Diversity-Multiplexing Tradeoff of Infinite Constellations in MIMO Fading Channels

by

Yair Yona

THESIS SUBMITTED TO THE SENATE OF TEL-AVIV UNIVERSITY
in partial fulfillment of the requirements for the degree of
"DOCTOR OF PHILOSOPHY"

March, 2014
Diversity-Multiplexing Tradeoff of Infinite Constellations in MIMO Fading Channels

by

Yair Yona

THESIS SUBMITTED TO THE SENATE OF TEL-AVIV UNIVERSITY
in partial fulfillment of the requirements for the degree of
"DOCTOR OF PHILOSOPHY"

Under the Supervision of Prof. Meir Feder

March, 2014
This work was carried out under the supervision of

Prof. Meir Feder
The past four years have been an incredible journey that provided me with some of the most significant experiences of my life. Throughout this period I was fortunate enough to have Meir Feder as my supervisor. Meir had both the open mindedness to go along and cooperate with my ideas, and also the vision and innovation that enabled to take this work a few steps forward. I will always be grateful to Meir for believing in me, giving me the opportunity to develop as a researcher, and sharing with me his wide knowledge and experience.

I also wish to thank Ram Zamir and Uri Erez for many interesting and insightful discussions both on my research and on information theory in general, and for always having their door open when I came.

Another thanks goes to Ofir Shalvi, Naftali Sommer and Joseph Boutrus for interesting discussions in the early stages of this research, that planted the seeds for this work.

I would also like to thank my dear past and present colleagues and friends in the 102 lab, who were not only the source for numerous exciting discussions, but also made this period fun and turned the lab into such a special place. Special thanks to Eli Haim, Ronen Dar, Or Ordentlich, Anatoly Khina, Oded Fishler, Yuval Domb, Yuval Lomnitz, Shlomi Vituri, Amir Ingber, Oz Harel, Idan Livni, Ayal Hitron, Ofer Shayevitz, Ohad Barak, Tal Philosof, Yuval Kochman, Idan Goldenberg, Sergey Tridenski and Oron Levi.

Finally, I wish to thank my wife, Ital, for her endless support and love, and to my beloved daughter Lior who brings light into my life.

This research was supported in part by the Yitzhak and Chaya Weinstein Research Institute for Signal Processing.
Abstract

The optimal diversity-multiplexing tradeoff (DMT) is a fundamental relation between rate and reliability in multiple-input multiple-output (MIMO) fading channels, for high signal to noise ratios. For each multiplexing gain the optimal DMT equals to the best diversity order that can be attained. In addition, it sets a unified framework that enables to compare the performance of different transmission schemes designed for multiple-antenna channels.

Infinite constellations (IC’s) are countable sets in the Euclidean space characterized by their density which essentially equals to the average number of points within a unit volume. When considering Poltyrev’s setting the encoder transmits a finite subset of the IC that satisfies a power constraint, while the receiver decodes over the IC without taking into consideration the power constraint. In this work we extend Poltyrev’s setting from the additive white Gaussian noise channel to MIMO fading channels, in order to find the best DMT that IC’s can attain.

In the first part of this work we investigate the natural connection between the IC dimension and the best DMT it can achieve in the point-to-point Rayleigh MIMO flat-fading channel. We derive an upper bound on the DMT of any IC of a certain dimension, given a certain number of transmit and receive antennas. Then, by considering the ”optimal“ dimensions for which the upper bound coincides with the optimal DMT, we prove that IC’s in general and lattices using regular lattice decoding in particular achieve the optimal DMT of finite constellations. This analysis provides another geometrical interpretation to the optimal DMT.

In the second part of this work we analyze the DMT of IC’s for the multiple-access (MAC) channel with $K$ users, where each user transmits using $M$ antennas to a receiver with $N$ antennas. Interestingly, unlike the point-to-point case, we show that for the MAC channel IC’s attain the optimal DMT only when $1 \leq K \leq \max\left(1, \frac{N-M+1}{M}\right)$, i.e. user limited regime. On the other hand when $K > \max\left(1, \frac{N-M+1}{M}\right)$ IC’s are suboptimal. Understanding the reasons for optimality and suboptimality of IC’s for the MAC channel provides a geometrical explanation why for $1 \leq K \leq \max\left(1, \frac{N-M+1}{M}\right)$ the optimal DMT of the MAC channel equals to the optimal DMT of a point-to-point channel with $M$ transmit and $N$ receive antennas.

In the third part of this work we apply low density lattice codes (LDLC’s) to the MIMO fading channel, where LDLC’s are lattice codes characterized by the sparseness of the inverse of their generator matrix. First, we derive a max-product algorithm for LDLC’s. For the additive white Gaussian noise channel we reveal an interesting connection between the passed messages of the max-product and sum-product algorithms. In addition, numerical results show that the max-product algorithm attains better block error rate than the sum-product algorithm for small dimensional LDLC’s. Then, we extend the max-product algorithm to general MIMO channels by using an extended tree assumption. When the lattice symbols have Gaussian a-priori probabilities we get an interesting connection between the passed messages and minimum-mean square error estimation. Finally, we design a very sparse LDLC and also propose a transmission scheme for the MIMO channel with two transmit and two receive antennas. The combination of the sparse LDLC and the transmission scheme enables to attain performance comparable to state of the art codes, while maintaining low computational complexity.
## Contents

1 **Introduction**

2 **Optimal DMT of Infinite Constellations (IC’s) in Point to Point Channels**
   2.1 Introduction ................................................................. 7
   2.2 Basic Definitions .......................................................... 9
   2.3 Upper Bound on the Diversity Order ................................. 12
   2.4 Attaining the Best Diversity Order ................................... 17
      2.4.1 The Transmission Scheme ........................................... 18
      2.4.2 The Effective Channel ............................................... 18
      2.4.3 Upper Bound on The Error Probability ........................... 20
      2.4.4 Achieving the Optimal DMT ....................................... 21
      2.4.5 Power Spreading ..................................................... 24
      2.4.6 Averaging Arguments ............................................... 25
   2.5 Discussion ................................................................. 26
      2.5.1 Lattice Constellations Vs. Full Dimension Lattice Based Finite Constellations . . . 26
      2.5.2 Geometrical Interpretation of the Optimal DMT for IC’s ........... 28
      2.5.3 Example for the Case $M = N = 2$ ............................... 31
      2.5.4 The Relation Between the Multiplexing Gains of an IC and a Finite Constellation . . . 32

3 **On the DMT of IC’s in MAC Channels**
   3.1 Introduction ................................................................. 33
   3.2 Basic Definitions .......................................................... 35
      3.2.1 Channel Model .......................................................... 35
      3.2.2 Infinite Constellations .............................................. 36
      3.2.3 Additional Notations ................................................. 37
   3.3 Upper Bound on the Best Diversity-Multiplexing Tradeoff .......... 38
      3.3.1 Upper Bound on the Diversity-Multiplexing-Tradeoff ............. 38
      3.3.2 Characterizing the Optimal Symmetric DMT ........................ 40
      3.3.3 Comparison to Finite Constellations .............................. 44
      3.3.4 Discussion: Convexity Vs. Non-Convexity of the Optimal DMT .... 45
   3.4 Attaining the Optimal DMT for $N \geq (K + 1) M - 1$ ............... 48
<table>
<thead>
<tr>
<th>Section</th>
<th>Title</th>
<th>Page</th>
</tr>
</thead>
<tbody>
<tr>
<td>A.5</td>
<td>Proof of Theorem 2.4</td>
<td>101</td>
</tr>
<tr>
<td>A.6</td>
<td>Proof of Lemma A.2</td>
<td>104</td>
</tr>
<tr>
<td>A.7</td>
<td>Proof of Theorem 2.5</td>
<td>105</td>
</tr>
<tr>
<td>A.8</td>
<td>Proof of Corollary 2.3</td>
<td>107</td>
</tr>
<tr>
<td>A.9</td>
<td>Proof of Corollary 2.5</td>
<td>107</td>
</tr>
<tr>
<td>B.1</td>
<td>Proof of Lemma 3.2</td>
<td>111</td>
</tr>
<tr>
<td>B.2</td>
<td>Proof of Lemma 3.3</td>
<td>112</td>
</tr>
<tr>
<td>B.3</td>
<td>Proof of Lemma 3.4</td>
<td>113</td>
</tr>
<tr>
<td>B.4</td>
<td>Proof of Lemma 3.5</td>
<td>114</td>
</tr>
<tr>
<td>B.5</td>
<td>Proof of Theorem 3.4</td>
<td>118</td>
</tr>
<tr>
<td>B.6</td>
<td>Proof of Lemma 3.6</td>
<td>123</td>
</tr>
<tr>
<td>B.7</td>
<td>Proof of Theorem 3.5</td>
<td>124</td>
</tr>
<tr>
<td>B.8</td>
<td>Final Part of the Proof of Theorem 3.5</td>
<td>126</td>
</tr>
<tr>
<td>B.9</td>
<td>Proof of Theorem 3.6</td>
<td>130</td>
</tr>
<tr>
<td>B.10</td>
<td>Proof of Lemma 3.8</td>
<td>133</td>
</tr>
<tr>
<td>B.11</td>
<td>Proof of Theorem 3.7</td>
<td>134</td>
</tr>
<tr>
<td>B.12</td>
<td>Proof of Lemma B.1</td>
<td>138</td>
</tr>
<tr>
<td>B.13</td>
<td>Proof of Lemma B.2</td>
<td>142</td>
</tr>
<tr>
<td>B.14</td>
<td>Proof of Theorem 3.8</td>
<td>142</td>
</tr>
<tr>
<td>B.15</td>
<td>Proof of Corollary 3.2</td>
<td>144</td>
</tr>
</tbody>
</table>

References | 147 |
List of Figures

1.1 Optimal DMT for \( M = N = 2 \) ................................................................. 2

2.1 Upper bound on the DMT of any IC for different NDCU ............................. 16
2.2 \( d_{4,3}^{*,D}(0) \) as a function of the NDCU .................................................. 17
2.3 Example of the significance of reducing the NDCU ..................................... 29

3.1 Upper bound on the DMT of any IC for \( M = N = 2 \) and different NDCU .... 41
3.2 Illustration of Lemma 3.2 .......................................................................... 42
3.3 Illustration of Lemma 3.4 .......................................................................... 43
3.4 \( d^*(r) \) for the case \( M = K = 2 \) and \( N = 4 \) ........................................... 43
3.5 Upper bound on the optimal DMT of IC’s in the symmetric case for \( K = 2, M = 3, N = 6 \). 46
3.6 Illustration of the sub-optimality of the unconstrained multiple-access channel 47
3.7 Comparison between the optimal DMT of finite constellations and the upper bound on the optimal DMT of IC’s ................................................................. 48

4.1 Comparison between the max-product and sum-product algorithms .............. 70
4.2 The extended tree structure ...................................................................... 72
4.3 The extended bipartite graph ..................................................................... 74
4.4 Normalized word error rate for different block lengths for the AWGN channel 83
4.5 Performance of CLDLC in Rayleigh MIMO fading channel for \( M = N = 2 \) ........ 84
# List of Acronyms

<table>
<thead>
<tr>
<th>Acronym</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>ARQ</td>
<td>Automatic Retransmission Request</td>
</tr>
<tr>
<td>AWGN</td>
<td>Additive White Gaussian Noise</td>
</tr>
<tr>
<td>CDMA</td>
<td>Code-division Multiple-Access</td>
</tr>
<tr>
<td>CLDLC</td>
<td>Complex Low-Density Lattice Code</td>
</tr>
<tr>
<td>DMT</td>
<td>Diversity-Multiplexing Tradeoff</td>
</tr>
<tr>
<td>FC</td>
<td>Finite Constellations</td>
</tr>
<tr>
<td>IC</td>
<td>Infinite Constellation</td>
</tr>
<tr>
<td>IID</td>
<td>Independent and Identically Distributed</td>
</tr>
<tr>
<td>LAST</td>
<td>Lattice Space-Time</td>
</tr>
<tr>
<td>LDPC</td>
<td>Low-Density Parity Check</td>
</tr>
<tr>
<td>LDLC</td>
<td>Low-Density Lattice Code</td>
</tr>
<tr>
<td>MAC</td>
<td>Multiple Access</td>
</tr>
<tr>
<td>MAP</td>
<td>Maximum A-Posteriori</td>
</tr>
<tr>
<td>ML</td>
<td>Maximum Likelihood</td>
</tr>
<tr>
<td>MIMO</td>
<td>Multiple-Input Multiple-Output</td>
</tr>
<tr>
<td>MMSE</td>
<td>Minimum-Mean Square Error</td>
</tr>
<tr>
<td>NDCU</td>
<td>Number of Dimensions Per Channel Use</td>
</tr>
<tr>
<td>NLD</td>
<td>Normalized Logarithmic Density</td>
</tr>
<tr>
<td>NVD</td>
<td>Non-Vanishing Determinant</td>
</tr>
<tr>
<td>OFDM</td>
<td>Orthogonal Frequency-Division Multiplexing</td>
</tr>
<tr>
<td>PDF</td>
<td>Probability Density Function</td>
</tr>
<tr>
<td>QAM</td>
<td>Quadrature Amplitude Modulation</td>
</tr>
<tr>
<td>SISO</td>
<td>Single-Input Single-Output</td>
</tr>
<tr>
<td>SNR</td>
<td>Signal-to-Noise Ratio</td>
</tr>
<tr>
<td>TDMA</td>
<td>Time-Division Multiple-Access</td>
</tr>
<tr>
<td>V-BLAST</td>
<td>Vertical Bell Labs Space-Time Architecture</td>
</tr>
<tr>
<td>VNR</td>
<td>Volume-to-Noise Ratio</td>
</tr>
<tr>
<td>WER/NWER</td>
<td>Word Error Rate/Normalized Word Error Rate</td>
</tr>
</tbody>
</table>
### List of Symbols

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\mathbb{C}$</td>
<td>The set of complex numbers</td>
</tr>
<tr>
<td>$\text{cube}_l(a)$</td>
<td>$l$-complex dimensional cube</td>
</tr>
<tr>
<td>$M(S, a)$</td>
<td>The number of points within $S \cap \text{cube}_l(a)$</td>
</tr>
<tr>
<td>$\gamma, \gamma(S)$</td>
<td>The density of the IC $S$</td>
</tr>
<tr>
<td>$\mu, \mu(S, \rho^{-1})$</td>
<td>VNR of IC $S$ and additive noise with variance $\rho^{-1}$</td>
</tr>
<tr>
<td>$V(x)$</td>
<td>The Voronoi region of IC point $x$</td>
</tr>
<tr>
<td>$r_{\text{eff}}(x)$</td>
<td>The effective radius of IC point $x$</td>
</tr>
<tr>
<td>$d_{\text{min}}^{(\text{lattice})}$</td>
<td>Lattice minimal distance</td>
</tr>
<tr>
<td>$r_{\text{packing}}$</td>
<td>Lattice packing radius</td>
</tr>
<tr>
<td>$M$</td>
<td>Number of transmit antennas</td>
</tr>
<tr>
<td>$N$</td>
<td>Number of receive antennas</td>
</tr>
<tr>
<td>$L$</td>
<td>$\text{min} (M, N)$</td>
</tr>
<tr>
<td>$H$</td>
<td>The fading channel matrix</td>
</tr>
<tr>
<td>$H_{\text{ex}}$</td>
<td>The extended fading channel matrix</td>
</tr>
<tr>
<td>$r$</td>
<td>Multiplexing gain</td>
</tr>
<tr>
<td>$d$</td>
<td>Diversity order</td>
</tr>
<tr>
<td>$\gamma_{\text{tr}}/\gamma_{\text{rc}}$</td>
<td>Density in the transmitter/receiver</td>
</tr>
<tr>
<td>$\mu_{\text{tr}}/\mu_{\text{rc}}$</td>
<td>VNR in the transmitter/receiver</td>
</tr>
<tr>
<td>$P_e(H, \rho)$</td>
<td>Average error probability for a certain channel realizations</td>
</tr>
<tr>
<td>$\overline{P_e}(\rho)$</td>
<td>Average error probability over all channel realizations</td>
</tr>
<tr>
<td>$D$</td>
<td>NDCU</td>
</tr>
<tr>
<td>$d_{M,N}^{(\text{FC})}(r)$</td>
<td>Optimal DMT of the point-to-point channel</td>
</tr>
<tr>
<td>$d_{M,N}^{(\text{IC})}(r)$</td>
<td>Upper bound on the optimal DMT of any IC in a point-to-point channel</td>
</tr>
<tr>
<td>$d_{M,N}^{(D)}(r)$</td>
<td>Upper bound on the DMT of any IC with $D$ NDC</td>
</tr>
<tr>
<td>$K$</td>
<td>Number of users in a MAC channel</td>
</tr>
<tr>
<td>$d_{K,M,N}^{(\text{FC})}(r)$</td>
<td>The optimal DMT of the MAC channel in the symmetric case</td>
</tr>
<tr>
<td>$d_{K,M,N}^{(\text{IC})}(r_1, \ldots, r_K)$</td>
<td>Upper bound on the optimal DMT of any IC in the MAC channel in the symmetric case</td>
</tr>
<tr>
<td>$d_{K,M,N}^{(D)}(r)$</td>
<td>Upper bound on the optimal DMT of any IC in the MAC channel in the symmetric case</td>
</tr>
</tbody>
</table>
\[ f_X(x) \quad \text{The PDF of the random vector } X \]
\[ f_{Y|X}(y|x) \quad \text{The PDF of the random vector } Y \text{ given } X \]
\[ \odot \quad \text{Conv-sup} \]
Chapter 1

Introduction

Wireless communication has an immense impact on everyday life. Today it can be found almost everywhere, both indoors (Wifi, bluetooth) and outdoors (cellular communication). Vast amount of devices and applications ranging from cloud storage services to medical equipments rely heavily on the portability provided via the wireless networks. In order to satisfy these demands researchers and engineers need to face two main challenges characterizing the wireless channel: Fading and interference [3].

A mean to overcome these obstacles is employing multiple antennas at the transmitter and the receiver ends. On one hand multiple antennas enable to increase the transmitted signal reliability for a certain rate, and on the other hand to increase the potential rate for a certain level of reliability. A measure for the reliability level of a wireless systems is the diversity order which represents how fast the average error probability decays with the signal to noise ratio (SNR). If the average error probability decays for large SNR as $\text{SNR}^{-d}$ we state that the diversity order is $d$. In addition, for large values of SNR it may be meaningful to ask what is the number of degrees of freedom that a transmission scheme utilizes. In this respect when the rate increases as $r \cdot \log(\text{SNR})$ we state that the transmission scheme has multiplexing gain $r$, i.e. it utilizes $r$ degrees of freedom of the channel. As a naive example for increasing the diversity order via multiple antennas for a certain multiplexing gain, consider a channel with a single transmit antenna and several receive antennas. Assuming the paths are independent and identically distributed (i.i.d), the probability to receive a strong fading that leads to large error probability decreases with the number of antennas, and so the diversity order equals to the number of paths. Moreover, multiple antennas add spatial or polarized degrees of freedom to the classical time and frequency degrees of freedom. For instance, the channel capacity of an ergodic channel with $M$ transmit and $N$ receive antennas, where the paths have i.i.d Rayleigh fading distribution, is approximately $\min(M, N) \cdot \log(\text{SNR})$ for large values of SNR [28], [8]. Also for the non-ergodic case where the transmitter does not know the channel and it is drawn once at the beginning of the transmission, increasing the number of antennas potentially enables to increase the multiplexing gain for a certain value of diversity order.

Understanding the benefits of multiple antennas in wireless channels arises the following fundamental question: What is the tradeoff between reliability and rate. For large SNR values Zheng and Tse [35] formulated and answered this question and presented for multiple-input multiple-output (MIMO) fading
channels the optimal diversity-multiplexing tradeoff (DMT). Essentially, for each multiplexing gain the optimal DMT equals to the best diversity order that can be attained. The tradeoff shows that increasing the multiplexing gain decreases the diversity order, i.e. increasing the rate decreases the reliability. It also provides unified framework that enables to compare transmission schemes designed for multiple antenna channels, by comparing their performance to the optimal DMT.

One of the basic scenarios that also has practical significance is the quasi-static Rayleigh flat-fading channel. In this model it is assumed that the channel is perfectly known at the receiver and that it is not known at the transmitter. The flat-fading assumption enables to model the effect of the fading by a matrix multiplying the transmitted signal, where the matrix columns represent the transmit antennas and the rows represent the receive antennas. An example for a transmission method for which the flat-fading assumption holds is orthogonal frequency-division multiplexing (OFDM) [29]. The channel is drawn once at the beginning of the transmission and remains constant. Zheng and Tse [35] found the optimal DMT for this setting and showed that it consists of a piecewise linear function connecting the points \((M - l) \cdot (N - l), \quad l = 0, \ldots, \min(M, N)\). Essentially, this result shows that the maximal diversity order equals \(M \cdot N\), which is the number of independent paths, and the maximal multiplexing gain that enables to attain a certain level of reliability is \(\min(M, N)\). The optimal DMT for the case \(M = N = 2\) is presented in Figure 1.1.

The optimal DMT was also characterized for various other channels such as the non-coherent channel [34] and the automatic retransmission request (ARQ) channel [5]. One result of special interest in our work is the optimal DMT of multiple-access (MAC) channel [30]. In this setting \(K\) distributes users, each with \(M\) antennas, are transmitting to a receiver with \(N\) antennas, and the goal is to decode the message from each user. Interestingly the optimal DMT of the MAC channel is directly related to the optimal DMT of the point-to-point channel. For instance when the number of receive antennas \(N \geq (K + 1) M - 1\) the optimal DMT for the symmetric case, where all users transmit at the same multiplexing-gain, equals to the optimal DMT of a point-to-point channel with \(M\) transmit and \(N\) receive antennas.

Coding schemes for the MIMO channel are coined space-time codes [25]. Space represents coding over the transmit antennas and time refers to the channel uses. The literature on space-time codes is very rich.
and different schemes aim at different targets. For instance the classical orthogonal designs [24] aim for maximizing the diversity order, even at the cost of losing degrees of freedom. On the other hand schemes such as the vertical Bell Labs space-time architecture (V-BLAST) [9] aim for utilizing the available degrees of freedom at the cost of reduced diversity order. There are also many schemes aimed at attaining the optimal DMT. Many of these coding schemes use lattices as an underlying code [6], [7], [2], [19], where lattices are the Euclidean analogue of algebraic codes, and essentially are infinite symmetric structures closed under addition. The general idea is that the encoder obtains the codewords by considering the intersection of the infinite lattice with a shaping region that satisfies a power constraint.

A subtle but very important point is that these schemes attain the optimal DMT only by considering the power constraint at the receiver. For instance, this is done by performing maximum-likelihood (ML) decoding over the finite codebook consisting of the intersection of the lattice and the shaping region. Another possibility is performing minimum-mean square error (MMSE) estimation (that takes into consideration the power constraint) on the received signal and then finding the lattice point at the receiver, closest to the estimation. Performing this operation can be translated to minimization over the lattice, of a metric with additive term that equals to the “potential” lattice point norm [13], making lattice points that do not satisfy the power constraint less likely. Indeed decoding without taking into consideration the power constraint leads to suboptimal performance as shown in [23] for fixed lattices and also in our work for any sequence of lattices.

In this work we take a different approach and extend Poltyrev’s setting [20] for infinite constellations (IC’s) from the additive white Gaussian noise (AWGN) channel to MIMO fading channels, in order to find the best DMT that IC’s can attain. IC’s are infinite sets in the Euclidean space characterized by their density which represents the average number of points within a unit volume. The logarithm of the density, normalized by the dimension, is coined normalized logarithmic density (NLD) and can be viewed as the analogue of the rate of finite constellations. In Poltyrev’s setting the encoder transmits a finite subset of the IC that satisfy the power constraint, while the receiver decodes over the IC without taking the power constraint into consideration, i.e. performs ML decoding over the IC given the observation. This approach enables to separate the encoding and decoding problems. The analysis of the performance of the decoder is cleaner and does not require to take into consideration the shaping region boundaries. A special case of IC’s that are structured and symmetric are lattices. Lattices gain a special interest since their symmetry may be utilized in some cases for efficient decoding. For lattices Poltyrev’s setting is translated to performing regular lattice decoding, i.e. performing ML decoding over the lattice given the observation.

Considering IC’s for the DMT problem is very natural for several reasons. First of all the DMT is asymptotic in nature and therefore the multiplexing gain represents degrees of freedom utilized by the transmission scheme. For IC’s in general and lattices in particular degrees of freedom are translated directly to geometrical dimensions. In addition, omitting the power constraint at the receiver enables to observe the problem of finding the optimal DMT geometrically and as a result to give a new geometrical interpretation to the optimal DMT. Finally, characterizing the cases for which the DMT of IC’s coincides with the optimal DMT of finite constellations, and the cases where IC’s are suboptimal, and also understanding the reasons for this suboptimality, allows to better understand the fading channel, and the limitations on the assump-
tions that can be made at the receiver. As an illustrative example consider the straight lines constituting the optimal DMT. For lattices these straight lines comes out naturally since finding the DMT of a certain lattice can be viewed simply as a scaling problem. Therefore, changing the multiplexing gain would simply mean changing the scaling of the lattice. As a result, it turns out that the DMT of a lattice of certain dimension, as a function of the multiplexing gain, is a straight line.

In the first part of this work we consider the point-to-point Rayleigh flat-fading channel. For this case we prove that IC’s in general and lattices using regular lattice decoding in particular attain the optimal DMT of finite constellations for any $M$ and $N$. We begin by extending the definitions of multiplexing gain and diversity order for IC’s. In addition, we define a new measure, the number of dimensions per channel use (NDCU) that equals to the IC dimension divided by the number of channel uses. Then, for each NDCU we derive an upper bound on the DMT of any sequence of IC’s. Essentially, this upper bound introduces a new tradeoff between the IC NDCU and the best DMT that may be attained. We use this tradeoff to find the NDCU’s for which the upper bound coincides with the optimal DMT of finite constellations. Finally, for each of these “optimal” NDCU’s we propose a transmission scheme matching the channel, and show that it attains the corresponding straight line of the optimal DMT. As an example consider the optimal DMT for the case $M = N = 2$ presented in Figure 1.1. In this case we show that the first line of the optimal DMT, in the range $0 \leq r \leq 1$, is attained only when the NDCU equals $\frac{4}{3}$, and we also present a transmission scheme that attains it. The second line in the range $1 \leq r \leq 2$ is attained only when the NDCU equals 2.

In the second part of this work we analyze the DMT of IC’s for the MAC channel. The result is rather surprising. Unlike the point-to-point case, we show that for the MAC channel the optimal DMT is attained only when $N \geq (K + 1) M - 1$, i.e. user limited regime. On the other hand when $N < (K + 1) M - 1$ we show that IC’s do not attain the optimal DMT. Interestingly, for the case $N \geq (K + 1) M - 1$ the optimal DMT equals to the DMT of a point-to-point channel with $M$ transmit and $N$ receive antennas. The analysis of the transmission scheme that attains the optimal DMT for IC’s, explains geometrically why the optimal DMT matches the point-to-point case.

