = -\frac{\partial}{\partial x} \left[ \left( \psi x - \frac{\beta_i}{V_r} \right) q \right] + \frac{1}{2} \Phi^2(t) \frac{\partial^2}{\partial x^2} \left( x^2 q \right)
$$

(2.94)

for $t > t_{i_0}, x > 0$, where $q(x, t)$ is the pdf of $\tilde{V}_i(t)/V_r$, with the initial condition and boundary conditions

$$
\tilde{q}(x, t_0 \mid x_0, t_{i_0}) = \delta (x - 1), \quad \tilde{q}(0, t \mid x_0, t_{i_0}) = 0.
$$

We solve (2.93) for different retirement periods, and different ratios of the parameter $\beta_i/V_r$. The Internal Rate of Return (IRR) of a retirement period of $n$ years is the return $r$ that satisfies

$$
\sum_{i=1}^{t} \frac{\beta_i}{(1 + r)^i} = V_r.
$$

The results are summarized in the tables below.
<table>
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<tr>
<th>Initial Pension</th>
<th>Retirement Period</th>
<th>IRR</th>
<th>Survival Probability</th>
</tr>
</thead>
<tbody>
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<td>Yearly Consumption</td>
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<td>8 Years</td>
<td>1.45%</td>
</tr>
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<td>9 Years</td>
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<td>2.12%</td>
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<tr>
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<td>2.92%</td>
</tr>
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<td></td>
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<td>16 Years</td>
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</tr>
<tr>
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<td>17 Years</td>
<td>4.17%</td>
</tr>
<tr>
<td></td>
<td></td>
<td>18 Years</td>
<td>4.66%</td>
</tr>
</tbody>
</table>

Table 2.3: Survival probabilities for consumption periods of 7.5, 10 and 12 years of uninvested pension

<table>
<thead>
<tr>
<th>Initial Pension</th>
<th>Retirement Period</th>
<th>IRR</th>
<th>Survival Probability</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yearly Consumption</td>
<td>12.5 Years</td>
<td>13 Years</td>
<td>0.56%</td>
</tr>
<tr>
<td></td>
<td>14 Years</td>
<td>1.54%</td>
<td>64.46%</td>
</tr>
<tr>
<td></td>
<td>15 Years</td>
<td>2.37%</td>
<td>42.61%</td>
</tr>
<tr>
<td></td>
<td>16 Years</td>
<td>3.06%</td>
<td>26.14%</td>
</tr>
<tr>
<td></td>
<td>17 Years</td>
<td>3.65%</td>
<td>16.68%</td>
</tr>
<tr>
<td></td>
<td>18 Years</td>
<td>4.15%</td>
<td>11.4%</td>
</tr>
<tr>
<td></td>
<td>19 Years</td>
<td>4.58%</td>
<td>8.12%</td>
</tr>
<tr>
<td></td>
<td>20 Years</td>
<td>4.96%</td>
<td>5.84%</td>
</tr>
<tr>
<td></td>
<td>15 Years</td>
<td>15 Years</td>
<td>0.00%</td>
</tr>
<tr>
<td></td>
<td>20 Years</td>
<td>2.91%</td>
<td>28.93%</td>
</tr>
<tr>
<td></td>
<td>25 Years</td>
<td>4.38%</td>
<td>3.48%</td>
</tr>
<tr>
<td></td>
<td>30 Years</td>
<td>5.21%</td>
<td>0.43%</td>
</tr>
<tr>
<td></td>
<td>16.25 Years</td>
<td>20 Years</td>
<td>2.06%</td>
</tr>
<tr>
<td></td>
<td>25 Years</td>
<td>3.63%</td>
<td>9.61%</td>
</tr>
<tr>
<td></td>
<td>30 Years</td>
<td>4.52%</td>
<td>1.08%</td>
</tr>
<tr>
<td></td>
<td>35 Years</td>
<td>5.06%</td>
<td>0.09%</td>
</tr>
</tbody>
</table>

Table 2.4: Survival probabilities for consumption periods of 12.5, 15 and 16.25 years of uninvested pension
2.8.6 Mean first passage time of the consumption process

The mean first passage Time (MFPT), is given by [30]

\[ E[\tau | V_r, \tau > t_0] = \int_{t_0}^{\infty} \Pr \{ \tau > t | V_r, t_0 \} \, dt = \int_{t_0}^{\infty} \int_{0}^{\infty} \tilde{p}(\tilde{v}, t | V_r, t_0) \, d\tilde{v} \, dt. \]  

(2.95)

The results are summarized in the table below

<table>
<thead>
<tr>
<th>Pension/Consumption</th>
<th>MFPT</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.50 Years</td>
<td>8.27 Years</td>
</tr>
<tr>
<td>10.00 Years</td>
<td>11.29 Years</td>
</tr>
<tr>
<td>12.00 Years</td>
<td>13.86 Years</td>
</tr>
<tr>
<td>12.50 Years</td>
<td>14.53 Years</td>
</tr>
<tr>
<td>15.00 Years</td>
<td>18.16 Years</td>
</tr>
<tr>
<td>16.25 Years</td>
<td>20.15 Years</td>
</tr>
</tbody>
</table>

Table 2.5: Mean first passage times for different consumption rates of uninvested pensions.