In the third part of this work we take a more practical approach and design low density lattice codes (LDLC’s) [22] for the MIMO fading channel. LDLC’s are lattice codes designed directly in the Euclidean space, that have sparse “parity-check matrix” defined as the inverse of the lattice generator matrix. They were shown to attain good performance close to the channel capacity of the AWGN channel by using an iterative sum-product algorithm [22] which has a linear computational complexity as a function of the lattice dimension. We begin by deriving a max-product algorithm for LDLC’s transmitted over a single-input single-output (SISO) channels, aimed at minimizing the block error rate. For the AWGN channel we reveal an interesting connection between the sum-product and max-product algorithms. In this case the passed messages can be represented by Gaussians, where the Gaussians representing the messages are identical in both algorithms. However, the algorithms’ messages are different. In the sum-product algorithm the message equals to the sum of these Gaussians, where in the max-product algorithm the message equals to a maximization over these Gaussians. Numerical results show improvement in the block error rate compared to the sum-product algorithm, for small dimensional LDLC’s.

Next we extend the max-product algorithm for general MIMO channels using an extended tree assump-
tion. When assuming Gaussian a-priori probabilities for each lattice symbol, we get an interesting connection between the passed messages and minimum mean square error (MMSE) estimation. This algorithm sets the key for efficient decoding of LDLC’s in MIMO channels. Then, we design a very sparse LDLC for the MIMO channel with two transmit and two receive antennas. We propose a transmission scheme that uses reduced dimensionality of $\frac{4}{3}$ NDCU in order to facilitate the decoding process. The general idea is to use the reduced dimensionality in order to “protect” a subset of the lattice symbols more than the other lattice symbols, and make sure that the protected symbols take place in the parity check equations with less protected symbols. We utilize the very sparse LDLC structure and the reduced dimensionality to present a very efficient approximation of the max-product algorithm. Numerical results show that this scheme attains performance comparable to state of the art coding schemes.

The outline of this thesis is as follows. We begin by showing that IC’s attain the optimal DMT of the MIMO Rayleigh fading point-to-point channel in Chapter 2. Then, in Chapter 3 we extend the analysis to the MAC channel and show that IC’s attain the optimal DMT only in a user limited regime. In Chapter 4 we derive max-product algorithms for LDLC’s in point-to-point and MIMO channels. Finally, Chapter 5 summarizes the results and presents topics for further research.
Chapter 2

Optimal DMT of Infinite Constellations (IC’s) in Point to Point Channels

2.1 Introduction

The use of multiple antennas in wireless communication has certain inherent advantages. On one hand, using multiple antennas in fading channels allows to increase the transmitted signal reliability, i.e. diversity. For instance, diversity can be attained by transmitting the same information on different paths between transmitting-receiving antenna pairs with i.i.d Rayleigh fading distribution. The number of independent paths used is the diversity order of the transmitted scheme. On the other hand, the use of multiple antennas increases the number of degrees of freedom available by the channel. In [28],[8] the ergodic channel capacity was obtained for multiple-input multiple-output (MIMO) systems with $M$ transmit and $N$ receive antennas, where the paths have i.i.d Rayleigh fading distribution. It was shown that for large signal to noise ratios (SNR), the capacity behaves as $C(SNR) \approx \min(M, N) \log(SNR)$. The multiplexing gain is the number of degrees of freedom utilized by the transmitted scheme.

For the quasi-static Rayleigh flat-fading channel, Zheng and Tse [35] characterized the dependence between the diversity order and the multiplexing gain, by deriving the optimal tradeoff between diversity and multiplexing, i.e. for each multiplexing gain the maximal diversity order was found. They showed that the optimal diversity-multiplexing tradeoff (DMT) can be attained by ensemble of i.i.d Gaussian codes, given that the block length is greater or equal to $N + M - 1$. For this case, the tradeoff curve takes the form of the piecewise linear function that connects the points $(N - l)(M - l), l = 0, 1, \ldots, \min(M, N)$.

Space-time codes are coding schemes designed for MIMO systems e.g. see [25],[24] [7] and references therein. The design of space-time codes in these works pursue various goals such as maximizing the diversity order, maximizing the multiplexing gain, or achieving the optimal DMT. El Gamal et al [6] were the first to show that lattice coding and decoding achieve the optimal DMT. They presented lattice space-time (LAST) codes. These space time codes are subsets of an infinite lattice, where the lattice dimensionality equals to the number of degrees of freedom available by the channel, i.e. $\min(M, N)$, multiplied by the number of channel uses. By using a random ensemble of nested lattices, common randomness, minimum mean
square error (MMSE) estimation followed by lattice decoding and modulo lattice operation, they showed that LAST codes can achieve the optimal DMT. It is worth mentioning that the MMSE estimation and the modulo operation take in a certain sense into account the finite code book.

There has been an extensive research on explicit coding schemes, based on lattices, which are DMT optimal. Such an explicit coding schemes that attain the optimal DMT for any number of transmit and receive antennas were presented in [7]. In addition it was shown in [7] that $M$ channel uses are sufficient to obtain the optimal DMT. Another step towards finding explicit space-time coding schemes that attain the optimal DMT with low computational complexity was made by Jalden and Elia [13]. They considered explicit coding schemes based on the intersection between an underlying lattice and a shaping region. They showed that for the cases where these coding schemes attain the optimal DMT using maximum-likelihood (ML) decoding, they also attain it when using MMSE estimation at the receiver, followed by lattice decoding. The MMSE estimation relies on the power constraint, i.e. the shaping region boundaries. In addition, it was shown in [13] that by applying lattice reduction methods, the optimal DMT is attained when using suboptimal linear lattice decoders that require linear complexity as a function of the rate. This result applies to wide range of explicit space-time codes such as golden-codes [2], perfect space-time codes [19] and in general cyclic division algebra based space-time codes [7], and as this codes are approximately universal [27] it also applies to every statistical characterization of the fading channel. Note that these schemes take into consideration the finiteness of the codebook in the decoder. In this chapter we refer to regular lattice decoding as decoding over the infinite lattice without taking into consideration the finiteness of the codebook.

The work in [6] also includes for the case $N \geq M$ a lower bound on the diversity order of LAST codes shaped into a sphere when regular lattice decoder is employed at the receiver. For sufficiently large block length it is shown that $d(r) \geq (N - M + 1)(M - r)$ where $r$ is the multiplexing gain and the lattice dimension per channel use is $M$. Taherzadeh and Khandani showed in [23] that this is also an upper bound on the diversity order of any LAST code shaped into a sphere and decoded with regular lattice decoding. These results show that LAST codes together with regular lattice decoding are suboptimal compared to the optimal DMT of power constrained constellations.

Infinite constellations (IC’s) are structures in the Euclidean space that have no power constraint. In [20], Poltyrev analyzed the performance of IC’s over the additive white Gaussian noise (AWGN) channel. In this chapter we first extend the definitions of diversity order and multiplexing gain to the case where there is no power constraint. We also introduce a new term: the number of dimensions per channel use (NDCU), which is essentially the IC dimension divided by the number of channel uses. Then we extend the methods used in [20] in order to derive an upper bound on the diversity of any IC with certain NDCU, as a function of the multiplexing gain. It turns out that for a given NDCU the diversity is a straight line as a function of the multiplexing gain, that depends on the number of transmit and receive antennas. This analysis holds for any $M$ and $N$, and also applies to lattices with regular lattice decoding. We also find the NDCU for which the upper bounds coincide with the optimal DMT of finite constellations. Finally, we show that each segment of the optimal DMT is attained by a sequence of lattices with a corresponding NDCU, when using regular lattice decoder, i.e. for each point in the DMT of [35] there exists a lattice sequence of certain dimension that achieves it with regular lattice decoding. Hence, the best DMT IC’s may attain for any
NDCU is characterized in this chapter, and also it is proved that lattices can achieve the optimal DMT when regular lattice decoder is employed at the receiver, by adapting their dimensionality. It is important to note that when the IC is a lattice, we show that the multiplexing gain of infinite lattices and finite constellations coincide.

This chapter gives a framework for designing lattices for multiple-antenna channels using regular lattice decoding. It also shows the fundamental and natural connection between the IC dimension and its optimal diversity order. For instance, it is shown that for the case $M = N = 2$, the maximal diversity order of 4 can be achieved (with regular lattice decoding) by a lattice that has at most $\frac{4}{3}$ NDCU. On the other hand the Alamouti scheme [1], that also has maximal diversity order of 4, utilizes only a single dimension per channel use in this set up. Hence, there is still a room to improve by a $\frac{1}{3}$ of a dimension per channel use. In addition, while in [6], [13], the MMSE estimation improves the channel in such a manner that enables the lattice decoder to attain the optimal DMT, this chapter shows that when considering regular lattice decoding, reducing the lattice dimensionality takes the role of MMSE estimation in the sense of improving the channel such that the optimal DMT is obtained. Finally, the analysis in this chapter gives another geometrical interpretation to the optimal DMT.

The outline of this chapter is as follows. In Section 2.2 basic definitions for the fading channel and IC’s are given. Section 2.3 presents for each channel realization a lower bound on the average decoding error probability of any IC, and an upper bound on the DMT of any IC. An upper bound on the error probability of ensemble of IC’s for each channel realization, a transmission scheme that attains the optimal DMT, and some averaging arguments on how the optimal DMT is attained by IC’s, are all presented in Section 2.4. Discussion on the results, that addresses the difference between lattice constellations and full dimension lattice based finite constellations, followed by a geometrical interpretation to the optimal DMT, and a discussion on the relation between the multiplexing gains of an IC and a finite constellation, is presented in Section 2.5. This discussion presents an intuitive interpretation to our results and relies mainly on the basic definitions given in Section 2.2.

### 2.2 Basic Definitions

We refer to the countable set $S = \{s_1, s_2, \ldots \}$ in $\mathbb{C}^n$ as infinite constellation (IC). Let $\text{cube}_l(a) \subset \mathbb{C}^n$ be a (probably rotated) $l$-complex dimensional cube ($l \leq n$) with edge of length $a$ centered around zero. An IC $S_l$ is $l$-complex dimensional if there exists rotated $l$-complex dimensional cube $\text{cube}_l(a)$ such that $S_l \subset \lim_{a \to \infty} \text{cube}_l(a)$ and $l$ is minimal. $M(S_l, a) = |S_l \cap \text{cube}_l(a)|$ is the number of points of the IC $S_l$ inside $\text{cube}_l(a)$. In [20], the $n$-complex dimensional IC density for the AWGN channel was defined as the upper limit (the limit supremum) of the ratio $\gamma_G = \lim \sup_{a \to \infty} \frac{M(S_l, a)}{a^{2n}}$ and the volume to noise ratio (VNR) was given as $\mu_G = \frac{1}{2\pi e \sigma^2}$.

The Voronoi region of a point $x \in S_l$, denoted as $V(x)$, is the set of points in $\lim_{a \to \infty} \text{cube}_l(a)$ closer to $x$ than to any other point in the IC. The effective radius of the point $x \in S_l$, denoted as $r_{\text{eff}}(x)$, is the radius
of the \( l \)-complex dimensional ball that has the same volume as the Voronoi region, i.e. \( r_{\text{eff}}(x) \) satisfies

\[
|V(x)| = \frac{\pi^{l/2} r_{\text{eff}}^l(x)}{\Gamma(l + 1)}. \tag{2.1}
\]

A complex lattice \( \Lambda \) is an IC that constitutes a discrete set in \( \mathbb{C}^n \), closed under addition. The Voronoi regions of all lattice points are identical and satisfy

\[
|V(x)| = \gamma_{\Lambda}^{-1} \quad \forall x \in \Lambda. \tag{2.2}
\]

Hence, for large dimension the VNR of a lattice, \( \mu_G \), approaches the ratio \( \frac{|\gamma_{\Lambda}|^2}{\pi e} \) where \( r_{\text{eff}} \) is the lattice effective radius. Regular lattice decoder finds the closest lattice point to an observation \( y \in \mathbb{C}^n \), i.e. regular lattice decoder finds the solution to the optimization problem

\[
\arg\min_{x \in \Lambda} \| y - x \|. \tag{2.3}
\]

Note that these definitions can be also extended in a straightforward manner to an IC that constitutes a real lattice in \( \mathbb{R}^{2n} \). For instance when the first \( n \) entries of each lattice point are transmitted on the real part of the IC, and the second \( n \) entries of each lattice point are transmitted on the imaginary part of the IC.

We consider a quasi static flat-fading channel with \( M \) transmit and \( N \) receive antennas. We assume for this MIMO channel perfect channel knowledge at the receiver and no channel knowledge at the transmitter. The channel model is as follows:

\[
y_t = H \cdot x_t + \rho^{-\frac{1}{2}} n_t \quad t = 1, \ldots, T \tag{2.4}
\]

where \( x_t, t = 1, \ldots, T \) is the transmitted signal, \( n_t \sim \mathcal{C}\mathcal{N}(0, \frac{2}{\pi e} I_N) \) is the additive noise where \( \mathcal{C}\mathcal{N} \) denotes complex-normal, \( I_N \) is the \( N \)-dimensional unit matrix, and \( y_t \in \mathbb{C}^N \). \( H \) is the fading matrix with \( N \) rows and \( M \) columns where \( h_{i,j} \sim \mathcal{C}\mathcal{N}(0,1), 1 \leq i \leq N, 1 \leq j \leq M \), and \( \rho^{-\frac{1}{2}} \) is a scalar that multiplies each element of \( n_t \), where \( \rho \) plays the role of average SNR at the receive antenna, for power constrained constellations that satisfy \( \frac{1}{T} \sum_{t=1}^{T} E\{\|x_t\|^2\} \leq \frac{2}{\pi e} \).

We also define the extended vector \( \mathbf{x} = \{x_1^\dagger, \ldots, x_T^\dagger\}^\dagger \). Suppose \( \mathbf{x} \in S_t \subset \mathbb{C}^{MT} \), where \( S_t \) is an IC with density \( \gamma_{\text{tr}} = \lim_{a \to \infty} \sup_{a \in S_t} \frac{M(S_t,a)}{a^{2T}} \) (\( a^{2T} \) is the volume of cube \( a^T \)). By defining \( H_{\text{ex}} \) as an \( NT \times MT \) block diagonal matrix, where each block on the diagonal equals \( H_{\text{ex}} = \rho^{-\frac{1}{2}} \cdot \{x_1^\dagger, \ldots, x_T^\dagger\}^\dagger \in \mathbb{C}^{NT} \) and \( y_{\text{ex}} \in \mathbb{C}^{NT} \), we can rewrite the channel model in (2.4) as

\[
y_{\text{ex}} = H_{\text{ex}} \cdot x + n_{\text{ex}}. \tag{2.5}
\]

In the sequel we use \( L \) to denote \( \min(M,N) \). We define as \( \sqrt{\lambda_i}, 1 \leq i \leq L \) the real valued, non-negative singular values of \( H \). We assume \( \sqrt{\lambda_L} \geq \cdots \geq \sqrt{\lambda_1} > 0 \). Our analysis is done for large values of \( \rho \) (large VNR at the transmitter). We state that \( f(\rho) \geq g(\rho) \) when \( \lim_{\rho \to \infty} \frac{\ln(f(\rho))}{\ln(g(\rho))} \leq \cdot \frac{\ln(g(\rho))}{\ln(f(\rho))} \), and also define \( \leq, \geq \) in a similar manner by substituting \( \leq \) with \( \geq \), = respectively.

We now turn to the IC definitions at the transmitter. We define the NDCU as the IC dimension divided
by the number of channel uses. We denote the NDCU by $D$. Let us consider a $D \cdot T$-complex dimensional sequence of IC’s $S_{D,T}(\rho)$, where $D \leq L$, and $T$ is the number of channel uses. First we define $\gamma_{\text{tr}} = \rho^{rT}$ as the density of $S_{D,T}(\rho)$ at the transmitter. The IC multiplexing gain is defined as

$$MG(r) = \lim_{\rho \to \infty} \frac{1}{T} \log \rho (\gamma_{\text{tr}} + 1) = \lim_{\rho \to \infty} \frac{1}{T} \log (\rho^{rT} + 1). \quad (2.6)$$

Note that $MG(r) = \text{max}(0, r)$, i.e. for $0 \leq r \leq D$ the multiplexing gain is $r$. Roughly speaking, $\gamma_{\text{tr}} = \rho^{rT}$ gives us the number of points of $S_{D,T}(\rho)$ within the $D \cdot T$-complex dimensional region $\text{cube}_{D,T}(1)$. In order to get the multiplexing gain, we normalize the exponent of the number of points within $\text{cube}_{D,T}(1)$, $rT$, by the number of channel uses - $T$. Note that the IC multiplexing gain, $r$, can be directly translated to finite constellation multiplexing gain $r$ by considering the IC points within a shaping region. For more details see 2.5.4. The VNR at the transmitter is

$$\mu_{\text{tr}} = \frac{\gamma_{\text{tr}}^\beta T}{2\pi e \sigma^2} = \rho^{1-\frac{2}{\gamma}} \quad (2.7)$$

where $\sigma^2 = \frac{\rho^{-1}}{2\pi e}$ is each dimension noise variance. Now we can understand the role of the multiplexing gain for IC’s. The AWGN variance decreases as $\rho^{-1}$, where the IC density increases as $\rho^{rT}$. When $r = 0$ we get constant IC density as a function of $\rho$, where the noise variance decreases, i.e. we get the best error exponent. In this case the number of points within $\text{cube}_{D,T}(1)$ remains constant as a function of $\rho$. On the other hand, when $r = D$, we get VNR $\mu_{\text{tr}} = 1$, and from [20] we know that it inflicts average error probability that is bounded away from zero. In this case, the increase in the number of IC points within $\text{cube}_{D,T}(1)$ occurs at maximal rate.

Now we turn to the IC definitions at the receiver. First we define the set $H_{ex} \cdot \text{cube}_{D,T}(a)$ as the multiplication of each point in $\text{cube}_{D,T}(a)$ with the matrix $H_{ex}$. In a similar manner $S'_{D,T} = H_{ex} \cdot S_{D,T}$. The set $H_{ex} \cdot \text{cube}_{D,T}(a)$ is almost surely $D \cdot T$-complex dimensional (where $D \leq L$) and in this case $M(S_{D,T}, a) = |S_{D,T} \cap \text{cube}_{D,T}(a)| = |S'_{D,T} \cap (H_{ex} \cdot \text{cube}_{D,T}(a))|$. We define the receiver density as

$$\gamma_{\text{rc}} = \limsup_{a \to \infty} \frac{M(S_{D,T}, a)}{\text{Vol}(H_{ex} \cdot \text{cube}_{D,T}(a))}$$

i.e., the upper limit of the ratio of the number of IC points in $H_{ex} \cdot \text{cube}_{D,T}(a)$, and the volume of the parallelepiped $H_{ex} \cdot \text{cube}_{D,T}(a)$. Based on the majorization property of a matrix singular values [14], we get that the volume of the set $H_{ex} \cdot \text{cube}_{D,T}(a)$ is smaller than $a^{2D \cdot T} \cdot \lambda_{L}^T \cdots \lambda_{L-B+1}^T \cdot \lambda_{L-B}^\beta$, assuming $D = B + \beta$ where $B \in \mathbb{N}$ and $0 < \beta \leq 1$, i.e. the volume is smaller than the multiplication of the $B + 1$ strongest singular values, raised to the power of the maximal amount of channel uses each can take place in. Hence we get

$$\gamma_{\text{rc}} \geq \rho^{rT} \lambda_{L}^T \cdots \lambda_{L-B+1}^T \cdot \lambda_{L-B}^\beta \quad (2.8)$$

and the receiver VNR is

$$\mu_{\text{rc}} \leq \rho^{1-\frac{2}{\gamma}} \cdot \lambda_{L}^\beta \cdots \lambda_{L-B+1}^\beta \cdot \lambda_{L-B}^\beta. \quad (2.9)$$

Note that for $N \geq M$ and $D = M$ we get $\gamma_{\text{rc}} = \rho^{rT} \cdot \prod_{i=1}^{M} \lambda_{i}^T$ and $\mu_{\text{rc}} = \rho^{1-\frac{2}{\gamma}} \cdot \prod_{i=1}^{M} \lambda_{i}^{\beta}$. The average
decoding error probability over the IC points of \( S_{D,T}(\rho) \), for a certain channel realization \( H \), is defined as

\[
\overline{P}_e(H, \rho) = \limsup_{a \to \infty} \frac{\sum_{x' \in S'_{D,T} \cap (H_{\text{ex\cube}}_{D,T}(a))} P_e(x', H, \rho)}{M(S_{D,T}, a)}
\]  

(2.10)

where \( P_e(x', H, \rho) \) is the error probability associated with \( x' \). The average decoding error probability of \( S_{D,T}(\rho) \) over all channel realizations is \( \overline{P}_e(\rho) = E_H\{\overline{P}_e(H, \rho)\} \). Hence the diversity order equals

\[
d = -\lim_{\rho \to \infty} \log_\rho(\overline{P}_e(\rho))
\]  

(2.11)

### 2.3 Upper Bound on the Diversity Order

In this section we derive an upper bound on the diversity order of any IC with NDCU \( D \) and any value of \( T, M \) and \( N \). In Theorem 2.1 we derive for each channel realization a lower bound on the error probability of any IC with \( D \) NDCU. In Theorem 2.2 we derive an upper bound on the DMT of any sequence of IC’s with \( D \) NDCU. Finally in Corollary 2.2 we show that by choosing the correct NDCU, the upper bound coincides with the optimal DMT of finite constellations.

As in [35] and [6], we also define \( \lambda_i = \rho^{-\alpha_i} \), \( 1 \leq i \leq L \). When the entries of the channel matrix \( H \) are all i.i.d with PDF \( \mathcal{CN}(0, 1) \), the PDF of its singular values is of the form \( \rho^{-\sum_{i=1}^{L}(|N-M|+2i-1)\alpha_i} \) for large \( \rho \) [35], where following the definitions above \( 0 \leq \alpha_L \leq \cdots \leq \alpha_1 \). \(^1\) By assigning in (2.8), (2.9) respectively , we can write

\[
\gamma_{\text{rc}} \geq \rho^{T(r+\sum_{i=0}^{B-1} \alpha_{L-i}+\beta \alpha_{L-B})}
\]

and

\[
\mu_{\text{rc}} \leq \rho^{1-\frac{1}{D}(r+\sum_{i=0}^{B-1} \alpha_{L-i}+\beta \alpha_{L-B})}.
\]

**Theorem 2.1.** For any \( D \cdot T \)-complex dimensional IC \( S_{D,T}(\rho) \) with transmitter density \( \gamma_{\text{tr}} = \rho^{rT} \) and channel realization \( \underline{\alpha} = (\alpha_1, \ldots, \alpha_L) \), we have the following lower bound on the average decoding error probability for \( 0 \leq r \leq D \)

\[
\overline{P}_e(H, \rho) > \frac{C(D \cdot T)}{4} e^{-\mu_{\text{rc}} A(D \cdot T)+(D \cdot T-1)\ln(\mu_{\text{rc}})}
\]

where \( A(D \cdot T) = e \cdot \Gamma(D \cdot T+1) \frac{1}{\sqrt{\pi}} \) and \( C(D \cdot T) = \frac{\Gamma(D \cdot T+\frac{1}{2})}{\Gamma(D \cdot T)} \).

**Proof.** We divide the proof into two parts. In the first part we prove the result for lattices, that constitute a symmetric structure for which the Voronoi regions of different lattice points are identical. In the second part we prove the result for general IC’s with receiver density \( \gamma_{\text{rc}} \). As the second part of the proof is somewhat more involved, we defer it to appendix A.1. Note that we could have used the tighter bounds of [12], but these bounds are not needed for DMT. Instead we derive coarser and more simplified upper bounds, which are sufficient for our purposes.

\(^1\)A generalization of the Rayleigh fading channel is the Jacobi fading channel. The optimal DMT for this channel was derived in [4].
We begin by proving the result for lattices. Lattices constitute a discrete subgroup of the Euclidean space, with the ordinary vector addition operation. Consider a $D \cdot T$-complex dimensional lattice, $S'_{D,T}(\rho)$, at the receiver with density $\gamma_{rc}$. The lattice points have identical Voronoi regions up to a translation. Hence, the volume of each Voronoi region equals

$$|V(x)| = \frac{1}{\gamma_{rc}} \quad \forall x \in S'_{D,T}(\rho).$$

According to the definition of the effective radius in (2.1), we get that $r_{\text{eff}}(x) = r_{\text{eff}}(\gamma_{rc}) = \left(\frac{\Gamma(D\cdot T+1)}{\gamma_{rc} \pi^{D\cdot T}}\right)^{\frac{1}{D\cdot T}}$, $\forall x \in S'_{D,T}(\rho)$. Note that in lattices the maximum-likelihood (ML) decoding error probability is identical for all lattice points, i.e. the average and maximal error probabilities are identical. It has been proven in [20], [26] that the error probability of any lattice point at the receiver fulfills

$$P_e^{S'_{D,T}} > \Pr(\|\tilde{n}_{\text{ex}}\| \geq r_{\text{eff}}(\gamma_{rc}))$$

where $P_e^{S'_{D,T}}$ is the ML decoding error probability of any lattice point, and $\tilde{n}_{\text{ex}}$ is the effective noise in the $D \cdot T$-complex dimensional hyperplane where $S'_{D,T}(\rho)$ resides. We find an explicit expression for the lower bound

$$\Pr(\|\tilde{n}_{\text{ex}}\| \geq r_{\text{eff}}(\gamma_{rc})) = \Pr(\|\tilde{n}_{\text{ex}}\| \geq r_{\text{eff}}(\gamma_{rc})) > \frac{\int_{r_{\text{eff}}}^{r_{\text{eff}}+\sigma^2} e^{-\frac{x^2}{2\sigma^2}} dx}{\frac{1}{\sqrt{2\pi} \Gamma(D\cdot T)}} \geq \frac{r_{\text{eff}}^{2D\cdot T-2} e^{-\frac{r_{\text{eff}}^2}{2\sigma^2}}}{\frac{1}{\sqrt{2\pi} \Gamma(D\cdot T)}}.$$

By assigning $r_{\text{eff}}^2 = \left(\frac{2\Gamma(D\cdot T+1)}{\gamma_{rc} \pi^{D\cdot T}}\right)^{\frac{1}{D\cdot T}}$ we get

$$P_e^{S'_{D,T}} > C(D \cdot T) \cdot e^{-\frac{\gamma_{rc} \pi^{D\cdot T}}{2\pi^2} A(D\cdot T) + (D\cdot T-1) \ln\left(\frac{\gamma_{rc} \pi^{D\cdot T}}{2\pi^2}\right)}$$

and by assigning $\mu_{rc} = \frac{\gamma_{rc} \pi^{D\cdot T}}{2\pi^2}$ we get

$$P_e^{S'_{D,T}} > \frac{C(D \cdot T)}{4} \cdot e^{-\mu_{rc} [A(D\cdot T) + (D\cdot T-1) \ln(\mu_{rc})]}.$$

Note that in (2.12) we lower bounded the error probability with $r_{\text{eff}}(\gamma_{rc})$ instead of $r_{\text{eff}}(\gamma_{rc})$, and also in (2.13) we multiplied by $\frac{1}{4}$, in order to be consistent with the general lower bound for IC’s shown in appendix A.1. For lattices we have $P_e(H, \rho) = P_e^{S'_{D,T}}$. Essentially what we have shown here is a scaled sphere packing bound.²

Next, we would like to use this lower bound to average over the channel realizations and get an upper bound on the diversity order.

²Note that while Theorem 2.1 refers to $D \cdot T$-complex dimensional IC’s, the lower bound derived in this theorem applies to any $2D \cdot T$-real dimensional IC.
The diversity order of any $D \cdot T$-complex dimensional sequence of IC’s $S_{D,T}(\rho)$, with $D$ NDCU, is upper bounded by

$$d_{D,T}(r) \leq d_{M,N}^{*,D}(r) = M \cdot N(1 - \frac{r}{D})$$

for $0 < D \leq \frac{M \cdot N}{N + M - T}$, and

$$d_{D,T}(r) \leq d_{M,N}^{*,D}(r) = (M - l)(N - l) \frac{D}{D - l}(1 - \frac{r}{D})$$

for $(M-l+1)(N-l+1) \leq D \leq l - 1 < D \leq \frac{(M-l)(N-l)}{N + M - 1 - 2l} + l$ and $l = 1, \ldots, L - 1$. In all of these cases $0 \leq r \leq D$.

Proof. For any IC with VNR $\mu_{rc}$, assigning $\mu_{le}^* > \mu_{rc}$ in the lower bound from Theorem 2.1 also gives a lower bound on the error probability

$$\overline{P}_e(H, \rho) > \frac{C(D \cdot T)}{4} e^{-\mu_{le}^* \cdot A(D \cdot T) + (D \cdot T - 1) \ln(\mu_{le}^*)}.$$ 

It results from the fact that inflating the IC into an IC with VNR $\mu_{le}^*$ must decrease the error probability, where

$$\frac{C(D \cdot T)}{4} e^{-\mu_{le}^* \cdot A(D \cdot T) + (D \cdot T - 1) \ln(\mu_{le}^*)}$$

is a lower bound on the error probability of any IC with VNR $\mu_{le}^*$. Hence, for the case $\mu_{rc} < 1$ we can lower bound the error probability by assigning $1$ in the lower bound and get $\frac{C(D \cdot T)}{4} e^{-A(D \cdot T)}$, i.e. for $\mu_{rc} < 1$ the average decoding error probability is bounded away from $0$ for any value of $\rho$. We can give the event $\mu_{rc} = 1$ the interpretation of an outage event.

We would like to set a lower bound for the error probability for each channel realization $\alpha$, which we denote by $P_e^{LB}(\rho, \alpha)$. We know that $\mu_{rc} \leq \rho \frac{1}{M} (r + \sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B})$. For the case $\sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} < D - r$, we take

$$P_e^{LB}(\rho, \alpha) = \frac{C(D \cdot T)}{4} e^{-L(\rho, \alpha) \cdot A(D \cdot T) + (D \cdot T - 1) \ln(L(\rho, \alpha))}$$

where $L(\rho, \alpha) = \rho^{1 - \frac{1}{M} (r + \sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B})} > 1$. For the case $\sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} \geq D - r$ we get that $\mu_{rc} \leq 1$, and we take

$$P_e^{LB}(\rho, \alpha) = \frac{C(D \cdot T)}{4} e^{-A(D \cdot T)}.$$ 

In order to find an upper bound on the diversity order, we would like to average $P_e^{LB}(\rho, \alpha)$ over the channel realizations. In our analysis we consider large values of $\rho$, and so we calculate

$$\overline{P}_e(\rho) > \int_{\alpha \geq 0} P_e^{LB}(\rho, \alpha) \cdot \rho^{-\sum_{i=0}^{L} ((M-I)-2i) \alpha_i} d\alpha \quad (2.14)$$

where $\alpha \geq 0$ signifies the fact that $\alpha_1 \geq \cdots \geq \alpha_L \geq 0$. By defining $A = \{\alpha | \sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} < D - r\}$.
\[ D - r; \alpha \geq 0 \} \text{ and } \mathcal{A} = \{ \alpha \mid \sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} \geq D - r; \alpha \geq 0 \} \] we can split (2.14) into 2 terms

\[ P_c(\rho) \geq \int_{\alpha \in \mathcal{A}} \rho \cdot \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \, d\alpha + \int_{\alpha \in \mathcal{A}} \rho \cdot \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \, d\alpha. \]

Hence

\[ P_c(\rho) \geq \int_{\alpha \in \mathcal{A}} \rho \cdot \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \, d\alpha. \]

(2.15)

In a similar manner to [35], [6], for very large \( \rho \), we approximate the average value by finding the most dominant exponential term in the integral. For this we would like to find the minimal value of

\[ \lim_{\rho \to -\infty} - \log \rho \cdot \rho \cdot \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \]

for the case \( \alpha \in \mathcal{A} \). For \( \alpha \in \mathcal{A} \), we get that \( \rho \cdot \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \) is bounded away from 0 for any value of \( \rho \). Hence, in order to find the most dominant error event we would like to find \( \min_{\alpha} \sum_{i=1}^{L} |N-M|+2i-1\alpha_i \) given that \( \alpha \in \mathcal{A} \). The minimal value is achieved at the boundary, i.e. for \( \alpha \) satisfying \( \sum_{i=0}^{B-1} \alpha \leq D - r \), \( \alpha \geq 0 \). Hence, for any \( D \leq L \) we state that

\[ d_{D,T}(r) \leq \min_{\alpha} \sum_{i=1}^{L} |N-M|+2i-1\alpha_i, \quad 0 \leq r \leq D \]

(2.17)

where \( \sum_{i=0}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} = D - r \) and \( \alpha_1 \geq \cdots \geq \alpha_L \geq 0 \). Basically this optimization problem is a linear programming problem whose solution is as follows. For \( 0 < D \leq \frac{MN}{N+M-1} \) the solution is \( \alpha_i = 1 - \frac{r}{D} \), \( i = 1, \ldots, L \). For \( \frac{M-1+i(M-1)}{N+M-1+2(i-1)} + l - 1 < D \leq \frac{(M-l)(N-l)}{N+M-1-2l} + l \) and \( l = 1, \ldots, L - 1 \) the solution is \( \alpha_L = \cdots = \alpha_{L-l+1} = 0 \) and \( \alpha_{L-l} = \cdots = \alpha_1 = \frac{lD-1}{D-r} \). The desired upper is attained by substituting the optimal values of \( \alpha \) in (2.17). The detailed solution for the optimization problem is presented in appendix A.2.

From Theorem 2.2 we get an upper bound on the diversity order by assuming transmission of the \( D \cdot T \) complex dimensions over the \( B+1 \) strongest singular values. This assumption is equivalent to assuming beamforming which may improve the coding gain, but does not increase the diversity order. This assumption allows us to derive a lower bound on the average decoding error probability. However, we still get maximal diversity order of \( MN \) in this case.

Let us consider as an illustrative example the case of \( M = N = 2 \). In this case, for \( 0 < D \leq \frac{4}{3} \) we get \( d_{2,2}^{D}(r) = 4(1 - \frac{r}{2}) \). For \( \frac{4}{3} < D \leq 2 \) we get \( d_{2,2}^{D}(r) = \frac{D}{D-1}(1 - \frac{r}{2}) \). In both cases \( 0 \leq r \leq D \). For this set up we have two singular values and so \( \alpha_1 \geq \alpha_2 \geq 0 \). The optimization problem is of the form \( \min_{\alpha_2 \geq 0} \alpha_1 + 3\alpha_2 \), where for \( 0 < D \leq 1 \) the constraint is \( \beta \alpha_2 = D - r \), and for \( 1 < D \leq 2 \) the constraint is \( \alpha_2 + \beta \alpha_1 = D - r \). For the case \( 0 < D < \frac{4}{3} \) the optimization problem solution is \( \alpha_1 = \alpha_2 = 1 - \frac{r}{2} \), i.e. in this case the most dominant error event occurs when both singular values are very small. For the case \( D = \frac{4}{3} \) the constraint is of the form \( \alpha_2 + \frac{\alpha_1}{3} = \frac{4}{3} - r \), and the optimization problem solution is achieved for both \( \alpha_1 = \alpha_2 = 1 - \frac{4r}{3} \) and \( \alpha_2 = 0, \alpha_1 = 4 - 3r \). For the case \( \frac{4}{3} < D \leq 2 \) the optimization problem
solution is achieved for \( \alpha_2 = 0, \alpha_1 = \frac{D-r}{D-1} \), i.e. one strong singular value and another very weak singular value.

Figure 2.1: The diversity order as a linear function of the multiplexing gain \( r \) for \( M = 4, N = 3 \) and \( D = 1, 2, 2.5 \) and 3.

**Corollary 2.1.** For \( 0 < D \leq \frac{MN}{N+M-1} \) we get \( d_{M,N}^{*,D}(0) = d_{M,N}^{*(FC)}(0) = MN \). For \( \frac{(M-l+1)(N-l+1)}{N+M-1-2(l-1)} + l - 1 < D \leq \frac{(M-l)(N-l)}{N+M-1-2l} + l, l = 1, \ldots, L - 1 \) we get \( d_{M,N}^{*,D}(l) = d_{M,N}^{*(FC)}(l) = (M-l)(N-l) \).

**Proof.** The proof is straightforward from \( d_{M,N}^{*,D}(r) \) properties.

From Corollary 2.1 we get that the range of \( D \) can be divided into segments, where for each segment we have a set of straight lines, that are all equal at a certain integer point. Note that at these points, we get the same values as the optimal DMT for finite constellations.

**Corollary 2.2.** In the range \( l \leq r \leq l + 1 \), the maximal possible diversity order is achieved at dimension \( D_l = \frac{(M-l)(N-l)}{N+M-1-2l} + l \) and equals

\[
d_{M,N}^{*,D_l}(r) = d_{M,N}^{*(FC)}(r) = (M-l)(N-l)\frac{D_l}{D_l - l}(1 - \frac{r}{D_l}) = (M-l)(N-l) - (r-l)(N+M-2\cdot l-1)
\]

where \( l = 0, \ldots, L - 1 \). This expression equals to the optimal DMT of finite constellations in this range.

**Proof.** The proof is straightforward from \( d_{M,N}^{*,D}(r) \) properties.

From Corollary 2.2 we can see that \( d_{M,N}^{*,D_l}(l) = (M-l)(N-l) \) and \( d_{M,N}^{*,D_l}(l+1) = (M-l-1)(N-l-1) \). We also know that \( d_{M,N}^{*,D_l}(r) \) is a straight line. Also, the optimal DMT for finite constellations consists of a straight line in the range \( l \leq r \leq l + 1 \), that equals \((N-l)(M-l)\) when \( r = l \) and \((M-l-1)(N-l-1)\) when \( r = l + 1 \). Hence, in the range \( l \leq r \leq l + 1 \) for \( D_l = \frac{(M-l)(N-l)}{N+M-1-2l} + l \), we get an upper bound that equals to the optimal DMT of finite constellations presented in [35]. Since for each \( l = 0, \ldots, L - 1 \), we have such \( D_l \), the solution for

\[
d_{M,N}^{*(FC)}(r) = \max_{0 \leq D \leq L} d_{M,N}^{*,D}(r) \quad 0 \leq r \leq L
\]
equals to the optimal DMT of finite constellations, i.e. \( d^{*,(IC)}_{M,N} (r) = d^{*,(FC)}_{M,N} (r) \).

Figure 2.1 illustrates the properties of \( d^{*,D}_{M,N} (r) \) following Corollaries 2.1, 2.2. We take the example of \( M = 4, N = 3 \). For \( 0 \leq D \leq 2 \) we get upper bounds that have diversity order 12 for \( r = 0 \). We can see that in the range \( 0 \leq r \leq 1 \), the upper bound of \( D = 2 \) is maximal and equals to the optimal DMT of finite constellations. In the range \( 2 < D \leq 2.5 \) we can see that the upper bounds have the same diversity order 6 at \( r = 1 \). In the range \( 1 \leq r \leq 2 \), the upper bound of \( D = 2.5 \) is maximal and equals to the optimal DMT of finite constellations in this range. For \( 2.5 < D \leq 3 \), the upper bounds equal to 2 at \( r = 2 \). In the range \( 2 < r \leq 3 \), the upper bound of \( D = 3 \) is maximal and again equals to the optimal DMT of finite constellations in this range.

Figure 2.2: \( d^{*,D}_{4,3} (0) \) as a function of the NDCU \( D \), for \( M = 4, N = 3 \).

Figure 2.2 presents the maximal diversity order that can be attained for different NDCU, for the case \( M = 4 \) and \( N = 3 \), i.e. the upper bound on the diversity order for \( r = 0 \), \( d^{*,D}_{4,3} (0) \), where \( 0 \leq D \leq 3 \). In the range \( 0 \leq D \leq 2 \) we get \( d^{*,D}_{4,3} (0) = 12 \). It coincides with the result presented in Figure 2.1, where we showed that in this range the straight lines have the same value for \( r = 0 \). Hence, for IC’s, one can use up to 2 NDCU without compromising the diversity order. Starting from \( D \geq 2 \), the tradeoff starts to kick-in and the maximal diversity order starts to reduce as we increase the NDCU. Also note that for \( D = 3 \) the diversity order is 6 when \( r = 0 \).

### 2.4 Attaining the Best Diversity Order

In this section we show that the optimal DMT of finite constellations is achievable by a sequence of IC’s in general and lattices using regular lattice decoding in particular. In Subsection 2.4.1 we present a transmission scheme for any \( M \) and \( N \) that transmits an IC with \( D_l = \frac{(M-l)(N-l)}{N+M-l-2l} + l \) and \( T_l = N + M - 1 - 2l, l = 0, \ldots, L - 1 \), where as previously defined \( L = \min(M, N) \) and \( D_l \) is chosen based on the results in Section 2.3. In Subsection 2.4.2 we present the effective channel induced by this transmission scheme. Following that we extend the methods presented in [20] and derive in Theorem 2.3 for each channel realization an upper bound on the average decoding error probability of ensemble of IC’s. By averaging the upper bound over the channel realizations, we show in Theorem 2.4 that the proposed transmission scheme attains the optimal DMT. In Theorem 2.5 we extend this result also to lattices when employing regular lattice decoder.
Finally, we discuss power spreading technique over the transmit antennas for the transmission scheme in Subsection 2.4.5, and give some averaging arguments on the existence of sequence of IC’s that attain the optimal DMT in Subsection 2.4.6.

2.4.1 The Transmission Scheme

The transmission matrix $G_l$, $l = 0, \ldots, L - 1$, has $M$ rows that represent the transmission antennas, and $T_l = N + M - 1 - 2 \cdot l$ columns that represent the number of channel uses.

We begin by describing the transmission matrix structure in general for any $M$ and $N$.

1. For $N \geq M$ and $D_{M-1} = \frac{M(N-M+1)}{N-M+1} = M$: the matrix $G_{M-1}$ has $N - M + 1$ columns (channel uses). In the first column transmit symbols $x_1, \ldots, x_M$ on the $M$ antennas, and in the $N - M + 1$ column transmit symbols $x_{M(N-M)+1}, \ldots, x_{M(N-M+1)}$ on the $M$ antennas.

2. For $M > N$ and $D_{N-1} = \frac{N(M-N+1)}{M-N+1} = N$: the matrix $G_{N-1}$ has $M - N + 1$ columns. In the first column transmit symbols $x_1, \ldots, x_N$ on antennas $1, \ldots, N$ and in the $M - N + 1$ column transmit symbols $x_{N(M-N)+1}, \ldots, x_{N(M-N+1)}$ on antennas $M - N + 1, \ldots, M$.

3. For $D_l$, $l = 0, \ldots, L - 2$: the matrix $G_l$ has $M + N - 1 - 2 \cdot l$ columns. We add to $G_{l+1}$, the transmission scheme of $D_{l+1}$, two columns in order to get $G_l$. In the first added column transmit $l + 1$ symbols on antennas $1, \ldots, l + 1$. In the second added column transmit different $l + 1$ symbols on antennas $M - l, \ldots, M$.

Example: $M = 4$, $N = 3$. In this case the transmission scheme for $D = 3, 2.5$ and $2$ ($G_2, G_1$ and $G_0$ respectively) is as follows:

\[
\begin{pmatrix}
  x_1 & 0 & x_7 & 0 & x_{11} & 0 \\
  x_2 & x_4 & x_8 & 0 & 0 & 0 \\
  x_3 & x_5 & 0 & x_9 & 0 & 0 \\
  0 & x_6 & 0 & x_{10} & 0 & x_{12}
\end{pmatrix}
\]

(2.18)

\[D_2 = \frac{6}{2}, \quad D_1 = \frac{10}{2}, \quad D_0 = \frac{12}{6}\]

2.4.2 The Effective Channel

Next we define the effective channel matrix induced by the transmission scheme. In accordance with the channel model from (2.4), the multiplication $H \cdot G_l$ yields a matrix with $N$ rows and $T_l$ columns, where each column equals to $H \cdot x_t$, $t = 1 \ldots T_l$, as in (2.4). We are interested in transmitting $D_l \cdot T_l$-complex dimensional IC with $D_l \cdot T_l$ complex symbols. Hence, in the proposed transmission scheme, $G_l$ has exactly $D_l \cdot T_l$ non-zero complex entries that represent the $D_l \cdot T_l$-complex dimensional IC within $\mathbb{C}^{M T_l}$. For each column of $G_l$, denoted by $g_i$, $i = 1 \ldots T_l$, we define the effective channel that $g_i$ sees as $\hat{H}_i$. It consists of the columns of $H$ that correspond to the non-zero entries of $g_i$, i.e. $H \cdot g_i = \hat{H}_i \cdot \hat{g_i}$, where $\hat{g_i}$ equals
the non-zero entries of $g_i$. As an example assume without loss of generality that the first $l_i$ entries of $g_i$ are not zero. In this case $\tilde{H}_i$ is an $N \times l_i$ matrix equals to the first $l_i$ columns of $H$. In accordance with (2.5), $H_{\text{eff}}^{(l)}$ is an $NT_l \times D_l \cdot T_l$ block diagonal matrix consisting of $T_l$ blocks. Each block corresponds to the multiplication of $H$ with different column of $G_l$, i.e. $\tilde{H}_i$ is the $i$'th block of $H_{\text{eff}}^{(l)}$. Note that in the effective matrix $NT_l \geq D_l \cdot T_l$.

We would like to elaborate on the structure of the blocks of $H_{\text{eff}}^{(l)}$. For this reason we denote the columns of $H$ as $h_i$, $i = 1, \ldots, M$.

1. The case where $N \geq M$. For this case the transmission scheme has $N + M - 1 - 2 \cdot l$ columns. The first $N - M + 1$ columns of $G_l, g_1, \ldots, g_{N-M+1}$, contain $M \cdot (N - M + 1)$ different complex symbols, i.e. there are no zero entries in these columns. Hence, in this case the first $N - M + 1$ blocks of $H_{\text{eff}}^{(l)}$ are

$$\tilde{H}_i = H \quad i = 1, \ldots, N - M + 1.$$  

(2.19)

After the first $N - M + 1$ columns we have $M - 1 - l$ pairs of columns. For each pair we have

$$\tilde{H}_{N-M+2k} = \{h_1, \ldots, h_{M-k}\}$$  

(2.20)

and

$$\tilde{H}_{N-M+2k+1} = \{h_{k+1}, \ldots, h_M\}$$  

(2.21)

where $k = 1, \ldots, M - 1 - l$.

2. The case where $M > N$. Again the transmission scheme has $N + M - 1 - 2 \cdot l$ columns. By the definition of the first $M - N + 1$ columns of $G_l$, we get that

$$\tilde{H}_i = \{h_i, \ldots, h_{N+i-1}\} \quad i = 1, \ldots, M - N + 1.$$  

(2.22)

We have additional $N - 1 - l$ pairs of columns in $G_l$. For each of these pairs we get

$$\tilde{H}_{M-N+2k} = \{h_1, \ldots, h_{N-k}\}$$  

(2.23)

and

$$\tilde{H}_{M-N+2k+1} = \{h_{M-N+k+1}, \ldots, h_M\}$$  

(2.24)

where $k = 1, \ldots, N - 1 - l$.

**Example:** consider $M = 4$, $N = 3$ as presented in (2.18). In this case $l = 0, 1, 2$ and we have $D_2 = 3$, $D_1 = 2.5$ and $D_0 = 2$ respectively.

1. $D_2 = 3$: $H_{\text{eff}}^{(2)}$ is generated from the multiplication of the $3 \times 4$ matrix $H$ with the first two columns of the transmission matrix. In this case $H_{\text{eff}}^{(2)}$ is a $6 \times 6$ block diagonal matrix, consisting of two blocks. Each block is a $3 \times 3$ matrix. We get that $\tilde{H}_1 = \{h_1, h_2, h_3\}$ and $\tilde{H}_2 = \{h_2, h_3, h_4\}$.  

19
\[ H^{(0)}_{\text{eff}} = \begin{pmatrix} h_1 & h_2 & h_3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & h_2 & h_3 & h_4 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & h_3 & h_4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & h_5 \end{pmatrix} \] (2.25)

2. \( D_1 = \frac{10}{4} = 2.5 \): \( H^{(1)}_{\text{eff}} \) is a 12 \( \times \) 10 block diagonal matrix consisting of 4 blocks. The first two blocks are identical to the blocks of \( H^{(2)}_{\text{eff}} \). The additional two blocks (multiplication with columns 3-4) are 3 \( \times \) 2 matrices. We get that \( \tilde{H}_3 = \{ h_3, h_4 \} \) and \( \tilde{H}_4 = \{ h_5 \} \).

3. \( D_0 = 2 \): \( H^{(0)}_{\text{eff}} \) consists of six blocks. In this case the last two blocks are 3 \( \times \) 1 vectors. We get that \( \tilde{H}_5 = h_4 \) and \( \tilde{H}_6 = h_4 \).

We present \( H^{(0)}_{\text{eff}} \) of our example in equation (2.25). Note that \( h_i \in \mathbb{C}^3 \) for 1 \( \leq \) \( i \) \( \leq \) 4, and \( \bar{0} \) is a 3 \( \times \) 1 vector.

From the sequential construction of the blocks of \( H^{(l)}_{\text{eff}} \) (2.19)-(2.21), (2.22)-(2.24) it is easy to see that when two columns of \( H \) occur in a certain block of \( H^{(l)}_{\text{eff}} \), the columns of \( H \) between them must also occur in the same block, i.e. if \( h_i, h_j \) occur in a certain block, then \( h_2, h_3, h_4 \) also occur in the same block. Next we prove a property of the transmission scheme \( G_l \), that relates to the number of occurrences of the columns of \( H \) in the blocks of \( H^{(l)}_{\text{eff}} \). For each set of columns in \( H \), we give an upper bound on the amount of its appearances in different blocks.

**Lemma 2.1.** Consider the transmission scheme \( G_l \), \( l = 0, \ldots, L - 1 \). In case 0 \( \leq \) \( i - j \) \( < \) \( L \), the columns \( h_j, \ldots, h_l \) may occur together in at most \( N - i + j \) blocks of \( H^{(l)}_{\text{eff}} \). In case \( i - j \) \( \geq \) \( L \) they can not occur together in any block of \( H^{(l)}_{\text{eff}} \).

**Proof.** See appendix A.3. \( \square \)

### 2.4.3 Upper Bound on The Error Probability

Next we would like to derive an upper bound on the average decoding error probability of ensemble of \( D_1 \cdot T_1 \)-complex dimensional IC, for each channel realization. We define \( |H^{(l)}_{\text{eff}} H^{(l)}_{\text{eff}}| = \rho^{-\sum_{i=1}^{D_1} \eta_i} \), where \( \rho^{-\eta_i} \) is the \( i \)th singular value of \( H^{(l)}_{\text{eff}} \), 1 \( \leq \) \( i \) \( \leq \) \( D_1 \cdot T_1 \). We also define \( \eta = (\eta_1, \ldots, \eta_{D_1 \cdot T_1})^T \). Note that \( N T_i \geq D_1 \cdot T_1 \).

**Theorem 2.3.** There exists a sequence of \( D_1 \cdot T_1 \)-complex dimensional IC’s, with channel realization \( H^{(l)}_{\text{eff}} \) and a receiver VNR \( \mu_{rc} = \rho^{1 - \frac{\sum_{i=1}^{D_1} \eta_i}{D_1 \cdot T_1}} \), that has an average decoding error probability

\[
\mathcal{T}_c(H^{(l)}_{\text{eff}}, \rho) = \mathcal{T}_c(\eta, \rho) \leq F(D_1 \cdot T_1) \rho^{-T_i(D_i - r) + \sum_{i=1}^{D_1} \eta_i} = F(D_1 \cdot T_1) \rho^{-T_i(D_i - r)} \cdot |H^{(l)}_{\text{eff}} H^{(l)}_{\text{eff}}|^{-1}
\]

where \( F(D_1 \cdot T_1) \) is a constant independent of \( \rho \), and \( \eta_i \geq 0 \) for every 1 \( \leq \) \( i \) \( \leq \) \( D_1 \cdot T_1 \).

**Proof.** We base our proof on the techniques developed by Poltyrev [20] for the AWGN channel. However, the channel considered here is colored. In spite of that, we show that what affects the average decoding error.
probability is the singular values product, which is encapsulated by the receiver VNR, \( \mu_{rc} \). This observation enables us to facilitate this colored channel analysis. The full proof in appendix A.4.

By averaging arguments we know that there exists a sequence of IC’s that satisfies these requirements.

### 2.4.4 Achieving the Optimal DMT

In this subsection we calculate the DMT of the proposed transmission scheme. We upper bound the determinant of the effective channel inverse, \( |H_{\text{eff}}^{(l)}H_{\text{eff}}^{(l)}|^{-1} \), based on the effective channel properties presented in Subsection 2.4.2. In Theorem 2.3 we showed that the upper bound on the error probability depends on this determinant. Hence, the upper bound on the determinant gives us a new upper bound on the average decoding error probability. We average the new upper bound over all channel realizations and get the DMT of the transmission scheme.