2.8.7 The Probability that the pension survives the pensioner

We assume that the time of death of a given pensioner is a random variable \( T \), with pdf \( f_T(t) \). The probability that the pension survives the pensioner, or the probability that the pensioner dies before he/she runs out of money, is given by

\[ \Pr \{ \tau > T \} = \int_{t_0}^{\infty} \Pr \{ \tau > T | T = t \} f_T(t | t_0) \, dt = \int_{t_0}^{\infty} S(t) f_T(t | t_0) \, dt. \]  

(2.96)

The distribution of life expectancy is taken from US Department of Health and Human Services (HHS), Centers for Disease Control and Prevention (CDC), and is based on US population [44]. See figure 4.25,table 4.1 (p.64-67). We compute the probabilities that the pension survives the pensioner, for the retirements ages 67 & 72, for different pension/consumption ratios.

<table>
<thead>
<tr>
<th>Initial Pension</th>
<th>Chance to Die</th>
</tr>
</thead>
<tbody>
<tr>
<td>Yearly Consumption</td>
<td>Before Pension is Consumed</td>
</tr>
<tr>
<td>7.5 Years</td>
<td>19.18%</td>
</tr>
<tr>
<td>10 Years</td>
<td>28.65%</td>
</tr>
<tr>
<td>12 Years</td>
<td>54.70%</td>
</tr>
<tr>
<td>12.5 Years</td>
<td>60.29%</td>
</tr>
<tr>
<td>15 Years</td>
<td>67.43%</td>
</tr>
<tr>
<td>16.25 Years</td>
<td>72.62%</td>
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</tbody>
</table>

Table 2.6: Pensions surviving 67 years old pensioners
<table>
<thead>
<tr>
<th>Initial Pension Yearly Consumption</th>
<th>Chance to Die Before Pension is Consumed</th>
</tr>
</thead>
<tbody>
<tr>
<td>7.5 Years</td>
<td>28.18%</td>
</tr>
<tr>
<td>10 Years</td>
<td>40.93%</td>
</tr>
<tr>
<td>12 Years</td>
<td>60.70%</td>
</tr>
<tr>
<td>12.5 Years</td>
<td>65.39%</td>
</tr>
<tr>
<td>15 Years</td>
<td>78.13%</td>
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<tr>
<td>16.25 Years</td>
<td>87.78%</td>
</tr>
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</table>

Table 2.7: Pensions surviving 72 years old pensioners
Chapter 3

Summary & Conclusions

In this dissertation we developed a stochastic model that is based on historical data. We find that CPI-adjusted salaries decrease over time, while CPI-adjusted market returns drift upwards. Assuming that past dynamics persist in the future, our model estimates that the pensioner is likely to accumulate 7.5 times his initial annual salary over 40 years of pension savings. Assuming 10% salary contribution, this means that the Super Trust’s CPI-adjusted average annual return is 2.85%. We find that there is only 19.18% chance for this pension to be sufficient, assuming retirement age at 67. In other words, a salaried employee, who worked his entire life without receiving any substantial promotions, bonuses or extra incomes, has about 80% chance to live his last 10-15 years in poverty. Raising the retirement age to 72 is expected to bring these chances down to about 70%. These results shed further light on the imminence and significance of the pension poverty problem.

In addition to the Super Trust’s projected performance, the investment strategy employed has economic implications that ought not be overlooked. Owning companies on a national scale might form a centrally planned economy, in which the government owns a portion of the means of production. This could lead to economic inefficiencies such as surplus and shortages of goods and services. The management of a state-scale conglomerate might be difficult and could create opportunities for corruption. Tax money is funnelled into the private sector in the USA by government contracts for projects and services, but not necessarily into direct subsidies for investors or in the form of preferential tax breaks. The latter is a common practice in Israel, though.

The contrast between the Ten Pillars Program and direct investment in the stock market can be summarized as follows. In a free Market, the management usually is partially compensated with the company’s shares in order to create incentive for improving the company’s performance. Also, compensation is often tied to stock performance. Furthermore, some of the compensation comes in the form of stock options. Thus the management is given incentive to generate profits and to be committed to the company’s success. However, in the Ten Pillars Program, the government is the shareholder of the pension money. It cannot offer shares or options of the pension money to its management, and the compensation is regulated and moderate. The management becomes a mere conduit of pension funds. It is not immediately clear why the management would be driven by profit. These settings, amongst
other factors, make the public sector less efficient than the private sector and therefore less suitable for managing the pension money.