The channel matrix \( H \) consists of \( N \cdot M \) i.i.d entries, where each entry has distribution \( h_{i,j} \sim \mathcal{CN}(0,1) \). Without loss of generality we consider the case where the columns of \( H \) are drawn sequentially from left to right, i.e. \( h_{1,j} \) is drawn first, then \( h_{2,j} \) is drawn et cetera. Column \( h_j \) is an \( N \)-dimensional vector. Given \( h_{\max(1,j-N+1)}, \ldots, h_{j-1} \), we can write

\[
h_j = \Theta(h_{\max(1,j-N+1)}, \ldots, h_{j-1}) \cdot \tilde{h}_j
\]

where \( \Theta(\cdot) \) is an \( N \times N \) unitary matrix. \( \Theta(\cdot) \) is chosen such that:

1. The first element of \( \tilde{h}_j \), \( \tilde{h}_{1,j} \), is in the direction of \( h_{j-1} \).
2. The second element, \( \tilde{h}_{2,j} \), is in the direction orthogonal to \( h_{j-1} \), in the hyperplane spanned by \( \{h_{j-1}, h_{j-2}\} \).
3. Element \( \tilde{h}_{\min(j,N)-1,j} \) is in the direction orthogonal to the hyperplane spanned by

\[
\{h_{\max(2,j-N+2)}, \ldots, h_{j-1}\}
\]

inside the hyperplane spanned by

\[
\{h_{\max(1,j-N+1)}, \ldots, h_{j-1}\}.
\]

4. The rest of the \( N - \min(j,N) + 1 \) elements are in directions orthogonal to the hyperplane

\[
\{h_{\max(1,j-N+1)}, \ldots, h_{j-1}\}.
\]

Note that \( \tilde{h}_{i,j}, 1 \leq i \leq N, 1 \leq j \leq M \) are i.i.d random variables with distribution \( \mathcal{CN}(0,1) \). Let us denote by \( h_{j,j-1,\ldots,j-k} \) the component of \( h_j \) which resides in the \( N-k \) subspace which is perpendicular to the
space spanned by \( \{h_{j-1}, \ldots, h_{j-k}\} \). In this case we get
\[
\|h_{j\perp_{1-j-k}}\|_2^2 = \sum_{i=k+1}^{N} |\tilde{h}_{i,j}|^2  \quad 1 \leq k \leq \min(j,N) - 1.
\] (2.26)

If we assign \( |\tilde{h}_{i,j}|^2 = \rho^{-\xi_{i,j}} \), we get that the probability density function (PDF) of \( \xi_{i,j} \) is
\[
f(\xi_{i,j}) = C \cdot \log \rho \cdot \rho^{-\xi_{i,j}} \cdot e^{-\rho^{-\xi_{i,j}}}
\] (2.27)
where \( C \) is a normalization factor. In our analysis we assume a very large value for \( \rho \). Hence we can neglect events where \( \xi_{i,j} < 0 \) since in this case the PDF (2.27) decreases exponentially as a function of \( \rho \). For a very large \( \rho \), \( \xi_{i,j} \geq 0 \), \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \), the PDF takes the following form
\[
f(\xi_{i,j}) \propto \rho^{-\xi_{i,j}} \quad \xi_{i,j} \geq 0.
\] (2.28)

In this case by assigning in (2.26) the vector \( \xi_j = (\xi_{1,j}, \ldots, \xi_{N,j})^T \), with PDF that proportional to \( \rho^{-\sum_{i=1}^{N} \xi_{i,j}} \), we get
\[
\|h_{j\perp_{1-j-k}}\|_2^2 = \rho^{-\min_{s\in\{k+1,\ldots,N\}} \xi_{s,j}} = \rho^{-a(k,\xi_j)}
\] (2.29)
where \( 1 \leq k \leq \min(j,L) - 1 \) and \( a(k,\xi_j) = \min_{s\in\{k+1,\ldots,N\}} \xi_{s,j} \). In addition
\[
\|\tilde{h}_{i,j}\|_2^2 = \rho^{-\min_{s\in\{1,\ldots,N\}} \xi_{s,j}} = \rho^{-a(0,\xi_j)}.
\] (2.30)

Note that
\[
a(\min(j,L) - 1, \xi_j) \geq \cdots \geq a(0, \xi_j) \geq 0.
\] (2.31)

Next we wish to quantify the contribution of a certain column in the channel matrix, \( \tilde{h}_{j} \), to the determinant \( |H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}| \). \( H_{\text{eff}}^{(l)} \) is a block diagonal matrix. Hence the determinant of \( |H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}| \) can be expressed as
\[
|H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}| = \prod_{i=1}^{T_l} |\tilde{H}_i^\dagger \tilde{H}_i|.
\] (2.32)

Assume \( \tilde{H}_i = (\tilde{h}_{i,1}, \ldots, \tilde{h}_{i,m}) \), i.e. \( \tilde{H}_i \) has \( m \) columns. In this case we can state that the determinant
\[
|\tilde{H}_i^\dagger \tilde{H}_i| = \|\tilde{h}_{i,1}\|^2 \|\tilde{h}_{i,2}\|^2 \cdots \|\tilde{h}_{i,m\perp_{m-1},\ldots,1}\|^2.
\]

Note that \( \tilde{H}_i \) also has more rows than columns. The columns of \( \tilde{H}_i \) are subset of the columns of the channel matrix \( H \). Hence we are interested in the blocks where \( \tilde{h}_{j} \) occurs. We know that the contribution of \( \tilde{h}_{j} \) to those determinants can be quantified by taking into account the columns to its left in each block. We consider two cases:

- The case \( N \geq M \). In this case we can see from (2.19)-(2.21) that \( \tilde{h}_{j} \) may occur with \( \{\tilde{h}_{1}, \ldots, \tilde{h}_{j-1}\} \) to its left in different blocks.
Theorem 2.4. There exists a sequence of $D_1 \cdot T_1$-complex dimensional IC’s with transmitter density $\gamma_{tr} = \rho^{T_1}$ and $T_1$ channel uses that has diversity order

$$d_{D_1,T_1}(r) \geq (M - l)(N - l) - (r - l)(N + M - 2 \cdot l - 1)$$

where $0 \leq r < D_1$ and $l = 0, \ldots, L - 1$. In the range $l \leq r \leq l + 1$ this lower bound coincides with the optimal DMT of finite constellations $d_{M,N}^{\ast,(FC)}(r)$.

Proof. The proof outline is as follows. The upper bound on the error probability from Theorem 2.3 depends on $|H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}|^{-1}$. We upper bound this determinant value and average over different realizations of $H_{\text{eff}}^{(l)}$ in order to find the diversity order of the transmission matrix $G_l$. We begin by lower bounding $|H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}|$. Based on the sequential structure of $G_l$, we lower bound the contribution of a certain column of $H$, $\tilde{h}_j$, $1 \leq j \leq M$ to the determinant. This gives us a new upper bound on the error probability for each channel realization. We average the new upper bound on the error probability, by averaging over $\tilde{h}_1, \ldots, \tilde{h}_M$. From this averaging we get the required DMT. The full proof is in appendix A.5.

The diversity order attained in Theorem 2.4 for $D_1, T_1$ coincides with the optimal DMT of finite constellations in the range $l \leq r \leq l + 1$. Hence, by considering $0 \leq l \leq L - 1$, we can attain the optimal DMT with $L$ sequences of IC’s.

We present as an illustrative example the case of $M = N = 2$. Let us consider the case where $l = 0$. In this case $D_0 = \frac{4}{3}$, and $T_0 = 3$, i.e. we transmit 4-complex dimensional IC. The transmission scheme diversity order in this case is $4 - 3r$, $0 \leq r \leq \frac{4}{3}$. In this case the effective channel matrix, $H_{\text{eff}}^{(0)}$, consists of three blocks: $\tilde{H}_1 = (\tilde{h}_1, \tilde{h}_2)$, $\tilde{H}_2 = \tilde{h}_1$ and $\tilde{H}_3 = \tilde{h}_2$. According to our definitions

$$|\tilde{H}_1^{\dagger} \tilde{H}_1| = \|\tilde{h}_1\|^2 \cdot \|\tilde{h}_{2,1}\|^2 = \rho^{-\min(\xi_{1,1},\xi_{2,1})} \cdot \rho^{-\xi_{2,2}}$$
and also \( \| \mathbf{h}_1 \|_2^2 = \rho^{\min(\xi_{1,1}, \xi_{2,1})} \), \( \| \mathbf{h}_2 \|_2^2 = \rho^{\min(\xi_{1,2}, \xi_{2,2})} \). In accordance with (A.38) we divide the integral into two terms. In the first term we solve the optimization problem

\[
\min_{\xi_{2,2} \in \mathcal{A}} (4 - 3r) - (\xi_{2,2} + 2 \cdot \min (\xi_{1,1}, \xi_{2,1}) + \min (\xi_{1,2}, \xi_{2,2})) + \sum_{i=1}^{2} \sum_{j=1}^{2} \xi_{i,j}.
\] (2.34)

One solution to this problem is \( \xi_{i,j} = 0 \) for \( 1 \leq i \leq 2, 1 \leq j \leq 2 \). In this case we get an exponential term that equals \( 4 - 3r \). For the second integral we solve the optimization problem

\[
\min_{\xi_{2,2} \in \mathcal{A}} \sum_{i=1}^{2} \sum_{j=1}^{2} \xi_{i,j}.
\]

In this case the optimization problem solution is \( \sum_{i=1}^{2} \sum_{j=1}^{2} \xi_{i,j} = 4 - 3r \). Hence, all together, we get a diversity order that equals \( 4 - 3r \), that coincides with the optimal DMT of finite constellations in the range \( 0 \leq r \leq 1 \).

In the next theorem we prove the existence of a sequence of lattices that has the same lower bound as in Theorem 2.4.

**Theorem 2.5.** There exists a sequence of \( 2D_l \cdot T_l \)-real dimensional lattices with transmitter density \( \gamma_{tr} = \rho^{T_l} \) and \( T_l \) channel uses, that attains a diversity order

\[
d_{D_l \cdot T_l}(r) \geq (M - l)(N - l) - (r - l)(N + M - 2 \cdot l - 1)
\]

where \( 0 \leq r \leq D_l \) and \( l = 0, \ldots, L - 1 \).

**Proof.** See appendix A.7

Note that we considered a \( 2D_l \cdot T_l \)-real dimensional lattice, where the lattice first \( D_l \cdot T_l \) dimensions are spread over the real part of the non-zero entries of \( G_l \), and the other \( D_l \cdot T_l \) dimensions of the lattice are spread on the imaginary part of the non-zero entries of \( G_l \). This does not necessarily yields a \( D_l \cdot T_l \)-complex dimensional lattice in the transmission scheme. Considering the \( 2D_l \cdot T_l \)-real dimensional lattice enables us to use the Minkowski-Hlawaka-Siegel Theorem [20],[11], and prove Theorem 2.5.

### 2.4.5 Power Spreading

For practical reasons, such as power peak to average ratio, one may prefer to have a transmission scheme that spreads the transmitted power equally over time and space. The transmitting matrix \( G_l \) contains exactly \( D_l \cdot T_l \) non-zero entries, where the rest of the entries are zero. In order to spread the power more equally over time and space we use the following unitary operations

\[ U_L G_l U_R. \]
$U_L$ is an $M \times M$ unitary matrix that spreads each column of $G_I$, i.e. spreads over space. $U_R$ is a $T_l \times T_l$ unitary matrix that spreads each raw of $G_I$, i.e. spreads over time. As the distribution of $H$ and $H \cdot U_L$ are identical, multiplying $U_L$ with $G_I$ gives exactly the same performance. Based on the notations from (2.4) we can state that

$$G_I \cdot U_R = (\underline{x}_1, \ldots, \underline{x}_{T_l})$$

where $(\underline{x}_1, \ldots, \underline{x}_{T_l})$ are the channel inputs. At the receiver we can state that the received signals are $(\underline{y}_1, \ldots, \underline{y}_{T_l})$. By multiplying with $U_R^\dagger$ we get

$$(\underline{y}_1, \ldots, \underline{y}_{T_l}) \cdot U_R^\dagger = G_I + (\underline{w}_1, \ldots, \underline{w}_{T_l}) U_R^\dagger$$

The distribution of $(\underline{w}_1, \ldots, \underline{w}_{T_l})$ is identical to the distribution of $(\underline{w}_1, \ldots, \underline{w}_{T_l}) U_R^\dagger$. Hence, multiplying $G_I$ with $U_R$ gives also exactly the same performance. For instance, in order to achieve full diversity and spread the power more uniformly, we take $G_0$ and duplicate its structure $s$ times to create the transmission scheme $G_0^{(s)}$. In this case the transmission matrix $G_0^{(s)}$ consists of $sD_0T_0$ complex non-zero entries, i.e. we transmit an $sD_0T_0$ complex dimensional IC within the $sMT_0$ complex space. $G_0^{(s)}$ is an $M \times sT_0$ dimensional matrix, that has exactly the same diversity order as $G_0$ (it duplicates the structure of $G_0$ $s$ times). Each row of $G_0^{(s)}$ has exactly $sN$ non-zero entries. We define $U_R^{(s)}$ as $sT_0 \times sT_0$ unitary matrix. For large enough $s$, the multiplication $G_0^{(s)} \cdot U_R^{(s)}$ spreads the power more uniformly over space and time, and still achieves full diversity.\(^3\)

### 2.4.6 Averaging Arguments

In this subsection we show that there exist $L$ sequences of lattices that attain the optimal DMT, where each sequence of the $L$ sequences attains a different segment on the optimal DMT curve. In addition we show that there exists a single IC that attains the optimal DMT by diluting its points and adapting its dimensionality.

As a consequence of Theorem 2.3 and Theorem 2.4 we can state the following

**Corollary 2.3.** Consider a sequence of $D \cdot T$-complex dimensional IC’s $S_{D,T}(\rho)$ with density $\gamma_{tr} = 1$, that attains diversity order $d$. This sequence of IC’s also attains diversity order $d(1 - \frac{r}{D})$ when the sequence density is scaled to $\gamma_{tr} = \rho^T$.

**Proof.** The proof is in appendix A.8. \(\square\)

**Corollary 2.4.** The optimal DMT is attained by exactly $L$ sequences of $2D_l \cdot T_l$-real dimensional lattices, $l = 0, \ldots, L - 1$, where each sequence attains different segment of the optimal DMT.

**Proof.** From Theorem 2.5 we know that there exists a $2D_l \cdot T_l$-real dimensional sequence of lattices with density $\gamma_{tr} = 1$ that attains diversity $(M - l)(N - l) + l(N + M - 2 \cdot l - 1)$. Hence, based on Corollary 2.3 we can scale this $2D_l \cdot T_l$-real dimensional sequence of lattices into a sequence of lattices with density $\gamma_{tr} = \rho^T$.

\(^3\)It can be shown that replacing $U_L$ and $U_R$ with any other two invertible matrices still yields transmission scheme that attains the optimal DMT. It extends the set of subspaces in $C^{MT}$ that attain the optimal DMT. It also alludes that alongside the proposed transmission matrix 2.4.1, there are many other options to attain the optimal DMT.
γ_{tr} = ρ^{T_1}, and a diversity order (M - l)(N - l) - (r - l)(N + M - 2l - 1), i.e. the sequence of lattices attains the optimal DMT line in the range l ≤ r ≤ l + 1. The optimal DMT is the maximal value of the L lines, for each 0 ≤ r ≤ L. Hence, there exist L sequences of lattices that attain the optimal DMT.

Next, we show that there exists a single sequence of IC’s that attains the optimal DMT. The optimal DMT consists of L segments of straight lines. Each segment is attained by reducing the IC’s dimensionality to the correct dimension, and diluting their points to get the desired density. Note that in Theorem 2.4 we showed that for each multiplexing gain, r, there exists a sequence of IC’s that attains the optimal DMT. On the other hand, in Corollary 2.5 we show that a single sequence of IC’s attains the optimal DMT for any r, by adapting its dimensionality and diluting its points. Also note that D_0T_0 > D_1T_1 > · · · > D_{L-1}T_{L-1}.

Corollary 2.5. There exists a single sequence of D_0T_0-complex dimensional IC’s, that attains the L segments of the optimal DMT:

(M - l)(N - l) - (r - l)(N + M - 2l - 1)  0 ≤ r ≤ D_l

where l = 0, · · · , L − 1. The l’th segment is attained by reducing the IC’s complex dimensionality to D_lT_l, and by diluting their points to get density γ_{tr} = ρ^{T_lr}.

Proof. See Appendix A.9.

2.5 Discussion

In this section we discuss the results presented in the chapter. We begin by explaining why full dimension lattice based coding schemes such as Golden-codes [2], perfect codes [19] and other cyclic-division algebra based space-time codes [7] which were shown to attain the optimal DMT, are sub-optimal when regular lattice decoder (2.3) is employed at the receiver. In addition, we explain why using the MMSE estimation at the receiver enables these schemes to attain the optimal DMT. Afterwards, based on our results, we give another geometrical interpretation for the optimal DMT. Finally, since in practice a finite codebook is transmitted, we show that a finite constellation with multiplexing gain r as defined in [35] can also be carved from a lattice with multiplexing gain r as defined for IC’s in (2.6).

2.5.1 Lattice Constellations Vs. Full Dimension Lattice Based Finite Constellations

In order to demonstrate that full dimension lattice based coding schemes with regular lattice decoding are sub-optimal let us consider Golden-codes transmitted over a channel with M = N = 2 where T = 2. For large ρ the channels singular values PDF is proportional to ρ^{-α_1−3α_2}, where α_1 ≥ α_2 ≥ 0. A Golden-code of a certain rate is carved from a 4-complex dimensional lattice. We show that when performing regular lattice decoding at the receiver the maximal diversity order that can be attained for r = 0 is 2. This is in contrast to ML decoding or alternatively MMSE estimation followed by lattice decoding [6], [13] for which the maximal diversity order equals 4.
We begin by showing why the maximal diversity order of a Golden-code is 2 when performing regular lattice decoding. At the receiver, the squared effective radius of the effective lattice induced by the channel realization equals (2.1)
\[ r_{eff}^2 \doteq \rho - \frac{\alpha_1 + \alpha_2}{2} = \gamma_{rc}. \] (2.35)
For lattices \( r_{eff} \geq r_{\text{packing}} = \frac{d_{\text{min}}^{(\text{lattice})}}{2} \), where \( r_{\text{packing}}, d_{\text{min}}^{(\text{lattice})} \) are the packing radius and the minimal distance of the lattice respectively. Hence, we get
\[ \left( \frac{d_{\text{min}}^{(\text{lattice})}}{2} \right)^2 \doteq \rho - \frac{\alpha_1 + \alpha_2}{2}. \] (2.36)
When the squared minimal distance is in the order of the additive noise variance, \( \rho^{-1} \), the error probability will not decrease with \( \rho \). This will happen for instance when \( \alpha_2 = 0 \) and \( \alpha_1 = 2 \). This event occurs for large \( \rho \) with probability proportional to \( \rho^{-2} \). Hence, in this case the diversity order is 2. Note that for the 4-complex dimensional lattice we get (2.9)
\[ \mu_{rc} \doteq \frac{r_{eff}^2}{\rho^{-1}} = \rho^{-1} - \frac{\alpha_1 + \alpha_2}{2}. \] (2.37)
Therefore, the event where the squared effective radius is in the order of the noise variance is equivalent to \( \mu_{rc} \doteq 1 \) which is the outage event for lattices, presented in Theorem 2.2.

From equation (2.36) we get that the minimal distance for each channel realization of the entire lattice, induces diversity order 2. On the other hand, when the decoder only considers the words within the finite codebook, the non-vanishing determinant (NVD) property combined with the boundaries of the codebook lead to a lower bound on the minimal distance of the Golden-code for each channel realization, that is larger than the expression in (2.36), and enables to attain diversity order 4, [7].

The fact that considering the entire lattice leads to smaller minimal distance is not surprising since the multiplication of the transmitted lattice and the channel matrix leads to scaling of this lattice in the direction of the channel singular values. When considering the infinite lattice, the scaling may reduce the distance between points that were very far in the transmitted lattice. These points are not necessarily part of the finite codebook and therefore do not affect the minimal distance of the finite Golden-code but do affect the minimal distance of the lattice.

MMSE estimation followed by lattice decoding will also lead to diversity order 4. Translating the arguments presented in [6], [13] to our setting leads to VNR
\[ \tilde{\mu}_{rc} = \rho \frac{(1-\alpha_1)^+ + (1-\alpha_2)^+}{2} \] (2.38)
where \( (x)^+ = x \) for \( x \geq 0 \) and zero otherwise. This expression is larger than the expression in (2.37) and implies that the MMSE estimation, which takes into account the transmitted power, also improves the minimal distance for each channel realization. However, the improvement in VNR (and minimal distance) comes at the expense of a self additive noise that depends on the transmitted codeword. Under the assumption that
the transmitted codewords are not too far from the origin the variance of the effective noise is small enough to attain the optimal DMT. For instance the codewords of the Golden-code lie within a bounded shaping region, which enables to attain diversity order 4. Note that for the entire lattice, the farther the lattice point is from the origin, the larger is the effective noise variance. This eventually leads to poor error performance for lattice points far enough from the origin.

In this chapter we show that transmitting a lattice with NDCU $D = \frac{4}{3}$ and performing regular lattice decoding at the receiver leads to VNR

$$\mu_{\text{rc}} = \rho^{1 - \frac{\alpha_1}{4} - \frac{3\alpha_2}{4}}$$

which is also larger than (2.37) and enables to attain diversity order 4 (in fact it attains the optimal DMT in the range $0 \leq r \leq 1$). Hence, we can see from the results in this chapter that reducing the lattice dimensionality increases the lattice minimal distance to such an extent that enables to attain the optimal DMT when performing regular lattice decoding. In this sense reducing the lattice dimensionality takes the role of MMSE estimation. It is also interesting to note that MMSE estimation followed by lattice decoding yields good error performance for lattice points close enough to the origin (for instance lattice points within the shaping region), and bad performance for lattice points very far from the origin. On the other hand, regular lattice decoding yields the same performance for all lattice points inside or outside the shaping region. An illustrative example that shows how reduced dimension assists in increasing the minimal distance compared to full dimension lattice is presented in Figure 2.3.

2.5.2 Geometrical Interpretation of the Optimal DMT for IC’s

In this subsection we give a geometrical interpretation for the optimal DMT, based on allocation of lattice dimensions. This is a qualitative discussion and the exact results appear in Sections 2.3, 2.4.

First from our results we can see that for a sequence of lattices with certain NDCU the DMT is a straight line as a function of the multiplexing gain (see Corollary 2.3). It results from the fact that for lattices changing the multiplexing gain is equivalent to scaling each dimension by $\rho^{-\frac{r}{D}}$. Assume that the sequence of lattices attains diversity order $d$ for multiplexing gain $r = 0$, i.e. the error probability decays as $\rho^{-d}$. In this case scaling each dimension by $\rho^{-\frac{r}{D}}$ leads to error probability that decays as $\rho^{-d\left(1 - \frac{r}{D}\right)}$. This behavior results from the fact that the lattice decoder takes into consideration all the lattice points. Hence, the scaling merely replaces $\rho$ with $\rho^{1 - \frac{r}{D}}$ in the error probability expression. The optimal DMT is a piecewise linear function. We get that each line corresponds to a sequence of lattices with certain NDCU.

Next we wish to give the reasoning for the NDCU required to achieve each line of the optimal DMT. For simplicity let us consider the case $M = N = 3$. We begin by considering the straight line in the range $0 \leq r \leq 1$. In this range the optimal DMT equals $9 - 5 \cdot r$. We wish to show why the NDCU that enables to attain this straight line equals $\frac{9}{5}$. For large $\rho$ the channel singular values PDF is of the form of $\rho^{-\alpha_1 - 3\alpha_2 - 5\alpha_3}$, where $\alpha_1 \geq \alpha_2 \geq \alpha_3 \geq 0$. When the transmission scheme spreads over $T$ channel uses, the equivalent channel matrix, $H_{\text{ex}}$, presented in (2.5) has $3T$ singular values. Each singular value of $H$ occurs $T$ times in the singular values of $H_{\text{ex}}$. Assume each complex dimension of the lattice is transmitted on a certain singular value of $H_{\text{ex}}$. Let us denote by $T_i$ the number of dimensions transmitted on the singular value
(a) Finite constellation: In this case even when $h_2$ is small it is possible to decode.

(b) Full dimensional infinite constellation: In this case due to the finiteness of the constellation when $h_2$ is very small it is impossible to decode.

(c) Infinite constellation with reduced dimension: In this case even when $h_2$ is very small it is possible to decode.

Figure 2.3: Illustrative example for the case $M = 2, N = 2$ of the significance of reducing dimensions when considering regular lattice decoding. For this example we assume that the realization of $H$ is diagonal, where the diagonal elements are $h_1$ and $h_2$. 

29
that equals $\rho^{-\frac{\alpha_i}{r}}$, $1 \leq i \leq 3$. Note that $\sum_{i=1}^{3} T_i$ may be smaller than $3T$. According to this assumption a $(\sum_{i=1}^{3} T_i)$-complex dimensional lattice is transmitted over $T$ channel uses, and the NDCU is $D = \sum_{i=1}^{3} T_i$. The effective radius at the receiver equals

$$r_{eff}^2 = \rho \frac{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{1}^{\alpha_{1} + T_{2}^{\alpha_{2}} + T_{3}^{\alpha_{3}}}}{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{i}}. \quad (2.40)$$

and the VNR equals

$$\mu_{tc} = \rho \frac{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{1}^{\alpha_{1} + T_{2}^{\alpha_{2}} + T_{3}^{\alpha_{3}}}}{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{i}}. \quad (2.41)$$

We are interested in the probability of the outage event, i.e. the probability that $\rho^{-\frac{\alpha_i}{r}}$, $1 \leq i \leq 3$. Note that $\sum_{i=1}^{3} T_i$ may be smaller than $3T$. According to this assumption a $(\sum_{i=1}^{3} T_i)$-complex dimensional lattice is transmitted over $T$ channel uses, and the NDCU is $D = \sum_{i=1}^{3} T_i$. The effective radius at the receiver equals

$$r_{eff} = \rho \frac{T_{i}^{\alpha_{1} + T_{2}^{\alpha_{2}} + T_{3}^{\alpha_{3}}}}{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{i}}. \quad (2.42)$$

i.e. each singular value can not occur in more dimensions than the relative effect it has on the PDF of the singular values. The largest NDCU that fulfils (2.42) is $\frac{9}{5}$. In this case for $T = 5$ a $9$-complex dimensional lattice is transmitted, and the conditions are fulfilled with equality when $T_1 = 1$, $T_2 = 3$ and $T_3 = 5$. When $D < \frac{9}{5}$ the conditions in (2.42) are still fulfilled and therefore diversity order $9$ is still attained for $r = 0$. However, based on (2.40) we get for $r > 0$ that $r_{eff}$ decreases faster than the case of $D = \frac{9}{5}$. Hence, for $D < \frac{9}{5}$ the diversity order is smaller than $9 - 5 \cdot r$ when $0 < r \leq \frac{9}{5}$.

So far we have shown that choosing $D < \frac{9}{5}$ leads to sub-optimal DMT. Now, we wish to show that in the range $0 \leq r \leq 1$ the DMT is smaller than $9 - 5 \cdot r$ also when $D > \frac{9}{5}$. First, for $D > \frac{9}{5}$ the conditions in (2.42) are not met. Hence, in this case the diversity order is smaller than $9$ at $r = 0$. This can be shown rather easily due to the “mismatch” of the channel singular values and the lattice dimensions. For $r = 1$ and $D = \frac{9}{5}$ the diversity order equals $4$. Assume the best assignment of lattice dimensions would enable to choose $T_3 = T$. In this case $\mu_{tc}$ in (2.41) is affected equally if $r = 1$, $\alpha_3 = 0$ or $r = 0$, $\alpha_3 = 1$, i.e. the scaling inflicted by $r = 1$ decreases $r_{eff}$ in (2.40) as if the singular value $\rho^{-\frac{\alpha_3}{r}}$ equals $\rho^{-\frac{1}{r}}$. In both cases we get

$$\mu_{tc} = \rho \frac{T_{1}^{\alpha_{1} + T_{2}^{\alpha_{2}} + T_{3}^{\alpha_{3}}}}{\sum_{i=1}^{T} \sum_{j=1}^{T_i} T_{i}}. \quad (2.43)$$

The difference is that when $r = 1$, $\alpha_3 = 0$ the PDF of the singular values equals $\rho^{-\alpha_1 - 3\alpha_2}$ which leads to smaller diversity order than the case $r = 0$, $\alpha_3 = 1$. For large $\rho$ and $r = 1$, $\alpha_3 = 0$ is included in the most
dominant error event when $D \geq \frac{9}{4}$. Hence, diversity order of 4 is attained for $r = 1$ and $D > \frac{9}{3}$ when the following condition is met
\[
\frac{T_1}{T_1 + T_2} \leq \frac{1}{4} \tag{2.44}
\]
which is exactly the condition for attaining maximal diversity order of 4 when $r = 0$ in a channel with 2 transmit and 2 receive antennas. This condition is met as long as $D \leq \frac{7}{3}$. Hence, for $\frac{9}{3} < D \leq \frac{7}{3}$ the best diversity order is smaller than 9 when $r = 0$, and equals 4 when $r = 1$. Since for each $D$ the largest DMT is a straight line, the DMT for each $0 < D \leq \frac{7}{3}$ (except for $D = \frac{9}{3}$) in the range $0 \leq r \leq 1$ is smaller than $9 - 5 \cdot r$. We are left with the case $\frac{7}{3} < D \leq 2$. By applying similar arguments, only this time considering $r = 2$, it can be shown that in the range $0 \leq r < 2$ the largest DMT for any $\frac{7}{3} < D \leq 2$ is smaller than $7 - 3 \cdot r$. These arguments also show that in the range $2 \leq r \leq 3$ the optimal DMT equals $2 - r$. Hence, we get for $0 \leq r < 1$ that the optimal DMT equals $9 - 5 \cdot r$, where for $1 \leq r < 2$, $2 \leq r \leq 3$ the optimal DMT equals $7 - 3 \cdot r$ and $2 - r$ respectively.

### 2.5.3 Example for the Case $M = N = 2$

We wish to give an example of a transmission scheme for the case $M = N = 2$, that enables IC’s to attain the first segment of the optimal DMT $4 - 3 \cdot r$ in the range $0 \leq r \leq 1$. In order to attain the first segment we need an IC that spreads over $\frac{1}{3}$ NDCU. For instance transmitting a 4-complex dimensional IC over 3 channel uses. In a similar manner to the discussion in Subsection 2.5.2 we wish to design a transmission scheme that matches the PDF of the singular values of the channel at large $\rho$, i.e. matches $\lambda_1 \cdot \lambda_2 = \rho^{-\alpha_1 - 3\alpha_2}$, where $\alpha_1 \geq \alpha_2 \geq 0$. Therefore, we wish to construct a transmission scheme that allocates three complex dimensions of the IC to the stronger singular value $\lambda_2 = \rho^{-\alpha_2}$ and only a single complex dimension to the smaller singular value $\lambda_1 = \rho^{-\alpha_1}$.

Let us consider a 4-complex dimensional IC. We propose the following transmission scheme that combines regular transmission with the Alamouti scheme

\[
\begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
\begin{pmatrix}
    \frac{x_2}{\sqrt{2}} \\
    \frac{x_4}{\sqrt{2}}
\end{pmatrix}
\begin{pmatrix}
    -\frac{x_2}{\sqrt{2}} \\
    \frac{x_4}{\sqrt{2}}
\end{pmatrix}
\]

where $(x_1, x_2, x_3, x_4)$ is a point in the IC. After performing the same manipulations as in [1] on the symbols $x_3, x_4$ we get at the receiver
\[
\begin{pmatrix}
    y_1 \\
    y_2
\end{pmatrix}
= H \cdot \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix}
+ \begin{pmatrix}
    n_1 \\
    n_2
\end{pmatrix}
\tag{2.45}
\]
and
\[
\begin{pmatrix}
    \bar{y}_3 \\
    \bar{y}_4
\end{pmatrix}
= \sqrt{\frac{\sum_{k=1}^{2} \sum_{j=1}^{2} |h_{k,j}|^2}{2}} \cdot \begin{pmatrix}
    x_3 \\
    x_4
\end{pmatrix}
+ \begin{pmatrix}
    \bar{n}_3 \\
    \bar{n}_4
\end{pmatrix}
\tag{2.46}
\]
where $n_1, n_2, \bar{n}_3$ and $\bar{n}_4$ are independent with distribution $CN(0, \rho^{-1})$. Since $\sum_{k=1}^{2} \sum_{j=1}^{2} |h_{k,j}|^2 = \lambda_1 + \lambda_2 = \lambda_2$ we get from equation (2.46) that symbols $x_3$ and $x_4$ are scaled by the stronger singular value $\lambda_2$, where symbols $x_1, x_2$ are scaled by $\lambda_1$ and $\lambda_2$. Hence, the aforementioned transmission scheme positions
the IC in the 6-complex dimensional space such that three complex dimensions are scaled by $\lambda_2$ and only a single complex dimension is scaled by $\lambda_1$ for any channel realization. As a result, the channel has the smallest possible effect on a 4-complex dimensional IC.

It is important to note that the transmission scheme does not attain the optimal DMT for any IC, but (as we showed) there exists a sequence of IC’s that attains the optimal DMT when using, for instance, this transmission scheme. For example if we transmit the integer lattice, $\mathbb{Z}^n$, the transmission scheme will not attain diversity order 4 at $r = 0$.

2.5.4 The Relation Between the Multiplexing Gains of an IC and a Finite Constellation

In this chapter we defined the multiplexing gain of IC’s sequence as the rate at which the density of the IC’s increases (2.6), i.e., when $\gamma_{\text{tr}} = \rho^{rT}$ the multiplexing gain is $r$. We characterized the optimal DMT of IC’s based on this definition of the multiplexing gain. In practice a finite constellation is transmitted, even when performing regular lattice decoding at the receiver. Hence, in this subsection we show that finite constellation with multiplexing gain $r$ can be carved from a lattice with multiplexing gain $r$ (according to the definition given in (2.6)), while maintaining the same performance with regular lattice decoding at the receiver.

Consider a lattice $\Lambda$ with density $\gamma_{\text{tr}} = \rho^{rT}$. In this case for each lattice point the Voronoi region volume equals

$$|V(x)| = |V| = \gamma_{\text{tr}}^{-1} = \rho^{-rT} \forall x \in \Lambda.$$  

In [16] it has been shown that for any Jordan measurable bounded set $S$ with volume $|V(S)|$ there exists a translation $u$ such that

$$|(\Lambda + u) \cap S| \geq \frac{|V(S)|}{|V|}$$  

(2.47)

where $\Lambda + u$ is the translation of each lattice point by the constant $u$, and $|(\Lambda + u) \cap S|$ is the number of words of the translated lattice within the region $S$. Hence, for each lattice in a sequence with multiplexing gain $r$, there exists a translation such that the number of codewords within a sphere with volume 1 is larger or equal to $\rho^{rT}$, i.e. the rate is $r \log (\rho)$ where in this setting $\rho$ takes the role of SNR. Hence, it is possible to carve from the translated lattice sequence a finite constellation sequence with multiplexing gain $r$ according to the definitions of finite constellations. When performing regular lattice decoding the translation does not affect the performance. Hence, the results we presented in this chapter also apply when carving finite constellations with the corresponding multiplexing gain from the lattices sequence, and performing regular lattice decoding at the receiver.
Chapter 3

On the DMT of IC’s in MAC Channels

3.1 Introduction

Employing multiple antennas in a point-to-point wireless channel increases the number of degrees of freedom available for transmission. This is illustrated for the ergodic case in [28],[8], where $M$ transmit and $N$ receive antennas increase the capacity by a factor of $\min(M, N)$. The number of degrees of freedom utilized by the transmission scheme is referred to as multiplexing gain. Another advantage of employing multiple antennas is the potential increase in the transmitted signal reliability. The fact that multiple antennas increase the number of independent links between antenna pairs, enables the error probability to decrease, i.e. add diversity. If for high signal to noise ratio (SNR) the error probability is proportional to $\text{SNR}^{-d}$, then we state that the diversity order is $d$.

For the point-to-point setting, Zheng and Tse [35] characterized the optimal diversity-multiplexing trade-off (DMT) of the quasi-static Rayleigh flat-fading channel, i.e. for each multiplexing gain they found the best attainable diversity order. The optimal DMT is a piecewise linear function connecting the points $(M - l)(N - l)$, $l = 0, \ldots, \min(M, N)$. The transmission scheme in [35] uses random codes. Subsequent works presented more structured schemes that attain the optimal DMT. El Gamal et al. [6] showed by using probabilistic methods that lattice space-time (LAST) codes attain the optimal DMT by using minimum-mean square error (MMSE) estimation followed by lattice decoding. Later, explicit coding schemes based on lattices and cyclic-division algebra [7],[19] were shown to attain the optimal DMT by using maximum-likelihood (ML) decoding, and also by using MMSE estimation followed by lattice decoding [13]. A subtle but very important point is that these coding schemes take into consideration the finiteness of the codebook in the decoder. A question that remained open was whether lattices can achieve the optimal DMT by using regular lattice decoding, i.e. decoder that takes into account the infinite lattice without considering the shaping region or the power constraint. In order to answer this question an analysis of the performance of infinite constellations (IC’s) in multiple-input multiple-output (MIMO) fading channels is presented in Chapter 2. A new tradeoff is presented between the IC’s number of dimensions per channel use (NDCU), i.e. the IC dimensionality divided by the number of channel uses, and the best attainable DMT. By choosing the right NDCU, it is shown in Chapter 2 that IC’s in general and more specifically lattices using regular lattice decoding, attain the optimal DMT of finite constellations.
For the multiple-access channel, where a number of users transmit to a single receiver, the number of users in the network affects the multiplexing gain and the diversity order. For instance, for a network with $K$ users transmitting at the same rate, the number of available degrees of freedom for each user is $\min(M, \frac{N}{K})$. Tse, Viswanath and Zheng [30] characterized the optimal DMT of a network with $K$ users, where each user has $M$ transmit antennas and the receiver has $N$ antennas. For the symmetric case, where the users transmit at the same multiplexing gain $r$, i.e. $r_1 = \cdots = r_K = r$, the optimal DMT takes the following elegant form [30]:

- For $r \in \left[0, \min\left(\frac{N}{K+1}, M\right)\right]$ the optimal symmetric DMT equals to the optimal DMT of a point-to-point channel with $M$ transmit and $N$ receive antennas $d_{M,N}^{*(FC)}(r)$.
- For $r \in \left[\min\left(\frac{N}{K+1}, M\right), \min\left(M, \frac{N}{K}\right)\right]$ the optimal symmetric DMT equals to the optimal DMT of a point-to-point channel with all $K$ users pulled together $d_{K,M,N}^{*(FC)}(Kr)$.

Similar to the development in the point-to-point case, random codes were used in [30]. Later Nam and El Gamal [18] showed that a random ensemble of LAST codes attains the optimal DMT of the multiple-access channel using MMSE estimation followed by lattice decoding over the lattice induced by the $K$ users. An explicit coding scheme based on lattices and cyclic division algebra that attains the optimal DMT using ML decoding was presented in [17].

In this chapter we study the optimal DMT of lattices using regular lattice decoding, i.e. decoding without taking into consideration the power constraint, for the MIMO Rayleigh fading multiple-access channel. The result is rather surprising; unlike the point-to-point case where the tradeoff between dimensions and diversity enables to attain the optimal DMT, we show that for the multiple-access channel the optimal DMT is attained only when $N \geq (K + 1)M - 1$, i.e. user limited regime. On the other hand when the network is heavily loaded we show that IC’s or lattices using regular lattice decoding, can not attain the optimal DMT.

In the first part of this chapter an upper bound on the optimal symmetric DMT IC’s can achieve is derived. The upper bound is attained by finding for each multiplexing gain $r$, the NDCU for each user, that maximizes the diversity order. For the case $N < (K + 1)M - 1$ it is shown that the optimal DMT of IC’s does not coincide with the optimal DMT of finite constellations. Moreover, for $N < (K - 1)M + 1$ it is shown that the optimal DMT of IC’s in the symmetric case is inferior compared to the optimal DMT of finite constellations, for any value of $r$ except for the edges $r = 0, \frac{N}{K}$. On the other hand for the case $N \geq (K + 1)M - 1$, by choosing the correct NDCU for each user, it is shown that the upper bound on the optimal DMT of IC’s coincides with the optimal DMT of finite constellations $d_{M,N}^{*(FC)}(\max(r_1, \ldots, r_K))$.

In the second part of this chapter, a transmission scheme that attains the optimal DMT for $N \geq (K + 1)M - 1$ is presented. Each user in this scheme transmits according to the DMT optimal scheme for the point-to-point channel, presented in Chapter 2. By analyzing the receiver joint ML decoding performance, it is shown that this transmission scheme attains the optimal DMT of finite constellations. We wish to emphasize that the proposed transmission scheme is more involved than simply using orthogonalization between users, which in general is shown to be suboptimal for IC’s. The proposed transmission scheme requires $N + M - 1$ channel uses to attain the optimal DMT, which is smaller than $N + KM - 1$, the number of channel uses required in [30] (the dependence in the number of users lies in the fact that $N \geq (K + 1)M - 1$). Finally,
the algebraic analysis of the transmission scheme geometrically explains why for \( N \geq (K + 1)M - 1 \) the optimal DMT equals to the optimal DMT of the point-to-point channel of each user, i.e., why the optimal DMT equals \( d_{M,N}^{\ast,\text{FC}}(\max(r_1,\ldots,r_K)) \).

As a basic illustrative example of the results we consider the following two cases. In the first case assume a network with two users \((K = 2)\), where each user has a single transmit antenna \((M = 1)\), and a receiver with a single receive antenna \((N = 1)\). In this case the optimal DMT of finite constellations in the symmetric case \([30]\) equals \(1 - r\) for \( r \in \left[0, \frac{1}{3}\right] \), and \(2 - 4r\) for \( r \in \left[\frac{1}{3}, \frac{1}{2}\right] \). For IC’s it is shown in this setting that the optimal DMT for the symmetric case equals \(1 - 2r\) for \( r \in \left[0, \frac{1}{2}\right] \), which is strictly inferior except for \( r = 0, \frac{1}{2} \). In the second case, by merely adding another receive antenna, i.e. \( M = 1, N = K = 2 \), the optimal DMT of IC’s coincides with finite constellations optimal DMT \( d_{1,2}^{\ast,\text{FC}}(\max(r_1,r_2)) \).

The outline of the chapter is as follows. In Section 3.2 basic definitions for the fading multiple-access channel and IC’s are given. Section 3.3 presents an upper bound on the optimal DMT of IC’s, and shows the sub-optimality of IC’s for the case \( N < (K + 1)M - 1 \). Transmission scheme that attains the optimal DMT of finite constellations for the case \( N \geq (K + 1)M - 1 \) is presented in Section 3.4. Finally, in Section 3.5 we discuss the results in this chapter and present for the multiple-access channel a geometrical interpretation to the DMT of IC’s.

### 3.2 Basic Definitions

#### 3.2.1 Channel Model

We consider a \(K\)-user multiple access channel where each user has \(M\) transmit antennas, and the receiver has \(N\) antennas. We assume perfect knowledge of all channels at the receiver, and no channel knowledge at the transmitters. We also assume quasi static flat-fading channel for each user. The channel model is as follows:

\[
y_t = \sum_{i=1}^{K} H^{(i)} \cdot x^{(i)}_t + \rho^{-\frac{1}{2}} n_t \quad t = 1, \ldots, T \tag{3.1}
\]

where \(x^{(i)}_t\), \(t = 1, \ldots, T\) is user \(i\) transmitted signal, \(n_t \sim \mathcal{CN}(0, \frac{2}{\pi e} I_N)\) is the additive noise where \(\mathcal{CN}\) denotes complex-normal, \(I_N\) is the \(N\)-dimensional unit matrix, and \(y_t \in \mathbb{C}^N\). \(H^{(i)}\) is the fading matrix of user \(i\). It consists of \(N\) rows and \(M\) columns, where \(h^{(i)}_{l,j} \sim \mathcal{CN}(0,1), 1 \leq l \leq N, 1 \leq j \leq M\), are the entries of \(H^{(i)}\). The scalar \(\rho^{-\frac{1}{2}}\) multiplies each element of \(n_t\), where \(\rho\) can be interpreted as the average SNR of each user at the receive antennas, for power constrained constellations that satisfy \(\frac{1}{T} \sum_{t=1}^{T} E\{\|x^{(i)}_t\|^2\} \leq \frac{2}{2\pi e}\).

Next we wish to define an equivalent channel to (3.1). Let us define the extended transmission vector

\[
\tilde{x} = \left(\tilde{x}^{(1)}_1, \ldots, \tilde{x}^{(K)}_1, \ldots, \tilde{x}^{(1)}_T, \ldots, \tilde{x}^{(K)}_T\right)^\dagger \tag{3.2}
\]

i.e., first concatenate the users in each channel use, and then concatenate the vectors between channel uses. Now we define \(H = (H^{(1)}, \ldots, H^{(K)})\) which is an \(N \times KM\) matrix. By defining \(H_{ex}\) as an \(NT \times KM\)
block diagonal matrix, where each block on the diagonal equals \( H, \mathbf{u}_{\text{ex}} = \rho^{-\frac{1}{2}} \cdot \left( \mathbf{n}_1^\dagger, \ldots, \mathbf{n}_T^\dagger \right) \in \mathbb{C}^{NT} \) and \( y_{\text{ex}} \in \mathbb{C}^{NT} \), we can rewrite the channel model in (3.1)

\[
\mathbf{y}_{\text{ex}} = H \mathbf{u}_{\text{ex}} \cdot \Xi + \mathbf{u}_{\text{ex}}.
\]

Let \( L = \min(N, KM) \), and let \( \sqrt{\lambda_i}, 1 \leq i \leq L \) be the real valued, non-negative, singular values of \( H \). We assume \( \sqrt{\lambda}_L \geq \cdots \geq \sqrt{\lambda}_1 > 0 \). For large values of \( \rho \), we state that \( f(\rho) \leq g(\rho) \) when \( \lim_{\rho \to \infty} \frac{\ln(f(\rho))}{\ln(\rho)} \geq \frac{\ln(g(\rho))}{\ln(\rho)} \), and also define \( \leq \equiv \geq \) in a similar manner by substituting \( \geq \) with \( \leq \), respectively.

### 3.2.2 Infinite Constellations

Infinite constellation (IC) is a countable set \( S = \{s_1, s_2, \ldots \} \) in \( \mathbb{C}^n \). Let \( \text{cube}_l(a) \subset \mathbb{C}^n \) be a (probably rotated) \( l \)-complex dimensional cube \((l \leq n)\) with edge of length \( a \) centered around zero. We define an IC \( S_l \) to be \( l \)-complex dimensional if there exists rotated \( l \)-complex dimensional cube \( \text{cube}_l(a) \) such that \( S_l \subset \lim_{a \to \infty} \text{cube}_l(a) \) and \( l \) is minimal. \( M(S_l, a) = |S_l \cap \text{cube}_l(a)| \) is the number of points of the IC \( S_l \) inside \( \text{cube}_l(a) \). In [20], the \( n \)-complex dimensional IC density was defined as

\[
\gamma_G = \limsup_{a \to \infty} \frac{M(S_n, a)}{a^{2n}}
\]

and the volume to noise ratio (VNR) for the additive white Gaussian noise (AWGN) channel was given as

\[
\mu_G = \frac{\gamma_G}{\frac{1}{2\pi \sigma^2}}
\]

where \( \sigma^2 \) is the noise variance of each component.

We now turn to the IC definitions at the transmitters. We define the NDCU as the IC dimension divided by the number of channel uses. Let us consider user \( i \), where \( 1 \leq i \leq K \). We denote the NDCU by \( D_i \).

Let us consider a \( D_i T \)-complex dimensional sequence of IC’s \(- S_{D_i T}^{(i)}(\rho) \), where \( D_i \leq M, T \) is the number of channel uses, and \( \sum_{i=1}^{K} D_i \leq L \). First we define \( \gamma_{tr}^{(i)} = \rho^{r_{tr}} \) as the density of \( S_{K T}^{(i)}(\rho) \) at transmitter \( i \).

Similarly to the definitions in Chapter 2 the multiplexing gain of user’s \( i \) IC is defined as

\[
r_i = \lim_{\rho \to \infty} \frac{1}{T} \log_\rho (\gamma_{tr}^{(i)} + 1) = \lim_{\rho \to \infty} \frac{1}{T} \log_\rho (\rho^{r_{tr}} + 1), \quad 0 \leq r_i \leq D_i.
\]

The VNR at the transmitter of user \( i \) is

\[
\mu_{tr}^{(i)} = \frac{\gamma_{tr}^{(i)} - r_{tr}}{2\pi \sigma^2} = \rho^{1 - \frac{r_i}{D_i}}
\]

where \( \sigma^2 = \frac{\rho^{-1}}{2\pi \sigma} \) is each component’s additive noise variance. Now let us concatenate the users IC’s in accordance with (3.2). We denote \( D = \sum_{i=1}^{K} D_i \). The concatenation yields an equivalent \( DT \)-complex dimensional IC, \( S_{DT}^{(i)}(\rho) \), that has multiplexing gain \( \sum_{i=1}^{K} r_i \), density \( \gamma_{tr} = \rho^{(\sum_{i=1}^{K} r_i)} \), and VNR \( \mu_{tr} = \)
\(\rho^{1-\frac{i}{d}}\). In this case we get in (3.3) that the transmitted signal \(x \in S_{DT}(\rho) \subset \mathbb{C}^{KMT}\).

At the receiver we first define the set \(H_{ex} \cdot \text{cube}_{D,T}(a)\) as the multiplication of each point in \(\text{cube}_{D,T}(a)\) with the matrix \(H_{ex}\). In a similar manner, the IC induced by the channel at the receiver is \(S'_{DT} = H_{ex} \cdot S_{DT}\). The set \(H_{ex} \cdot \text{cube}_{D,T}(a)\) is almost surely \(D \cdot T\)-complex dimensional (where \(D \leq L\)). In this case

\[M(S_{DT}, a) = |S_{DT} \cap \text{cube}_{D,T}(a)| = |S'_{DT} \cap (H_{ex} \cdot \text{cube}_{D,T}(a))|.

We define the receiver density as

\[\gamma_{rc} = \limsup_{a \to \infty} \frac{M(S_{DT}, a)}{\text{Vol}(H_{ex} \cdot \text{cube}_{D,T}(a))}\]

i.e., the upper limit on the ratio of the number of IC points in \(H_{ex} \cdot \text{cube}_{D,T}(a)\), and the volume of \(H_{ex} \cdot \text{cube}_{D,T}(a)\). Note that for \(N \geq KM\) and \(D = KM\) we get \(\gamma_{rc} = \rho^{K\sum_{i=1}^{K} r_i} \prod_{i=1}^{KM} \lambda_i^{-T}\) and \(\mu_{rc} = \rho^{K\sum_{i=1}^{K} r_i} \prod_{i=1}^{KM} \lambda_i^{-TM}\). The joint decoder average decoding error probability, over the points of the effective IC \(S_{DT}(\rho)\), for a certain channel realization \(H\), is defined as

\[\overline{Pe}(H, \rho) = \limsup_{a \to \infty} \frac{\sum x' \in S_{DT} \cap (H_{ex} \cdot \text{cube}_{D,T}(a)) \text{Pe}(x', H, \rho)}{M(S_{DT}, a)}\] \hspace{1cm} \text{(3.6)}

where \(\text{Pe}(x', H, \rho)\) is the error probability associated with \(x'\). The average decoding error probability of \(S_{DT}(\rho)\) over all channel realizations is \(\overline{Pe}(\rho) = E_{H}(\overline{Pe}(H, \rho))\). The diversity order is defined as

\[d = -\lim_{\rho \to \infty} \log_{\rho}(\overline{Pe}(\rho)).\] \hspace{1cm} \text{(3.7)}

In practice finite constellations are transmitted even when performing regular lattice decoding at the receiver. Based on the results in [16] it was shown in Chapter 2 that finite constellation with multiplexing gain \(r\) can be carved from a lattice with multiplexing gain \(r\), while maintaining the same performance when regular lattice decoder is employed at the receiver. In our case it also applies to each of the users, i.e. carving finite constellations with multiplexing gains tuple \((r_1, \ldots, r_K)\) that satisfy the power constraint, from lattices with multiplexing gains tuple \((r_1, \ldots, r_K)\). At the receiver the performance is maintained by performing regular lattice decoding on the effective lattice.

### 3.2.3 Additional Notations

We further denote by \(d_{M,N}^{k}(\text{FC})\) the optimal DMT of finite constellations, and by \(d_{M,N}^{k}(\text{D})\) the upper bound on the optimal DMT of any IC with NDCU \(D\), both in a point to point channel with \(M\) transmit and \(N\) receive antennas. For the multiple access channel with \(K\) users, \(M\) transmit antennas for each user, and \(N\) receive antennas, we denote by \(d_{K,M,N}^{k}(\text{FC})\) the optimal DMT of finite constellations in the symmetric case, and by \(d_{K,M,N}^{k}(\text{IC})\), \(d_{K,M,N}^{k}(\text{IC})\) the upper bounds on the optimal DMT of the unconstrained multiple-access channel for the symmetric case, and for multiplexing gains tuple \((r_1, \ldots, r_K)\) respectively.