To conclude, there is an imminent urge for the structuring of a long-term investment scheme that secures the pensioner welfare, yet it is a complicated, large-scale problem. Based on numerical results, a symptomatic treatment of the pension problem can be achieved by raising the retirement age and increasing contributions. However, in order to achieve a systematic solution, researchers from the entire scientific spectrum need to contribute to the effort.
Chapter 4

Figures

Figure 4.1: (Left) Cyan S&P500 CPI-adjusted net income growth, $R(t)$. (Left) Black Simple moving average of $R(t)$. (Right) Cyan S&P500 CPI-adjusted returns. (Right) Black Simple moving average of S&P500 CPI-adjusted net earnings growth

Figure 4.2: (Left) Cyan S&P50 CPI-adjusted net income growth, $R(t)$. (Left) Black Simple moving average of $R(t)$. (Right) Cyan S&P50 CPI-adjusted returns. (Right) Black Simple moving average of S&P50 CPI-adjusted net earnings growth
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Figure 4.4: (Cyan) Stock returns drift as a function of growth between 1983 and 2002 (Black) Linear interpolant
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Figure 4.10: **(Left) Cyan** The leading quadratic coefficient, $r(t)$, between 1970 and 2011 **(Left) Black** The simple moving average of $r(t)$, with 5% window **(Middle) Cyan** The coefficient $r_2(t)$ between 1970 and 2011 **(Middle) Black** The simple moving average of the constant terms $r_2(t)$, with 5% window **(Right) Cyan** The coefficient $r_3(t)$ between 1970 and 2011 **(Right) Black** The simple moving average of the constant terms $r_3(t)$, with 5% window
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**(Black)** A polynomial fit of degree 18
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Figure 4.15: The amount of PSID-surveyed individuals entering work force, by year.
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Figure 4.20: **(Left) Blue** The leading quadratic coefficient, $\eta(t)$, between 1970 and 1992 **(Left) Green** The average of $\eta(t)$ **(Middle) Blue** The coefficient $\eta_2(t)$ between 1970 and 1992 **(Middle) Green** The average of $\eta_2(t)$ **(Right) Blue** The coefficient $\eta_3(t)$ between 1970 and 1992 **(Right) Green** The average of the constant terms $\eta_3(t)$
Figure 4.21: The joint pdf $p(v, s, t)$ for 25 years. Grid on $v$-axis has 720 points, spaced 0.03 apart, $s$-axis has 25 points, spaced 0.2 apart. $\Delta t = 0.1$

Figure 4.22: The joint pdf $p(v, s, t)$ for 25 years. (Left) $p(v, s, t)$ projected on the $v$-axis. (Right) $p(v, s, t)$ projected on the $s$-axis
Figure 4.23: The joint pdf $p(v, s, t)$ for 50 years. Grid on $v$-axis has 720 points, spaced 0.03 apart, $s$-axis has 25 points, spaced 0.2 apart. $\Delta t = 0.1$

Figure 4.24: The joint pdf $p(v, s, t)$ for 50 years. (Left) $p(v, s, t)$ projected on the $v$-axis. (Right) $p(v, s, t)$ projected on the $s$-axis
Figure 4.25: The pdf of the age of death, US 2003
<table>
<thead>
<tr>
<th>Age</th>
<th>Probability of dying between ages x to x+1</th>
<th>Number surviving to age x</th>
<th>Number dying ages between ages x to x+1</th>
<th>Person-years lived between ages x to x+1</th>
<th>Total number of person-years lived above age x</th>
<th>Expec. of life at age x</th>
</tr>
</thead>
<tbody>
<tr>
<td>0-1</td>
<td>0.006865</td>
<td>100,000</td>
<td>687</td>
<td>99,394</td>
<td>7,743,016</td>
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<td>99,313</td>
<td>47</td>
<td>99,290</td>
<td>7,643,622</td>
<td>77.0</td>
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<td>2-3</td>
<td>0.000337</td>
<td>99,267</td>
<td>33</td>
<td>99,250</td>
<td>7,544,332</td>
<td>76.0</td>
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<td>3-4</td>
<td>0.000254</td>
<td>99,233</td>
<td>25</td>
<td>99,221</td>
<td>7,445,082</td>
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<tr>
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<td>0.000194</td>
<td>99,208</td>
<td>19</td>
<td>99,199</td>
<td>7,345,861</td>
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<td>99,180</td>
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<td>99,134</td>
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<td>0.000954</td>
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<td>98,267</td>
<td>5,367,101</td>
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<td>97,800</td>
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<td>Age</td>
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<td>Number surviving to age x</td>
<td>Number dying ages between ages x to x+1</td>
<td>Person-years lived between ages x to x+1</td>
<td>Total number of person-years lived above age x</td>
<td>Expec. of life at age x</td>
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<td>------------------------------------------</td>
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Table 4.1: The distribution of life span in the USA, 2003 (CDC)
Bibliography


[38] Zelen, M., Norman C. Severo *Handbook of mathematical functions with formulas, graphs, and mathematical tables*, Dover Publications, Mineola, NY (1964)