We denote \(r_{max} = \max(r_1, \ldots, r_K)\), i.e. the maximal multiplexing gain in the multiplexing gains
tuple. In addition for any \( A \subseteq \{1, \ldots, K\} \) we define \( R_A = \sum_{a \in A} r_a \) and \( D_A = \sum_{a \in A} D_a \).

3.3 Upper Bound on the Best Diversity-Multiplexing Tradeoff

In this section we show that for \( N < (K+1)M-1 \) the DMT of the unconstrained multiple-access channel is suboptimal compared to the optimal DMT of finite constellations. On the other hand for \( N \geq (K+1)M-1 \), we derive an upper bound on the optimal DMT that coincides with the optimal DMT of finite constellations.

In Subsection 3.3.1 we lower bound the error probability of any IC for the multiple-access channel, by using lower bounds on the error probability of any IC in the point-to-point channel. We use these lower bounds to formulate an upper bound on the optimal DMT of IC’s for the multiple-access channel, in the form of an optimization problem. In Subsection 3.3.2 we solve this optimization problem for the symmetric case. We compare the optimal DMT of IC’s to the optimal DMT of finite constellations, and find the cases where IC’s are suboptimal in Subsection 3.3.3. Finally in Subsection 3.3.4 we give a convexity argument that shows for the symmetric case that whenever the optimal DMT is not a convex function IC’s are suboptimal.

3.3.1 Upper Bound on the Diversity-Multiplexing-Tradeoff

We lower bound the error probability of the unconstrained multiple-access channel in Lemma 3.1. Based on this lower bound we present in Theorem 3.2 an upper bound on the optimal DMT of IC’s.

Assume user \( i \) transmits \( D_i T \)-complex dimensional IC, with NDCU \( D_i \) and \( T \) channel uses. The following lemma lower bounds the average decoding error probability of the \( K \)-users \( Pe^{(D_1, \ldots, D_K, T)} (\rho, r_1, \ldots, r_K) \), where \((D_1, \ldots, D_K)\) is the tuple of NDCU, \( T \) is the number of channel uses and \((r_1, \ldots, r_K)\) is the tuple of multiplexing gains.

Lemma 3.1.

\[
Pe^{(D_1, \ldots, D_K, T)} (\rho, r_1, \ldots, r_K) \geq \max_{A \subseteq \{1, \ldots, K\}} \left( Pe^{(D_A, T)} (\rho, R_A) \right)
\]

where \( Pe^{(D_A, T)} (\rho, R_A) \) is the lower bound derived in Chapter 2 on the error probability of any IC with \( T \) channel uses, \( D_A = \sum_{a \in A} D_a \) NDCU, and multiplexing gain \( R_A = \sum_{a \in A} r_a \), in a point-to-point channel with \( |A| \cdot M \) transmit and \( N \) receive antennas.

Proof. By considering the extended channel model (3.3), we get that the \( K \) distributed transmitters transmit an effective \( \left( \sum_{i=1}^{K} D_i \right) T \)-complex dimensional IC, over \( T \) channel uses, with multiplexing gain \( \sum_{i=1}^{K} r_i \). The error probability of this IC is lower bounded by the lower bound on the error probability of any IC with NDCU \( \sum_{i=1}^{K} D_i \), \( T \) channel uses, and multiplexing gain \( \sum_{i=1}^{K} r_i \), in a point-to-point channel with \( KM \) transmit and \( N \) receive antennas. Such a lower bound on the error probability was derived in Chapter 2 for each channel realization (Chapter 2 Theorem 2.1), and then for the average over all channel realizations when \( \rho \) is large (Chapter 2 Theorem 2.2). Now consider the set \( A \subset \{1, \ldots, K\} \). In case a genie tells the receiver the transmitted messages of users \( \{1, \ldots, K\} \setminus A \), the optimal receiver attains an error probability that lower bounds the \( K \)-user optimal receiver error probability. Without loss of optimality, the optimal receiver can subtract them from the received signal, and get a new \( |A| \)-users unconstrained multiple-access
channel with NDCU \( \{D_a\}_{a \in A} \), \( T \) channel uses, and multiplexing gain \( \sum_{a \in A} r_a \). In a similar manner, the error probability of this \(|A|\)-users channel is lower bounded by the lower bound on the error probability of any IC with \( \sum_{a \in A} D_a \) NDCU, \( T \) channel uses, and multiplexing gain \( \sum_{a \in A} r_a \), derived in Chapter 2. Hence, the maximal lower bound on the error probability between all \( A \subseteq \{1, \ldots, K\} \) also sets a lower bound on the error probability. This concludes the proof.

Next we wish to formulate an upper bound on the DMT of IC’s in the \( K \)-user unconstrained multiple-access channel. We derive this bound based on the lower bound on the error probability presented in Lemma 3.1, and on an upper bound on the DMT of IC’s for the point-to-point channel, presented in Chapter 2. Let us begin by presenting the upper bound on the DMT for the point-to-point channel.

**Theorem 3.1 (Chapter 2 Theorem 2.2).** For any sequence of IC’s \( S_{D,T} (\rho) \) with \( D \) NDCU, in a point-to-point channel with \( M \) transmit and \( N \) receive antennas, the DMT \( d^{D,T}_{M,N} (r) \) is upper bounded by

\[
d^{D,T}_{M,N} (r) \leq d^*, D_{M,N} (r) = \frac{M \cdot N}{D} (D - r)
\]

for \( 0 \leq D \leq \frac{M \cdot N}{N + M - 1} \), and

\[
d^{D,T}_{M,N} (r) \leq d^*, D_{M,N} (r) = \frac{(M - l) (N - l)}{D - l} \cdot (D - r)
\]

for \( \frac{M \cdot N - (l - 1) l}{N + M - 1 - 2 (l - 1)} \leq D \leq \frac{M \cdot N - (l + 1) l}{N + M - 1 - 2 l} \), and \( l = 1, \ldots, \min (M, N) - 1 \). In all cases \( 0 \leq r \leq D \).

Based on Lemma 3.1 and Theorem 3.1 we formulate the following upper bound on the optimal DMT of the multiple-access channel.

**Theorem 3.2.** The optimal DMT of any sequence of IC’s with multiplexing gains tuple \((r_1, \ldots, r_K)\) is upper bounded by

\[
d^{*, (IC)}_{K,M,N} (r_1, \ldots, r_K) = \max_{(D_1, \ldots, D_K) \in D} \min_{A \subseteq \{1, \ldots, K\}} \left( d^{*, D}_{|A|; M,N} (R_A) \right)
\]

where \( D = \{ D_1, \ldots, D_K \mid 0 \leq D_i \leq M, \sum_{i=1}^{K} D_i \leq L \} \).

**Proof.** From Lemma 3.1 we get a lower bound on the error probability of any sequence of effective IC’s \( S_{\sum_{i=1}^{K} D_i T} (\rho) \), transmitted by the \( K \) users. The lower bound on the error probability can be translated to an upper bound on the diversity order. In addition the lower bound on the error probability depends on lower bounds for the point-to-point channel on the error probabilities. Hence, we can use the upper bound on the DMT in the point-to-point channel, presented in Theorem 3.1, to get the following upper bound on the DMT of a tuple of NDCU \((D_1, \ldots, D_K)\)

\[
\min_{A \subseteq \{1, \ldots, K\}} \left( d^{*, D}_{|A|; M,N} (R_A) \right).
\]

Maximizing over \((D_1, \ldots, D_K) \in D\) yields the upper bound on the optimal DMT.

39
3.3.2 Characterizing the Optimal Symmetric DMT

We wish to characterize an upper bound on the optimal DMT of IC’s in the symmetric case, i.e. \( r_1 = \cdots = r_K = r \). Later we will use this upper bound in order to show the sub-optimality of the unconstrained multiple-access channel in the case \( N < (K + 1) M - 1 \). In addition, we will show that this upper bound coincides with the optimal DMT of finite constellations in the case \( N \geq (K + 1) M - 1 \).

Lemmas 3.2, 3.3, 3.4, 3.5 present the relations between \( d^{*,i,D}_{i,M,N} (i \cdot r) \), \( i = 1, \ldots, K \) for different values of \( N \). We use these lemmas in order to upper bound the optimal DMT in the symmetric case in Theorem 3.4.

Based on Theorem 3.2 we can state that the optimal DMT for the symmetric case for \( K \) users is upper bounded by

\[
d^{*,(IC)}_{K,M,N} (r) = \max_{(D_1, \ldots, D_K) \in D} \min_{A \subseteq \{1, \ldots, K\}} \left( d^{*,D^A}_{|A|,M,N} (|A| \cdot r) \right) \quad 0 \leq r \leq \frac{L}{K}, \tag{3.8}
\]

i.e. we wish solve the aforementioned optimization problem for each \( 0 \leq r \leq \frac{L}{K} \). In order to solve this optimization problem we first solve a simpler optimization problem for each \( D_i = \cdots = D_K = D \), i.e. each user transmits \( D \) NDCU. In this case the upper bound in (3.8) takes a simpler form

\[
\max_D \min_{1 \leq i \leq K} \left( d^{*,i,D}_{i,M,N} (i \cdot r) \right) \tag{3.9}
\]

where \( 0 \leq D \leq \frac{L}{K} \). After solving this optimization problem, we will show that choosing \( D_1 = \cdots = D_K = D \) also yields the optimal solution for (3.8).

In order to solve the optimization problem in (3.9), we first need to present some properties on the relations between \( d^{*,i,D}_{i,M,N} (i \cdot r) \), \( 1 \leq i \leq K \). We begin by presenting a property on the behavior of \( d^{*,D}_{M,N} (\cdot) \) as a function of \( D \).

**Corollary 3.1** (Chapter 2 Corollary 2.1). For \( 0 \leq D \leq \frac{M,N}{N+M-1} \), we get

\[
d^{*,D}_{M,N} (0) = MN.
\]

For \( \frac{M,N-(l-1)l}{N+M+1-2(l-1)} \leq D \leq \frac{M,N-(l+1)(l+1)}{N+M+1-2l(l-1)} \), and \( l = 1, \ldots, \min (M,N) - 1 \) we get

\[
d^{*,D}_{M,N} (l) = (M-l) \cdot (N-l).
\]

A simple interpretation of Corollary 3.1 is that for the case \( 0 \leq D \leq \frac{M,N}{N+M-1} \) the straight lines \( d^{*,D}_{M,N} (\cdot) \) that represent the upper bounds on the DMT, all have the same “anchor” point at multiplexing gain \( r = 0 \), i.e. they all have diversity order \( MN \) for \( r = 0 \), and each line equals to zero for \( r = D \). On the other hand, for \( \frac{M,N-(l-1)l}{N+M+1-2l(l-1)} \leq D \leq \frac{M,N-(l+1)(l+1)}{N+M+1-2l(l-1)} \), and \( l = 1, \ldots, \min (M,N) - 1 \), the straight lines equal to \( (M-l) \cdot (N-l) \) for multiplexing gain \( r = l \), and again each line equals to zero for \( r = D \). Figure 3.1 illustrates this property for the case \( M = N = 2 \). Another property relates to the optimal DMT of finite constellations in the symmetric case.
Figure 3.1: Upper bound on the DMT of any IC for $M = N = 2$ and $D$ NDCU, in a point-to-point channel. Note that $d_{2,2}^{1/2}(r)$ and $d_{2,2}^{3/2}(r)$ are straight lines equal to $MN = 4$ for multiplexing gain $r = 0$, where $d_{2,2}^{1/2}(r)$ and $d_{2,2}^{2/2}(r)$ are straight lines equal to $(M - 1)(N - 1) = 1$ for multiplexing gain $r = 1$, in accordance with Corollary 3.1. In bold we get the optimal DMT of finite constellations.

**Theorem 3.3** ([30] Theorem 3). The optimal DMT of finite constellations in the symmetric case equals

$$d_{K,M,N}^{*(FC)}(r) = \begin{cases} d_{M,N}^{*(FC)}(r) & 0 \leq r \leq \min \left( \frac{N}{K+1}, M \right) \\ d_{K,M,N}^{*(FC)}(K \cdot r) & \min \left( \frac{N}{K+1}, M \right) \leq r \leq \min \left( \frac{N}{K}, M \right) \end{cases}$$

In order to solve the optimization problem in (3.9) we present several lemmas related to the inequalities between $d_{i-M,N}^{*(i \cdot D)}(i \cdot r)$ for $1 \leq i \leq K$. The proofs of these lemmas rely mainly on Corollary 3.1 and Theorem 3.3.

**Lemma 3.2.** For $N \geq (K + 1)M - 1$ we get

$$d_{M,N}^{*(i \cdot D)}(r) \leq d_{i-M,N}^{*(i \cdot D)}(i \cdot r) \quad 2 \leq i \leq K$$

for any $0 \leq r \leq D$ and $0 \leq D \leq M$.

*Proof.* The proof is in appendix B.1.

An example of Lemma 3.2 for the case $M = K = 2$ and $N = 4$ is illustrated in Figure 3.2.

**Lemma 3.3.** For $N < (K + 1)M - 1$ we get

$$d_{M,N}^{*(i \cdot D)}(r) \leq d_{i-M,N}^{*(i \cdot D)}(i \cdot r) \quad 2 \leq i \leq K - 1$$

for any $0 \leq D \leq \frac{1}{K}$ and $0 \leq r \leq D$.

*Proof.* The proof is in appendix B.2

41
From Lemmas 3.2, 3.3 we can see that the optimization problem in (3.9) involves only $d^*_{M,N} (r)$ and $d^*_{K \cdot M,N} (K \cdot r)$. Now we prove two more properties that will enable us to find the optimal DMT of IC’s in the symmetric case.

**Lemma 3.4.** For $N < (K - 1) M + 1$ we get

$$\max_{0 \leq D \leq \frac{N}{K}} \min_{1 \leq i \leq K} d^*_{i \cdot M,N} (i \cdot r) = d^*_{M,N} (r) = M \cdot N - M \cdot K \cdot r \quad 0 \leq r \leq \frac{N}{K}$$

*Proof.* The proof is in appendix B.3

From Lemma 3.4 we can see that for the multiple-access channel, when $N < (M - 1) K + 1$ the optimal DMT of IC’s is smaller than finite constellations optimal DMT for any value of $r$ except for $r = 0$ and $r = \frac{N}{K}$. Figure 3.3 illustrates Lemma 3.4 for the case $M = N = K = 2$. Next we wish to show the cases where $d^*_{M,N} (r)$ and $d^*_{K \cdot M,N} (K \cdot r)$ coincide.

The following lemma sets another building block in upper bounding the optimal DMT in the symmetric case when $N = (K - 1) M + 1 + l$, $l = 0, \ldots, 2M - 3$. It finds the NDCU that leads to the equality $d^*_{M,N} (r) = d^*_{K \cdot M,N} (K \cdot r)$ for any value of $r$, and also shows for which values of $r$ these straight lines are equal to the optimal DMT of finite constellations in a point-to-point channel.

**Lemma 3.5.** For $N = (K - 1) M + 1 + l < (K + 1) M - 1$, where $l = 0, \ldots, 2M - 3$, we get for NDCU per user $D_l$ that

$$d^*_{M,N} (r) = d^*_{K \cdot M,N} (K \cdot r) = d^* (r)$$

$$= MN - \left\lceil \frac{l}{2} \right\rceil \left( \left\lceil \frac{l}{2} \right\rceil + 1 \right) - 2 \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \frac{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor \right) - (N + M - 1 - l) r$$
Figure 3.3: Illustration of Lemma 3.4 for the case $M = N = K = 2$. For this case the optimal DMT in the symmetric case is smaller than the optimal DMT of finite constellations, for any value of $r$ except for $r = 0, 1$.

where $0 \leq r \leq D_l$. In addition

$$d^*_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d^* \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right)$$

and also

$$d^*_{K,M,N} \left( (K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor \right) = d^* \left( \frac{(K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor}{K} \right)$$

Proof. The proof is in appendix B.4.

An example that illustrates Lemma 3.5 for the case $M = K = 2$ and $N = 4$ is given in Figure 3.4.

Figure 3.4: $d^* (r)$ for the case $M = K = 2$ and $N = 4$, i.e. $l = 1$. Note that $d^* (1) = d^*_{4,2} (1) = d^*_{2,4}^{(FC)} (1)$ and $d^* \left( \frac{3}{2} \right) = d^*_{4,4} (3) = d^*_{4,4}^{(FC)} (3)$.
Now we are ready to characterize the upper bound on the optimal DMT of IC’s in the symmetric case. Recall that for

\[ N = (K - 1) M + 1 + l < (K + 1) M - 1, \quad l = 0, \ldots, 2M - 3 \]

\[ d^* (r) = MN - \lceil \frac{l}{2} \rceil \left( \left\lceil \frac{l}{2} \right\rceil + 1 \right) - 2 \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \left( \frac{l}{2} - \left\lfloor \frac{l}{2} \right\rfloor \right) - (N + M - 1 - l) r. \]

**Theorem 3.4.** The optimal DMT of any sequence of IC’s in the symmetric case is upper bounded by:

For \( N \geq (K + 1) M - 1 \)

\[ d^*_{IC} (K, M, N) (r) = d^*_{FC} (M, N) (r). \]

For \( N < (K - 1) M + 1 \)

\[ d^*_{IC} (K, M, N) (r) = M \cdot N - K \cdot M \cdot r. \]

For \( N = (K - 1) M + 1 + l < (K + 1) M - 1, \) where \( l = 0, \ldots, 2M - 3 \)

\[
\begin{align*}
    d^*_{IC} (K, M, N) (r) &= \begin{cases} 
        d^*_{FC} (M, N) (r) & 0 \leq r \leq \lfloor \frac{l}{2} \rfloor + 1 \\
        d^* (r) & \left\lfloor \frac{l}{2} \right\rfloor + 1 \leq r \leq \frac{(K-1)M+\left\lceil \frac{l+1}{2} \right\rceil}{K} \\
        d^*_{FC} (K r) & \frac{(K-1)M+\left\lceil \frac{l+1}{2} \right\rceil}{K} \leq r \leq \frac{l}{K} 
    \end{cases}
\end{align*}
\]

**Proof.** The proof is in appendix B.5.
For \( N = (K - 1) M + 1 + l < (K + 1) M - 1 \) where \( l = 0, \ldots, 2M - 3 \)
\[
d^*_{(IC)}(r) < d^*_{(FC)}(r) \quad \left\lfloor \frac{l}{2} \right\rfloor + 1 < r < \frac{(K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K}.
\]

**Proof.** The full proof is in appendix B.6. In a nutshell the proof is based on the properties of \( d^*_{MN}(r) \) derived in Corollary 3.1, and also on the results in Theorem 3.4. It is important to note that for \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \) we get that \( d^*_{K,M,N}(r) = d^*_{K,M,N}(r) \) because in this case \( \left\lfloor \frac{l}{2} \right\rfloor + 1 = \frac{(K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K} \).

The sub-optimality of \( d^*_{K,M,N}(r) \) for the case \( N < (K - 1) M + 1 \) is illustrated in Figure 3.3, where the sub-optimality for the case \( N = (K - 1) M + 1 + l \), where \( l = 0, \ldots, 2m - 3 \) is illustrated in Figure 3.4.

Now we are ready to present the cases where the upper bound on the optimal DMT of the unconstrained multiple-access channel coincides with the optimal DMT of finite constellations, and the cases where the optimal DMT of the unconstrained multiple-access channel is suboptimal compared to the optimal DMT of finite constellations.

**Theorem 3.5.** For \( N \geq (K + 1) M - 1 \) the optimal DMT of the unconstrained multiple-access channel is upper bounded by \( d^*_{MN}(\max(r_1, \ldots, r_K)) \) the optimal DMT of finite constellations. For \( N < (K + 1) M - 1 \) the best DMT that can be attained in the unconstrained multiple-access channel is inferior compared to the optimal DMT of finite constellations.

**Proof.** The full proof is in appendix B.7. The proof outline is as follows. Recall that in Theorem 3.2 we have shown that the optimal DMT of IC’s is upper bounded by
\[
d^*_{K,M,N}(r_1, \ldots, r_K) = \max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} \left( d^*_{DA}(r_A) \right).
\]

For \( N \geq (K + 1) M - 1 \) we show that this term is upper and lower bounded by \( d^*_{MN}(\max(r_1, \ldots, r_K)) \), which is the optimal DMT of finite constellations in this case.

For the case \( N < (K + 1) M - 1 \) we show that the optimal DMT is not attained by finding a set of multiplexing gain tuples \( (r_1, \ldots, r_K) \in B \) for which \( d^*_{(IC)}(r_1, \ldots, r_K) < d^*_{(FC)}(r_1, \ldots, r_K) \). Based on Lemma 3.6 we get for \( r_1 = \cdots = r_K = r \) that there exists a set of multiplexing gains for which \( d^*_{K,M,N}(r) < d^*_{K,M,N}(r) \), except for the case \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \), where \( s \geq 1 \) is an integer.

For this case showing that \( d^*_{2,s+1,3,s}(r_1, r_2) < d^*_{2,s+1,3,s}(r_1, r_2) \) is more involved and requires considering the case \( r_1 \neq r_2 \) (see appendix B.7 for the full proof). An illustrative example of the method of proof for this case is presented in Figures 3.5, 3.6.

### 3.3.4 Discussion: Convexity Vs. Non-Convexity of the Optimal DMT

It is interesting to note that the upper bound on the optimal DMT of IC’s in the symmetric case is a convex function, where the optimal DMT of finite constellations is not necessarily so. The convexity of the optimal DMT of IC’s can be shown rather easily by the following arguments. It is based on the fact that a function
that equals to the maximum between straight lines is a convex function. For \( N \geq (K + 1) M - 1 \) the optimal DMT of IC’s in the symmetric case is simply upper bounded by \( d_{M,N}^{r, (FC)} (r) \) which is a maximization between straight lines, and therefore is a convex function. For the case \( N < (K - 1) M + 1 \) the upper bound on the optimal DMT of IC’s in the symmetric case is a straight line. Finally, for \( N = (K - 1) M + 1 + l < (K + 1) M - 1 \), where \( l = 0, \ldots, 2M - 3 \), the upper bound on the optimal symmetric DMT of IC’s equals to the maximization between the first \( \lfloor \frac{K}{2} \rfloor + 1 \) straight lines constituting \( d_{M,N}^{r, (FC)} (r) \), \( d^*(r) \), and the last \( M - \lfloor \frac{K}{2} \rfloor \) straight lines constituting \( d_{K,M,N}^{r, (FC)} (K - r) \). This maximization also yields a convex function.

On the other hand the optimal DMT of finite constellations in the symmetric case is not necessarily a convex function. See Figure 3.4 for illustration. In fact the optimal DMT is not a convex function whenever \( N < (K - 1) M + 1 \), or when \( N = (K - 1) M + 1 + l < (K + 1) M - 1 \) and \( \lfloor \frac{K}{2} \rfloor + 1 \neq \frac{(K - 1) M + \lfloor \frac{K}{2} \rfloor}{K} \) where \( l = 0, \ldots, 2M - 3 \). It results from the following arguments. For \( N < (K - 1) M + 1 \) we get \( \frac{MN}{N + M - 1} > \frac{N}{K} \), and so \( d_{M,N}^{r, (FC)} \left( \frac{N}{K} \right) > 0 \). In addition \( d_{M,N}^{r, (FC)} \left( \frac{N}{K} \right) = d_{M,N}^{r, (FC)} (r) \) for \( 0 \leq r \leq \min \left( 1, \frac{N}{K+1} \right) \). Based on these facts and on the facts that \( d_{M,N}^{r, (FC)} \) is a piecewise linear function and \( d_{K,M,N}^{r, (FC)} \left( \frac{N}{K} \right) = 0 \), we get that \( d_{K,M,N}^{r, (FC)} (r) \) is not a convex function. For the case \( N = (K - 1) M + 1 + l < (K + 1) M - 1 \), where \( l = 0, \ldots, 2M - 3 \), we know that

$$ d_{K,M,N}^{r, (IC)} (r) = d^*(r) < d_{K,M,N}^{r, (FC)} \left( \lfloor \frac{K}{2} \rfloor + 1 \right) < \frac{(K - 1) M + \lfloor \frac{K}{2} \rfloor}{K}. $$

Since \( d^*(r) \) is a straight line it necessarily means that \( d_{K,M,N}^{r, (FC)} (r) \) is not a convex function whenever \( \lfloor \frac{K}{2} \rfloor + 1 \neq \frac{(K - 1) M + \lfloor \frac{K}{2} \rfloor}{K} \). For the case \( \lfloor \frac{K}{2} \rfloor + 1 = \frac{(K - 1) M + \lfloor \frac{K}{2} \rfloor}{K} \) we get \( d_{K,M,N}^{r, (FC)} (r) = d_{K,M,N}^{r, (IC)} (r) \), and so in this case the optimal DMT of finite constellations in the symmetric case is also a convex function. Finally, for the case \( N \geq (K + 1) M - 1 \) the optimal DMT in the symmetric case equals \( d_{M,N}^{r, (FC)} \) and as aforementioned it is a convex function. Therefore, we can state that whenever the optimal DMT of finite constellations in the
In this case transmitting the information itself over the space-time code enables to obtain the optimal DMT from the symbols required for transmission. The most notable example of such a transmission scheme is to the multiple-access channel, i.e. a transmission scheme that enables to separate the space-time code. 

The symmetric case is not a convex function, IC’s are suboptimal with multiplexing gain smaller than 2, where they should rotate around anchor point with multiplexing gain 2.

Figure 3.6: Illustration of the sub-optimality of the unconstrained multiple-access channel for $M = 3$, $N = 6$ and $K = 2$. In this example we take $r_1 = r_0 + \epsilon = \frac{13}{6} + \frac{1}{6}$ and $r_2 = r_0 - \epsilon = \frac{13}{6} - \frac{1}{6}$, where $r_0 = \frac{13}{6}$. In this case the optimal diversity order of finite constellations equals 

$$
\min \left( d_{3,6}^{s,(FC)}(r_1), d_{3,6}^{s,(FC)}(r_2), d_{6,6}^{s,(FC)}(r_1 + r_2) \right).
$$

From the figure it can be seen that the minimum is obtained for $d_{3,6}^{s,(FC)}(r_1 + r_2) = d_{6,6}^{s,(FC)}(2r_0) = 3$. On the other hand IC’s diversity order equals 

$$
\min \left( d_{3,6}^{s,D_1}(r_1), d_{3,6}^{s,D_2}(r_2), d_{6,6}^{s,D_1+D_2}(2r_0) \right).
$$

In this example we choose $D_1 = \frac{8}{3} + \frac{1}{6}$, $D_2 = \frac{8}{3} - \frac{1}{6}$. In this case we get 

$$
d_{6,6}^{s,D_1+D_2}(2r_0) = d_{6,6}^{s,\frac{16}{3}}(2r_0) = 3, d_{3,6}^{s,D_1}(r_1) = d_{3,6}^{s,\frac{12}{3}}(r_1) = 3 \text{ and } d_{3,6}^{s,D_2}(r_2) = d_{3,6}^{s,\frac{15}{6}}(r_2) = \frac{5}{2} < 3.
$$

Hence, in this case the diversity order of IC’s is smaller than the optimal diversity order of finite constellations. It results from the fact that for $0 < D \leq \frac{8}{3}$ the straight lines $d_{3,6}^{s,D}(r)$ rotate around anchor points with multiplexing gain smaller than 2, where they should rotate around anchor point with multiplexing gain 2.

symmetric case is not a convex function, IC’s are suboptimal.

Finally, a question that may arise is whether it is possible to find an extension of orthogonal designs [24] to the multiple-access channel, i.e. a transmission scheme that enables to separate the space-time code from the symbols required for transmission. The most notable example of such a transmission scheme is the Alamouti scheme [1] for the case of two transmit antennas and a single receive antenna. For example, in this case transmitting the information itself over the space-time code enables to obtain the optimal DMT $d_{2,1}^{s,(FC)}(r)$ regardless of the constellation size. For the multiple-access channel, if we examine the optimal DMT of finite constellations for the symmetric case, for $M = 2$, $K = 2$ and $N = 1$ we get

$$
\begin{align*}
\fi
\end{align*}$

47
which imply that in the range $0 \leq r \leq \frac{1}{3}$ each user can obtain the same performance as the Alamouti scheme. However, our results show that for this setting we get $N = 1 < (K - 1) M + 1 = 3$. Therefore, the optimal DMT of IC’s for the symmetric case is upper bounded by

$$d_{2,1}^{*,(IC)}(r) = d_{2,1}^{*,(FC)}(2r)$$

which is strictly smaller than $d_{2,1}^{*,(FC)}(r)$ except for $r = 0$, as illustrated in Figure 3.7. This leads us to the conclusion that for the multiple-access channel, the signals required for transmission affect the performance and can not be separated from the space-time code. This is due to the fact that when the constellation size is infinite, the performance is sub-optimal. Hence, in this sense there is no extension of orthogonal designs to the multiple-access channel.

**Figure 3.7:** Comparison between the optimal DMT of finite constellations in the symmetric case and the upper bound on the optimal DMT of IC’s for the case $M = K = 2$ and $N = 1$. Note that in the range $0 \leq r \leq \frac{1}{3}$ finite constellations attain the Alamouti performance, while IC’s can not obtain it. This illustrates that for the multiple-access channel the constellation and the space-time code can not be separated.

### 3.4 Attaining the Optimal DMT for $N \geq (K + 1) M - 1$

In this section we show that the upper bound on the DMT of the unconstrained multiple-access channel, derived in Section 3.3, is achievable for the case $N \geq (K + 1) M - 1$ by a sequence of IC’s in general and lattices in particular. Essentially, we show for $N \geq (K + 1) M - 1$ that IC’s attain DMT that equals to $d_{K,M,N}^{*,(FC)}(r_1,\ldots,r_K) = d_{M,N}^{*,(FC)}(\max(r_1,\ldots,r_K))$.

We begin by showing in Subsection 3.4.1 that simple orthogonal transmission approaches such as time-division multiple-access (TDMA) or code-division multiple-access (CDMA) will result in sub-optimal performance for $N \geq (K + 1) M - 1$. Then, we introduce in Subsection 3.4.2 the transmission scheme for each user, followed by presentation of the effective channel induced by the transmission scheme in Subsection 3.4.3. We derive in Subsection 3.4.4 for each channel realization an upper bound on the error probability of the ML decoder of an ensemble of $K$ IC’s. Finally, in Subsection 3.4.5 we average this upper bound over the channel realizations, and show that the optimal DMT is attained for $N \geq (K + 1) M - 1$.  

48
3.4.1 Orthogonal Transmission is Sub-optimal

In this subsection we show the sub-optimality of transmission methods that create at the receiver orthogonalization between different independent streams, for any channel realization. The advantage of these transmission schemes is their simplicity. By assigning the IC’s or lattices correctly in the space, they enable to consider each stream independently and reduce the decoding problem to the point-to-point scenario. Such an approach is very natural when considering IC’s in general and lattices in particular, as it involves assigning the streams with dimensions or subspaces that remain orthogonal at the receiver for each channel realization. The IC related to a certain stream lies within the assigned subspace. We show for \( N \geq (K + 1) M - 1 \) that such transmission method is sub-optimal as it requires each user to give up too many dimensions to create the orthogonalization.

At the receiver, orthogonal transmission scheme enables each independent stream to lie within a subspace orthogonal to the other streams, for each channel realization. In order for a transmission scheme to fulfill this property, the streams must be assigned with orthogonal subspaces already at the transmitter, i.e., must be assigned with orthogonal subspaces in \( \mathbb{C}^{M T} \) assuming there are \( T \) channel uses. Hence, orthogonal transmission schemes require the partition of at most \( M \) NDCU between all users. On the other hand, \( N \geq (K + 1) M - 1 \) leads to \( N \geq K \cdot M \), and so potentially the \( K \) users could transmit together up to \( KM \) dimensions per channel use, but not orthogonally. The optimal DMT for the symmetric case for \( N \geq (K + 1) M - 1 \) is \( d^{*\text{FC}}_{M,N}(r) \). From Corollary 3.1 and Theorem 3.4 we know that in the range \( M - 1 \leq r \leq M \) the optimal DMT is obtained only when each user transmits \( M \) NDCU, i.e., the \( K \) users must transmit together \( KM \) dimensions per channel use. Hence, orthogonal transmission is not provided with enough dimensions per channel use to obtain the last line of the optimal DMT. This leads to its sub-optimality.

As a first example we consider an orthogonal transmission scheme that takes the natural partition to \( K \) streams induced by the multiple-access channel. In order to obtain orthogonalization for this case, a different user is transmitting in each channel use, while the others wait for their turn to transmit. This transmission method is referred to as TDMA. Let us consider the symmetric case where each user transmits with multiplexing gain \( r \). Assuming there are \( T \) channel uses, in the symmetric case each user transmits only on \( \frac{T}{K} \) channel uses. In this case each user can obtain the point-to-point performance of a channel with \( M \) transmit and \( N \) receive antennas, using \( \frac{T}{K} \) channel uses. However, in order for each user to obtain multiplexing gain \( r \) per channel use, it must transmit at multiplexing gain \( K r \) in the \( \frac{T}{K} \) channel uses, which leads to DMT performance of \( d^{*\text{FC}}_{M,N}(K r) \). This shows the sub-optimality of TDMA.

Another transmission approach is assigning an independent stream for each transmit antenna. This is equivalent to considering a multiple-access channel with \( KM \) users, each with a single transmit antenna. Let us consider for example a multiple-access channel with \( M = 1, K \) users and \( N \geq K \). In this case the optimal DMT for the symmetric case equals \( d_{1,N}^{\text{FC}}(r) \). On the other hand for CDMA each user is assigned with an orthogonal subspace in \( \mathbb{C}^{T} \), assuming there are \( T \) channel uses. In this way each stream can obtain the performance of a point-to-point channel with a single transmit antenna and \( N \) receive antennas. However, for the orthogonalization to hold each user is assigned with \( \frac{T}{K} \) dimensional subspace, which must be orthogonal to the other users subspaces. Hence, in order for each user to obtain multiplexing gain \( r \) per
channel use, he must transmit at multiplexing gain $Kr$ over the $\frac{T}{K}$ dimensional subspace. This leads to suboptimal DMT performance of $d_{1;N}^{(FC)}(Kr)$.

### 3.4.2 The Transmission Scheme

From Subsection 3.4.1 we get that an optimal transmission scheme must allow different users to lie in overlapping subspaces at the receiver, i.e. at the receiver the users can not reside in orthogonal subspaces. Essentially, in the proposed transmission scheme each user transmits as if the channel was a point-to-point channel with $M$ transmit and $N$ receive antennas. Hence, each user transmission matrix is identical to the transmission matrix presented in Chapter 2.

We denote the transmission matrix of user $i$ by $G^{(i)}_l$, where $l = 0, \ldots, M - 1$ and $i = 1, \ldots, K$. $G^{(i)}_l$ has $M$ rows that represent the transmission antennas, and $T_l = N + M - 1 - 2 \cdot l$ columns that represent the number of channel uses. $G^{(i)}_l$ transmits $D_l = \frac{NM - l(l+1)}{N+M-1-2l}$ NDCU in the following manner.

Consider a channel with $M$ transmit and $N$ receive antennas.

1. For $D_{M-1} = \frac{M(N-M+1)}{N-M+1} = M$: the matrix $G^{(i)}_{M-1}$ has $N - M + 1$ columns (channel uses). In the first column transmit symbols $x_1, \ldots, x_M$ on the $M$ antennas, and in the $N - M + 1$ column transmit symbols $x_{M(N-M)+1}, \ldots, x_{M(N-M+1)}$ on the $M$ antennas.

2. For $D_l, l = 0, \ldots, L - 2$: the matrix $G^{(i)}_l$ has $N + N - 1 - 2 \cdot l$ columns. We add to $G^{(i)}_l$, the transmission scheme for $D_{l+1}$, two columns in order to get $G^{(i)}_{l+1}$. In the first added column transmit $l + 1$ symbols on antennas $1, \ldots, l + 1$. In the second added column transmit different $l + 1$ symbols on antennas $M - l, \ldots, M$.

According to the definition of the transmission scheme we can see that the different users transmit the same NDCU. Let us denote the transmission scheme of the first $k$ users by

$$G^{(1,\ldots,k)}_l = \left(G^{(1)}_l, \ldots, G^{(k)}_l\right)^\dagger \quad k = 1, \ldots, K. \quad (3.10)$$

$G^{(1,\ldots,k)}_l$ is a $k \cdot M \times T_l$ matrix. Note that $G^{(1,\ldots,K)}_l$ transmits $k \cdot D_l \cdot T_l$ NDCU. Later in this section we show that $G^{(1,\ldots,K)}_l$ attains the optimal DMT in the range $l \leq r_{\text{max}} \leq l + 1$.

**Example:** $M = 2$, $N = 5$ and $K = 2$. In this case the transmission scheme for $D_0 = \frac{10}{6}$, $D_1 = \frac{5}{4}$ ($G^{(1,2)}_0$, $G^{(1,2)}_1$ respectively) is as follows:

$$G^{(1,2)}_1 = \begin{pmatrix} G^{(1)}_1 \\ G^{(2)}_1 \end{pmatrix} = \begin{pmatrix} x_1 & x_3 & x_5 & x_7 & | & x_{17} & 0 \\ x_2 & x_4 & x_6 & x_8 & | & 0 & x_{18} \\ \cdots & \cdots & \cdots & \cdots & | & \cdots & \cdots \\ x_9 & x_{11} & x_{13} & x_{15} & | & x_{19} & 0 \\ x_{10} & x_{12} & x_{14} & x_{16} & | & 0 & x_{20} \end{pmatrix}. \quad (3.11)$$

$$D_1 = \frac{5}{4} G^{(1,2)}_1$$

$$D_0 = \frac{10}{6} G^{(1,2)}_0$$

50
3.4.3 The Effective Channel

Next we define the effective channel matrix induced by the transmission scheme of the first \( k \) users \( G_l^{(1,...,k)} \), where \( k = 1, \ldots, K \). Let us denote the first \( k \) users transmission at time instance \( t \) by

\[
\vec{x}_t = (\vec{x}_t^{(1)}, \ldots, \vec{x}_t^{(k)})^\dagger \quad t = 1, \ldots, T_l.
\]

In accordance with the channel model from (3.1) we get

\[
y_t = H^{(1,...,k)} : \vec{x}_t \quad t = 1, \ldots, T_l.
\]

where \( H^{(1,...,k)} = (H^{(1)}, \ldots, H^{(k)}) \) is an \( N \times k \cdot M \) matrix. The multiplication \( H^{(1,...,k)} : G_l^{(1,...,k)} \) yields a matrix with \( N \) rows and \( T_l \) columns, where each column equals to \( H^{(1,...,k)} : \vec{x}_t \), \( t = 1 \ldots T_l \). Each user is transmitting \( D_l \cdot T_l \)-complex dimensional IC with \( D_l \cdot T_l \)-complex symbols, i.e. \( G_l^{(i)} \) has exactly \( D_l \cdot T_l \) non-zero values representing the \( D_l \cdot T_l \) complex-dimensional IC within \( \mathbb{C}^{M T_l} \). Together, the first \( k \) users transmit an effective \( k \cdot D_l \cdot T_l \)-dimensional complex IC within \( \mathbb{C}^{k \cdot M T_l} \). For each column of \( G_l^{(1,...,k)} \), denoted by \( \vec{g}_m^{(k)} \), \( m = 1 \ldots T_l \), we define the effective channel that \( \vec{g}_m^{(k)} \) sees as \( \hat{H}_m \). It consists of the columns of \( H^{(1,...,k)} \) that correspond to the non-zero entries of \( \vec{g}_m^{(k)} \), i.e. \( H^{(1,...,k)} : \vec{g}_m^{(k)} = \hat{H}_m \cdot \vec{g}_m^{(k)} \), where \( \vec{g}_m^{(k)} \) equals to the non-zero entries of \( \vec{g}_m^{(k)} \). As an example assume without loss of generality that only the first \( l_m \) entries of \( \vec{g}_m^{(k)} \) are not zero. In this case \( \hat{H}_m \) is an \( N \times l_m \) matrix that equals to the first \( l_m \) columns of \( H^{(1,...,k)} \). In accordance with (3.3), \( H_{\text{eff}}^{(i), k} \) is an \( N T_l \times k D_l \cdot T_l \) block diagonal matrix consisting of \( T_l \) blocks. Since each block in \( H_{\text{eff}}^{(i), k} \) corresponds to the multiplication of \( H^{(1,...,k)} \) with different column in \( G_l^{(1,...,k)} \), the blocks of \( H_{\text{eff}}^{(i), k} \) equal \( \hat{H}_m \), \( m = 1, \ldots, T_l \). Note that in the effective matrix \( N T_l \geq k \cdot D_l \cdot T_l \).

Next we elaborate on the structure of the blocks of \( H_{\text{eff}}^{(i), k} \). For this reason we denote the \( m \)'th column of \( H^{(1,...,k)} \) by \( h_m^{(k)} \), \( m = 1, \ldots, k \cdot M \). The transmission scheme has \( N + M - 1 - 2 \cdot l \) columns. The entries of the first \( N - M + 1 \) columns of \( G_l^{(1,...,k)} \cdot \vec{g}_1^{(k)}, \ldots, \vec{g}_{N-M+1}^{(k)} \) are all different from zero. Hence, the first \( N - M + 1 \) blocks of \( H_{\text{eff}}^{(i), k} \) are

\[
\hat{H}_m = H^{(1,...,k)} \quad m = 1, \ldots, N - M + 1.
\]

(3.12)

After the first \( N - M + 1 \) columns we have \( M - 1 - l \) pairs of columns. For each pair we have

\[
\hat{H}_{N-M+2v} = \hat{H}_{N-M+2v-1} \setminus \{ h_{M-(v-1)}^{M}, h_{M-(v-1)}^{2M}, \ldots, h_{M-(v-1)}^{k M} \}
\]

\[
= \{ h_1^{M-v}, h_{M+1}^{M-v}, \ldots, h_{M+M-v}^{k M}, h_{M+2M-v}^{M}, \ldots, h_{M+(k-1) M+1}^{k M}, \ldots, h_{M+kM-v}^{M} \}
\]

(3.13)

and

\[
\hat{H}_{N-M+2v+1} = \hat{H}_{N-M+2v-1}+1 \setminus \{ h_0^{v+M}, \ldots, h_{v+M}^{k M} \}
\]

\[
= \{ h_1^{M+v+1}, \ldots, h_{M}^{M+v+1}, \ldots, h_{M+1}^{M+v+1}, \ldots, h_{M+(k-1) M+1}^{k M+1}, \ldots, h_{M+kM-v}^{M+1} \}
\]

(3.14)
where \( v = 1, \ldots, M - 1 - l \).

**Example:** consider \( M = 2, N = 5 \) and \( K = 2 \) as presented in (3.11). In this case \( l = 0,1 \) and we have \( D_0 = \frac{10}{6} \) and \( D_1 = \frac{8}{4} = 2 \) respectively. In addition \( H^{(1,2)} = (H^{(1)}, H^{(2)}) = (h_1, h_2, h_3, h_4) \).

We begin with \( k = 1 \). In this case we get a point-to-point channel with 2 transmit and 5 receive antennas \( H^{(1)} = (h_1, h_2) \), which leads to the following effective channels

1. \( D_1 = 2 \): \( H^{(l=1), k=1}_{\text{eff}} \) is generated from the multiplication of the \( 5 \times 2 \) matrix \( H^{(1)} \) with the four columns of the transmission matrix \( G^{(1)}_l \). In this case \( H^{(1), 1}_{\text{eff}} \) is a \( 20 \times 8 \) block diagonal matrix, consisting of four blocks, where each block equals to \( H^{(1)} \).

2. \( D_0 = \frac{10}{6} \): \( H^{(l=0), k=1}_{\text{eff}} \) is a \( 30 \times 10 \) block diagonal matrix consisting of six blocks. The first four blocks are equal to \( H^{(1)} \). The additional two blocks (induced by columns 5-6 of \( G^{(1)}_0 \)) are vectors. We get that \( \hat{H}_5 = h_1 \) and \( \hat{H}_6 = h_2 \).

For the case \( k=2 \) the effective channel induced by \( G^{(1,2)}_l \) is as follows.

1. \( D_1 = 2 \): In this case the effective channel \( H^{(l=1), k=2}_{\text{eff}} \) is a \( 20 \times 16 \) matrix consisting of four blocks, where each block equals \( H^{(1,2)} = (H^{(1)}, H^{(2)}) \).

2. \( D_0 = \frac{10}{6} \): In this case the effective channel \( H^{(l=0), k=2}_{\text{eff}} \) is a \( 30 \times 20 \) matrix consisting of six blocks. The first four blocks equal to \( H^{(1,2)} \), where the other two blocks are \( \hat{H}_5 = (h_1, h_4) \) and \( \hat{H}_6 = (h_2, h_4) \).

We present \( H^{(0,2)}_{\text{eff}} \) of our example in equation (3.15).

Now we consider the rows of \( G^{(1,\ldots,k)}_l \). Each row of the transmission matrix is related to the column of \( H^{(1,\ldots,k)} \) that multiplies it, i.e. row \( j \) in \( G^{(1,\ldots,k)}_l \) corresponds to column \( h_{\hat{j}} \). In case there is a non zero entry of row \( j \) in column \( m \) of \( G^{(1,\ldots,k)}_l \), it means that \( h_{\hat{j}} \) occurs in \( \hat{H}_m \). In the next lemma we examine the number of occurrences of a certain column of \( H^{(1,\ldots,k)}_l \) in the blocks of \( H^{(l), k}_{\text{eff}} \).

**Lemma 3.7.** For any \( k = 1, \ldots, K \) consider column \( h_{a,M+b} \) in \( H^{(1,\ldots,k)}_l \), where \( a = 0, \ldots, k-1 \) and \( b = 1, \ldots, M \). In this case \( h_{a,M+b} \) occurs only in the first \( m = N-M+1+\min{(M-l-1,M-b)} \) blocks of \( H^{(l), k}_{\text{eff}} \).

**Proof.** Straight forward from the definition of the blocks of \( H^{(l), k}_{\text{eff}} \) in (3.12), (3.13) and (3.14).

### 3.4.4 Upper Bound on the Error Probability

In this subsection we derive for each channel realization an upper bound on the error probability of the joint ML decoder of \( K \) ensembles of IC’s transmitted on the unconstrained multiple-access channel, assuming each IC is \( D_l \cdot T_l \)-complex dimensional.
In accordance with the definitions in 3.4.3 we denote the effective channel of any set of users pulled together by $H_{\text{eff}}^{(l),(s)}$, where $s \subseteq \{1, \ldots, K\}$. We define $|H_{\text{eff}}^{(l),(s)}| = \rho^{-\sum_{i=1}^{\lfloor s \cdot D_1 \cdot T_1 \rfloor} \eta_i^{(s)}}$, where $ho^{-\frac{1}{2} \eta_i^{(s)}}$ is the $i$'th singular value of $H_{\text{eff}}^{(l),(s)}$, $1 \leq i \leq \lfloor s \cdot D_1 \cdot T_1 \rfloor$. We also define $\eta^{(s)} = (\eta_1^{(s)}, \ldots, \eta_\lfloor s \cdot D_1 \cdot T_1 \rfloor^{(s)})^T$.

Note that in our setting $NT_1 \geq K \cdot D_1 \cdot T_1$.

**Theorem 3.6.** Consider $K$ ensembles of $D_1 \cdot T_1$-complex dimensional IC’s transmitted on the unconstrained multiple-access channel with effective channel $H_{\text{eff}}^{(l),(s)}$ and densities $\gamma_{tr}^{(i)} = \rho^{T_{ri}}, i = 1, \ldots, K$. The average decoding error probability of the joint ML decoder is upper bounded by

$$
\overline{Pe}(H_{\text{eff}}^{(l),(s)}), \rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} \overline{Pe}(\eta^{(s)}, \rho) = \sum_{s \subseteq \{1, \ldots, K\}} D(\lfloor s \cdot D_1 \cdot T_1 \rfloor) \rho^{-T_1(\lfloor s \cdot D_1 - \sum_{i \in s} r_i \rfloor + \sum_{i=1}^{\lfloor s \cdot D_1 \cdot T_1 \rfloor} \eta_i^{(s)})}
$$

where $D(\lfloor s \cdot D_1 \cdot T_1 \rfloor)$ is a constant independent of $\rho$, and $\eta_i^{(s)} \geq 0$ for any $s \subseteq \{1, \ldots, K\}$ and any $1 \leq i \leq \lfloor s \cdot D_1 \cdot T_1 \rfloor$.

**Proof.** The proof is based on dividing the error event into events of error for different sets of users (disjoint events). Then we show that the upper bound on the error probability for the point-to-point channel derived in Chapter 2 can be used to upper bound the probability for each of these events. The full proof is in appendix B.9. \hfill \square

We wish to emphasize that the constraint of $\eta_i^{(s)} \geq 0$, for $i = 1, \ldots, \lfloor s \cdot D_1 \cdot T_1 \rfloor$ and for any $s \subseteq \{1, \ldots, K\}$ results from the fact that the same ensemble is upper bounded for any channel realization. In cases where it is possible to fit an ensemble to each channel realization, i.e. the case where the transmitter knows the channel, the upper bound applies also without this restriction.

### 3.4.5 Achieving the Optimal DMT

In this subsection we show that the transmission scheme proposed in 3.4.2 attains the optimal DMT for $N \geq (K + 1) M - 1$, $d_{M,N}^{(FC)} \left( \max (r_1, \ldots, r_K) \right)$. We base the proof on the upper bound on the error probability derived in Theorem 3.6. This upper bound consists of the sum of several terms, one for each $s \subseteq \{1, \ldots, K\}$. Each term depends on the determinant corresponding to its effective channel $|H_{\text{eff}}^{(l),(s)}|^T |H_{\text{eff}}^{(l),(s)}|^{-1}$. For each term (for each $s$) we upper bound this determinant in Lemma 3.8 (different bounds than the bounds used in Chapter 2) to get a new upper bound on the error probability. The upper bound is based on the fact that a determinant equals to the multiplication of the orthogonal elements of its columns (when the number of rows is larger than the number of columns). We average the upper bound over the channel realizations and show it attains the optimal DMT in Theorem 3.7, and also prove that the results apply to lattices when regular lattice decoder is employed at the receiver, in Theorem 3.8.

\footnote{Note that in 3.4.3 we considered the case of the first $k$ users where $k = 1, \ldots, K$. The extension to any $s \subseteq \{1, \ldots, K\}$ is straightforward.}
Each term in the upper bound in Theorem 3.6 can be viewed as the error probability of a point-to-point channel with $|s| \cdot M$ transmit antennas and $N$ receive antennas, while transmitting an $|s| \cdot D_l \cdot T_l$-complex dimensional IC in the method described in 3.4.2. We wish to emphasize that in this subsection we show that the terms corresponding to $|s| = 1$ attain the required optimal DMT since each user uses an optimal transmission scheme for the point-to-point channel with $M$ transmit and $N$ receive antennas. However, for the terms corresponding to $1 < |s| \leq K$ the effective transmission scheme is no longer optimal and does not necessarily attain the optimal DMT for a point-to-point channel with $|s| \cdot M$ transmit and $N$ receive antennas. In fact it does not even necessarily attain $d_{|s|,M,N}(|s| \cdot M)$. Hence, the challenge in this subsection is to upper bound the DMT of these terms and show that, although not optimal for the corresponding point-to-point channel, they attain the optimal DMT of the multiple-access channel for the case $N \geq (K + 1) M - 1$.

The average decoding error probability equals to the average over all channel realizations, i.e.

$$
\overline{Pe}(\rho) = E_H \left( \overline{Pe}\left( H_{\text{eff}}^{(1)}, \rho \right) \right). 
$$

(3.16)

Based on Theorem 3.6 we get the following upper bound on the average decoding error probability

$$
\overline{Pe}(\rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} E_H \left( D(|s| \cdot D_l \cdot T_l)\rho^{-T_l(|s| \cdot D_l \cdot T_l)} |H_{\text{eff}}^{(1)}(s) | \cdot |H_{\text{eff}}^{(1)}(s) |^{-1} \right). 
$$

(3.17)

Note that $E_H \left( |H_{\text{eff}}^{(1)}(s) | \cdot |H_{\text{eff}}^{(1)}(s) |^{-1} \right)$ equals to the mean value for any the users equals to the mean value for the first $k$ users. Therefore, by replacing $H_{\text{eff}}^{(1)}(s)$ with $H_{\text{eff}}^{(1)}|s|$ we can write (3.17) as follows

$$
\overline{Pe}(\rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} D(|s| \cdot D_l \cdot T_l)\rho^{-T_l(|s| \cdot D_l \cdot T_l)} \cdot E_H \left( |H_{\text{eff}}^{(1)}|s | \cdot |H_{\text{eff}}^{(1)}|s |^{-1} \right). 
$$

(3.18)

where $H_{\text{eff}}^{(1)}|s|$ is the effective channel of the first $|s|$ users, as defined in Subsection 3.4.3.

The channel matrix $H$ consists of $N \cdot K \cdot M$ i.i.d entries, where each entry has distribution $h_{i,j} \sim \mathcal{CN}(0,1)$, $1 \leq i \leq N$ and $1 \leq j \leq K \cdot M$. Without loss of generality we consider the case where the columns of $H$ are drawn sequentially from left to right, i.e. $h_{1j}$ is drawn first, then $h_{2j}$ is drawn et cetera. Column $h_j$ is an $N$-dimensional vector. Given $h_1, \ldots, h_{j-1}$, let us denote by $\tilde{h}_j \in \mathbb{C}^N$ the elements of the projection of $h_j$ on an orthonormal basis that depends on $h_1, \ldots, h_{j-1}$. We can write

$$
h_j = \Theta(h_1, \ldots, h_{j-1}) \cdot \tilde{h}_j
$$

(3.19)

where $\Theta(\cdot)$ is an $N \times N$ unitary matrix. $\Theta(\cdot)$ is chosen such that:

1. The first element of $\tilde{h}_j$, $\tilde{h}_{1,j}$, is in the direction of $h_{j-1}$.

2. The second element, $\tilde{h}_{2,j}$, is in the direction orthogonal to $h_{j-1}$, in the hyperplane spanned by $\{h_{j-1}, h_{j-2}\}$.

54
3. Element \( \tilde{h}_{j-1,j} \) is in the direction orthogonal to the hyperplane spanned by \( \{ \tilde{h}_2, \ldots, \tilde{h}_{j-1} \} \) inside the hyperplane spanned by \( \{ \tilde{h}_1, \ldots, \tilde{h}_{j-1} \} \).

4. The rest of the \( N - j + 1 \) elements are in directions orthogonal to the hyperplane \( \{ \tilde{h}_1, \ldots, \tilde{h}_{j-1} \} \).

Note that \( \tilde{h}_{i,j}, 1 \leq i \leq N, 1 \leq j \leq K \cdot M \) are i.i.d random variables with distribution \( \mathcal{CN}(0,1) \). Let us denote by \( h_{j,j-1,\ldots,j-k} \) the component of \( h_j \) which resides in the \( N - k \) subspace which is perpendicular to the space spanned by \( \{ h_{j-1}, \ldots, h_{j-k} \} \). In this case we get

\[
\|h_{j,j-1,\ldots,j-k}\|^2 = \sum_{i=k+1}^{N} |\tilde{h}_{i,j}|^2 \quad 1 \leq k \leq j - 1. \tag{3.20}
\]

If we assign \( |\tilde{h}_{i,j}|^2 = \rho^{-\xi_{i,j}} \), we get that the probability density function (PDF) of \( \xi_{i,j} \) is

\[
f(\xi_{i,j}) = C \cdot \log \rho - \rho^{-\xi_{i,j}} \cdot e^{-\rho^{-\xi_{i,j}}} \tag{3.21}
\]

where \( C \) is a normalization factor. In our analysis we assume a very large value for \( \rho \). Hence we can neglect events where \( \xi_{i,j} < 0 \) since in this case the PDF (3.21) decreases exponentially as a function of \( \rho \). For a very large \( \rho \), \( \xi_{i,j} \geq 0 \), \( 1 \leq i \leq N \) and \( 1 \leq j \leq K \cdot M \), the PDF takes the following form

\[
f(\xi_{i,j}) \propto \rho^{-\xi_{i,j}} \quad \xi_{i,j} \geq 0. \tag{3.22}
\]

In this case by assigning in (3.20) the vector \( \xi_{\tilde{j}} = (\xi_{1,j}, \ldots, \xi_{N,j})^T \) with PDF which is proportional to \( \rho^{-\sum_{i=1}^{N} \xi_{i,j}} \), we get

\[
\|h_{j,j-1,\ldots,j-k}\|^2 \geq \rho^{-\min_{z \in \{k+1, \ldots, N\}} \xi_{z,j}} \tag{3.23}
\]

where \( 1 \leq k \leq j - 1 \). In addition

\[
\|h_j\|^2 \geq \rho^{-\min_{z \in \{1, \ldots, N\}} \xi_{z,j}}. \tag{3.24}
\]

As presented in (3.18), in order to calculate the upper bound on the error probability we need to consider only the effective channel of the first \( |s| \) users, \( 1 \leq |s| \leq K \). Hence, in order to obtain an upper bound on the error probability we wish to lower bound the determinant \( |H^{(i)}_e| \cdot |H^{(i)}_e| \) by lower bounding the contribution of each column in the channel matrix \( H \) to the determinant. The following lemma presents a lower bound on the determinant.

**Lemma 3.8.**

\[
|H^{(i)}_e| \cdot |H^{(i)}_e| \geq \prod_{a=0}^{M-|s|-1} \prod_{b=1}^{M} \rho^{-\left(N-M+1+\min(M-l-1,M-b)\right)\cdot\min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b} \cdot \prod_{b'=2}^{M} \rho^{-\sum_{i=1}^{\min(M-l-1,b'-1)} \min_{z \in \{aM+b'-i, \ldots, N\}} \xi_{z,aM+b'}}. 
\]
The proof is in appendix B.10. Essentially, the term

\[(N - M + 1 + \min(M - l - 1, M - b)) \cdot \min_{z \in \{aM + b, \ldots, N\}} \xi_{z, aM + b}\]

indicates that in the lower bound column \(h_{aM + b}\) occurs \(N - M + 1 + \min(M - l - 1, M - b)\) times with \(h_1, \ldots, h_{aM + b - 1}\) to its left. Therefore, only the elements of \(h_{aM + b}\) which are orthogonal to this set of columns, \(\xi_{z, aM + b}\), where \(aM + b \leq z \leq N\) contribute to the lower bound.

The term

\[\min_{i=1}^{M - l - 1, b' - 1} \sum_{z \in \{aM + b' - i, \ldots, N\}} \xi_{z, aM + b'}\]

indicates that column \(h_{aM + b'}\) occurs \(\min(M - l - 1, b' - 1)\) times. However, this time we handle the contribution of the orthogonal elements more carefully. For \(1 \leq i \leq \min(M - l - 1, b' - 1)\) we consider the elements in \(h_{aM + b'}\) which are orthogonal to the set of columns \(h_1, \ldots, h_{aM + b' - i - 1}\).

Now we are ready to lower bound the transmission scheme DMT, based on the lower bound on the determinant in Lemma 3.8. Let us denote the maximal multiplexing gain by \(r_{\max} = \max(1, \ldots, K)\), and also assume \(l = \lfloor r_{\max} \rfloor\).

**Theorem 3.7.** Consider \(K\) sequences of ensembles of \(D_l \cdot T_l\)-complex dimensional IC’s transmitted over the unconstrained multiple-access channel, where each user transmits multiplexing-gain \(r_i\) using \(G^{(i)}\), \(i = 1, \ldots, K\). The DMT of this transmission scheme is lower bounded by \(d^{*(FC)}_{M,N}(r_{\max})\).

**Proof.** We use the upper bound on the error probability derived in Theorem 3.6, and the lower bound on the determinant (B.128) in order to give a new upper bound on the error probability. We average this upper bound over the channel realization, and show that for large \(\rho\) the diversity order of the most dominant error event is lower bounded by \(d^{*(FC)}_{M,N}(r_{\max})\). The full proof is in appendix B.11.

In Theorem 3.5 we have shown that for \(N \geq (K + 1)M - 1\) the DMT of any IC is upper bounded by \(d^{*(FC)}_{M,N}(r_{\max})\). On the other hand in Theorem 3.7 we have shown that there exist sequences of IC’s that attain DMT which is lower bounded by \(d^{*(FC)}_{M,N}(r_{\max})\). Hence, the transmission scheme must attain the optimal DMT.

In the next theorem we prove the existence of a sequence of lattices that attains the optimal DMT as in Theorem 3.7.

**Theorem 3.8.** For each tuple of multiplexing gains \((r_1, \ldots, r_K)\) there exist \(K\) sequences of \(2D_l \cdot T_l\)-real dimensional lattices transmitted over the unconstrained multiple access channel that attain diversity order of \(d^{*(FC)}_{M,N}(r_{\max})\), when regular lattice decoder is employed, where \(l = \lfloor r_{\max} \rfloor\).

**Proof.** See appendix B.14

Now we show that for each segment of the optimal DMT there exists a sequence of \(K\) lattices that attains it, i.e. the optimal DMT consists of \(M\) segments, each in the range \(l \leq r_{\max} \leq l + 1\) where \(l = 0, \ldots, M - 1\), and there are \(M\) sequences of lattices that attain it.
Corollary 3.2. For the case $N \geq (K + 1) M - 1$ each segment of the optimal DMT of the unconstrained multiple-access channel, $d^{*,(FC)}_{M,N}(r_{\text{max}})$, is attained by a sequence of $K$, $2D_{\lfloor r_{\text{max}} \rfloor} T_{\lfloor r_{\text{max}} \rfloor}$-real dimensional lattices.

**Proof.** See appendix B.15.

### 3.5 Discussion

In this section we discuss the results presented in the chapter. As an illustrative example we consider the case where there are two users, each with two transmit antennas, i.e. $K = M = 2$. We consider the symmetric case where $r_1 = r_2 = r$, and explain based on Theorem 3.4 why for $N = 2$, 4 IC’s are suboptimal. On the other hand based on Theorem 3.6 and Theorem 3.7 we explain why the optimal DMT is attained for $N \geq 5$. The analysis in this section is somewhat loosed and we refer the reader to Sections 3.3, 3.4 for the full analysis.

We begin by giving a short reminder to the behavior of lattices in a point-to-point channel when $M = N = 2$, as presented in Chapter 2. We consider in this discussion lattices although the results apply to IC’s in general. In this case, the optimal DMT equals $d^{\ast,(FC)}_{2,2}(r) = 4 - 3r$ in the range $0 \leq r \leq 1$, and in order to attain it the NDCU, $D$, must be equal to $\frac{4}{3}$. We wish to explain why when $D \neq \frac{4}{3}$ the optimal DMT is not attained in the range $0 \leq r \leq 1$. For lattices, obtaining multiplexing gain $r > 0$ requires scaling each dimension of the lattice by $\rho^{-\frac{r}{2D}}$. When $D < \frac{4}{3}$ diversity order of 4 may be attained for $r = 0$. However, the scaling is too strong and does not enable to attain the optimal DMT for any $r > 0$ (there are not enough degrees of freedom to attain the straight line $4 - 3r$). On the other hand when $D > \frac{4}{3}$, the lattice “fills” too much of the space and the channel induces error probability that does not enable to attain diversity order of 4 for $r = 0$, and therefore does not allow attaining the optimal DMT in the range $0 \leq r \leq 1$. Hence, choosing $D = \frac{4}{3}$ balances the effect of the scaling and the channel on the lattice and allows to attain the optimal DMT in the range $0 \leq r \leq 1$. We now follow this intuition to discuss the multiple-access channel.

#### 3.5.1 Why IC’s are Suboptimal for $N < (K + 1) M - 1$

The error event in the multiple-access channel can be divided into the disjoint error events of any subset of the users, as described in Theorem 3.6. Consider a certain subset of users $s \subseteq \{1, \ldots, K\}$. Due to the distributed nature of the multiple-access channel, the error probability for this subset is upper bounded by the error probability of a point-to-point channel with $|s| \cdot M$ transmit and $N$ receive antennas, i.e. corresponding to a point-to-point channel where the users in $s$ are pulled together. Hence, the DMT in the multiple-access channel is determined by the most probable error event. For the unconstrained multiple-access channel the problem is more involved as each IC has a certain NDCU. Assume user $i$ has $D_i$ NDCU, where $1 \leq i \leq K$. When considering the error event of users in $s$, we consider an IC with $\sum_{i \in s} D_i$ NDCU. The DMT in this error event is upper bounded by $d^{\ast,\sum_{i \in s} D_i}|s| M,N (|s| \cdot r)$, i.e. the bounds derived in Chapter 2 for the point-to-point channel. In case the dimensions of any subset of the users do not “align”, i.e. in case a certain subset of the users has NDCU that is too large or too small to attain the optimal DMT, we get sub-optimality.


this subsection we take as example the case $M = K = 2$ and explain why for $N = 2, 4$ the dimensions do not align, and therefore the optimal DMT is not attained.

Let us begin with the case $M = K = N = 2$. In this case the optimal DMT in the symmetric case equals

$$d_{K,M,N}^*(r) = d_{2,2,2}^*(FC) (r) = \begin{cases} 
    d_{2,2}^*(FC) (r) & 0 \leq r \leq \frac{2}{3} \\
    d_{4,2}^*(FC) (2r) & \frac{2}{3} < r \leq 1 \\
    4 - 3r & 0 \leq r \leq \frac{2}{3} \\
    6 - 6r & \frac{2}{3} < r \leq 1.
\end{cases} \quad (3.25)$$

On the other hand the optimal DMT of IC’s in this case is upper bounded by $d_{2,2}^{*(IC)} (r) = 4 (1 - r)$, which is smaller than the optimal DMT for any $0 < r < 1$. Let us explain the reason for the sub-optimality. First, note that in the symmetric case we must choose $D_1 = D_2$ to maximize the IC’s DMT, i.e. the users have the same NDCU. Since $N = 2$ each user can not transmit more than one NDCU, where in Chapter 2 it was shown that each user needs to transmit $\frac{4}{3}$ NDCU in order to attain $d_{2,2}^{*(FC)} (r)$ in the range $0 \leq r \leq \frac{2}{3}$. In addition, the maximal diversity order each user may attain is 4 since $M = N = 2$, and also $d_{2,2}^{*(FC)} (r)$ is a straight line. Hence, even when transmitting one dimension per channel use the DMT must be smaller than $6 - 6r$. Therefore, in this case the dimension mismatch manifest itself in the fact that $N$ is too small even to attain the first line of $d_{2,2}^{*(FC)} (r)$. This sub-optimality is presented in Figure 3.3.

For $K = M = 2$ and $N = 4$ it was shown in Theorem 3.4 for the symmetric case that IC’s are suboptimal in the range $1 < r < \frac{3}{2}$. In this range the DMT of IC’s is upper bounded by $7 - 4r$, attained when $D_1 = D_2 = \frac{7}{4}$. The dimension mismatch manifests itself in this example both in error events of a single user, and the error event of both users. For error events of a single user the optimal DMT is $d_{2,4}^{*(FC)} (r)$ which is also the optimal DMT of the multiple-access channel in the range $1 \leq r \leq \frac{N}{K+1} = \frac{4}{5}$. The NDCU required to attain $d_{2,4}^{*(FC)} (r)$ for $1 \leq r \leq 2$ is $2$ which is larger than $D_1 = D_2 = \frac{7}{4}$. Therefore, for the single user error events the scaling of the IC of each user is too strong and does not enable to attain the optimal DMT. On the other hand, for the two users error event the optimal DMT is $d_{2,2}^{*(FC)} (2r)$ which is also the optimal DMT in the range $\frac{3}{4} \leq r \leq 2$. The effective IC of the two users pulled together has NDCU $D_1 + D_2 = \frac{7}{2}$, which is too large compared to what is required to attain $d_{2,2}^{*(FC)} (2r)$ in the range $1 < r < \frac{3}{2}$. Hence, for this error event we get that the effective IC fills too much of the space and so the channel does not enable to attain the optimal DMT.

### 3.5.2 Why IC’s Attain the Optimal DMT for $N \geq (K + 1) M - 1$

For the case where $N \geq (K + 1) M - 1$ there is no longer a dimension mismatch. However, the condition that there is no dimension mismatch is merely a necessary condition in order to attain the optimal DMT. Hence, in this subsection we will explain why the optimal DMT is attained based on the transmission scheme presented in Subsection 3.4.2 and on the effective channel presented in 3.4.3.

We consider as an example the case $M = K = 2$ and $N = 5$. We show why in this case the single user performance $d_{2,5}^{*(FC)} (r_{\text{max}})$ is attained. For simplicity we will focus on the symmetric case. Essentially, we show in this example that IC’s attain the first DMT line, $10 - 6r$, which coincides with the optimal DMT $d_{2,5}^{*(FC)} (r)$ in the range $0 \leq r \leq 1$. The transmission scheme is $G_{0}^{(1,2)}$ presented in (3.11). Note
receive antennas. For this lattice the NDCU equals
pulled together, i.e. an error event of a lattice transmits over a point-to-point channel with
two users is also upper bounded by the optimal DMT in the range $0 \leq r \leq 1$. What is left to show is that the DMT of the error event of the
two users is also upper bounded by the optimal DMT in the range $\frac{r}{\sqrt{M-1}}$. We will show that for $r = 0$ this
lattice attains diversity order 10. This will lead to DMT $10 - 6r$ since the lattice DMT is a straight line and $D_1 + D_2 = \frac{10}{3}$.

At the receiver, the effective radius of the lattice of the two users pulled together at $r = 0$ is

$$r_{\text{eff}}^2 = |V|^\frac{1}{(D_1 + D_2)T} = \frac{T}{(D_1 + D_2)^2} = |H_{\text{eff}}(l=0,K)^\dagger H_{\text{eff}}(l=0,K)|^\frac{-1}{(D_1 + D_2)T}$$

where $|V| = r_{\text{rc}}^{-1}$ is the volume of the Voronoi region of the effective lattice at the receiver. Recall that for lattices $r_{\text{eff}} \geq r_{\text{packing}} = \frac{d_{\text{min}}}{2}$, where $r_{\text{packing}}$ and $d_{\text{min}}$ are the packing radius and the minimal
distance of the lattice respectively. We are interested in the event where $r_{\text{eff}}^2$ is in the order of the additive
noise variance $\rho^{-1}$. In this case $(d_{\text{min}}^2)^2$ is in the order of the noise variance or worse, and so the error
probability does not reduce with $\rho$. In Subsection 3.4.5 it is shown that this event is the dominant error event in
determining the DMT of the transmission scheme. From (3.26) we get that $H_{\text{eff}}(l=0,K)$ determines the
effective radius at the receiver. From (3.11) and the description of the effective channel in Subsection 3.4.3 we get that $H_{\text{eff}}(l=0,K)$ is a block diagonal matrix, where 4 of its blocks equal $H \in \mathbb{C}^{5 \times 4}$. For large $\rho$, the most probable error event ($r_{\text{eff}}^2 = \rho^{-1}$) occurs when the determinant of $H$ reduces with $\rho$, and the determinants of
the rest of the blocks in $H_{\text{eff}}(l=0,K)$ remain constant with $\rho$. Note that if $|H^\dagger H| = \rho^{-\alpha}$, then most likely that the smallest singular value of $H$ equals $\rho^{-\alpha}$ and the rest of the singular values remain constant [35]. In this
case we get $|H^\dagger H| \approx \rho^{-\alpha}$ with a PDF proportional to $\rho^{-(5-4+1)\alpha} = \rho^{-2\alpha}$. By assigning $(D_1 + D_2)T = 20$ and $|H_{\text{eff}}(l=0,K)^\dagger H_{\text{eff}}(l=0,K)| = |H^\dagger H|^{\frac{4}{3}} = \rho^{-4\alpha}$ in (3.26) we get that

$$r_{\text{eff}}^2 = |H^\dagger H|^{-\frac{4}{3}} \approx \rho^{-\frac{2\alpha}{3}}$$

(3.27)

with a PDF proportional to $\rho^{-2\alpha}$. Hence, $r_{\text{eff}}^2 = \rho^{-1}$ when $\alpha = -5$. Based on Subsection 3.4.5 we get for large $\rho$ that this is the most dominant error event, and by assigning $\alpha = 5$ we get that it happens with probability $\rho^{-10}$. Therefore, in this case diversity order of 10 is attained.

For general $N = (K + 1)M - 1$ each user transmits an optimal transmission scheme of a point-to-point
channel with $M$ transmit and $N$ receive antennas. Since the users do not cooperate, in the worst case we get that $H_{\text{eff}}(l=0,K)$ has $N - M + 1$ blocks that equal $H \in \mathbb{C}^{N \times K \cdot M}$. For large $\rho$ we get that $|H^\dagger H| = \rho^{-\alpha}$ with a PDF proportional to $\rho^{-(N-K\cdot M+1)\alpha}$. In this case $(\sum_{i=1}^{K} D_i)T = K \cdot M \cdot M$ and so we get

$$r_{\text{eff}}^2 = |H^\dagger H|^{-\frac{N-M+1}{\sum_{i=1}^{K} D_i}T} \approx \rho^{\frac{(N-M+1)\alpha}{K \cdot M \cdot N}}.$$  

(3.28)

Since $N = (K + 1)M - 1$ we have enough equations at the receiver to get $N - M + 1 = K \cdot M$ and
\( N - K \cdot M + 1 = M \). Hence, by substituting in (3.28) we get
\[
r_{\text{eff}}^2 = \rho^{-\frac{N}{M}}
\]
with a PDF proportional to \( \rho^{-(N-KM+1)\alpha} = \rho^{-M\alpha} \). Therefore, we get for \( \alpha = N \) that \( r_{\text{eff}}^2 = \rho^{-1} \) with probability \( \rho^{-MN} \), which leads to a diversity order \( MN \) at \( r = 0 \). In addition \( \sum_{i=1}^{K} D_i = \frac{KMN}{N-M+1} \) and so the first line of the optimal DMT is attained. Note that we considered the case of the \( K \) users pulled together. For any error event of the users in \( s \subseteq (1, \ldots, K) \), the diversity order will be larger or equal to \( MN \) at \( r = 0 \).

In summary, since the users do not cooperate we get in the worst case \( N - M + 1 \) occurrences of \( H \) in the blocks of \( H_{\text{eff}}^{(l=0):K} \). However, when \( N \geq (K+1)M-1 \) there is a sufficient amount of receive antennas to compensate for the impact of \( H \) on \( r_{\text{eff}}^2 \), by decreasing the probability that \( H \) has small determinant.
Chapter 4

Practical Implementation of Lattice Codes: LDLC’s with Max-Product Decoding Algorithm

4.1 Introduction

Low density lattice codes (LDLC’s) [22] are lattices characterized by the sparseness of the inverse of their generator matrix. Under the tree assumption a sum-product algorithm for LDLC’s was derived [22] directly in the Euclidean space. For LDLC’s of dimension $n = 100,000$, this message passing decoding algorithm attains up to 0.6 dB from the additive white Gaussian noise (AWGN) channel capacity. In addition to its good performance, the iterative decoding algorithm has linear complexity as a function of the block length. However, the decoder presented in [22] samples and quantizes the passed messages, which result in a large storage requirement and relatively large (although linear in the block length) computational complexity. Efficient implementations of the sum-product algorithm, that significantly reduce both the computational complexity and the storage requirement were presented in [32], [15]. These works take a parametric approach in the representation of the passed messages, and essentially attain the same performance as the straightforward implementation of the sum-product algorithm. An extension of LDLC’s to complex lattices, coined complex low density lattice codes (CLDLC’s) was presented in [33]. CLDLC’s attain better performance than LDLC’s for small dimensional lattices, using the iterative parametric decoder.

The max-product algorithm, presented in [21] for algebraic codes, is a message passing decoding algorithm aimed at minimizing the word error rate (WER). Under the tree assumption this decoding algorithm yields blockwise maximum-likelihood (ML) decoding. For lattices, ML lattice decoding gives the most likely lattice point in the infinite lattice for a certain observation, i.e. decoding without taking into consideration any shaping region boundaries.

In the first part of this chapter we take a factor graph approach to derive a max-product algorithm for LDLC’s in point-to-point channels, where the derivation is done directly in the Euclidean space. For lattices that hold the tree assumption we get the exact ML lattice decoding, and for general LDLC’s we get
approximation of the blockwise ML lattice decoding.

For the AWGN channel we reveal an interesting connection between the sum-product and max-product algorithms of LDLC’s. The messages are composed of Gaussians in both algorithms; interestingly, these Gaussians are identical. However, the messages are different. While in the sum-product algorithm we sum these Gaussians to get a Gaussian-mixture, in the max-product algorithm for each message we take the maximal value between these Gaussians in each point, i.e. the passed message is the maximal envelope of these Gaussians.

The parametric approach provides an efficient way to implement the sum-product algorithm [32], while maintaining the same performance as the decoder presented in [22]. In this case each Gaussian is represented by three parameters: mean value, variance and amplitude. Each message is represented by a list of its Gaussians parameters. As there is an infinite number of Gaussians in each message, one of the cornerstones in the parametric algorithm is consolidating these Gaussians into a finite parametric list of Gaussians in an efficient way, while retaining good performance. In this chapter we also adapt the parametric algorithm to approximate the max-product algorithm. We show that extending the parametric approach to the max-product algorithm does not increase the computational complexity compared to the parametric sum-product algorithm.

Numerical results show that for small dimensional LDLC’s the max-product algorithm improves the WER compared to the sum-product algorithm. Improving the WER for small dimensional LDLC’s is desired since in practical systems it may reduce the number of packet retransmissions. As expected, we show that for either large dimensions or very small noise variance values, the performance of both algorithms is similar.

In the second part of this chapter we derive an iterative max-product algorithm for general multiple-input multiple-output (MIMO) channels. Under an extended tree assumption the max-product algorithm leads to maximum a-posteriori (MAP) decision that minimizes the word error rate (WER). The extended tree assumption takes into account the connection between symbols, induced by the MIMO channel. Under the assumption that the a-priori probabilities of the symbols are Gaussians, we get that the passed messages in the iterative algorithm consist of Gaussians. Interestingly, each Gaussian in a message sent from a lattice symbol, has a mean value that equals to the minimum mean square error (MMSE) estimation of this symbol given the observation. In addition, the variance value of this Gaussian equals to the MMSE. The a-priori probabilities for the MMSE estimation, are the Gaussians in the incoming messages to the channel node, that correspond to this Gaussian.

More specifically we consider the quasi static Rayleigh flat-fading channel with two transmit and two receive antennas. This channel attracts attention both from practical and theoretical aspects. The benchmark for performance in this channel is set by the Golden code [2] which is a full-rate code satisfying the non-vanishing determinant property. Another popular transmission scheme for this channel is the Alamouti scheme [1], which offers low complexity decoding at the expense of degraded performance.

We propose a transmission scheme for CLDLC’s for this channel, that uses \( \frac{4}{3} \) number of dimensions per channel use (NDCU) to transmit a very sparse CLDLC of degree 2, where the NDCU is the lattice dimensionality divided by the number of channel uses. The general idea is to reduce dimensionality in order to protect a subset of the lattice symbols by reducing the probability that the observations of these symbols
are not good. Then, the information on the “protected” symbols is used in the max-product algorithm to find the other symbols via the lattice “parity check” equations, i.e. the rows of the inverse of the CLDLC generator matrix.

The reduced dimensionality of the transmission scheme and the fact that a very sparse CLDLC is transmitted, enables to derive a very efficient parametric algorithm that approximates the max-product algorithm for the MIMO channel with two transmit and two receive antennas. For rate of 8 bits per channel use CLDLC decoded using the parametric algorithm has a gap of 1.7 dB from the performance of Golden code decoded using maximum-likelihood (ML) decoder and outperforms the Alamouti scheme by 5 dB, for WER of $10^{-4}$. For rate of 16 bits per channel use CLDLC using the parametric algorithm outperforms the Alamouti scheme by 9 dB and has a gap of 3.5 dB from the performance of tilted QAM [31] decoded using ML decoder, for WER of $10^{-4}$. We wish to emphasize that the a-priori probabilities of the lattice symbols enable the decoder to take into account the power constraint. However, in practice the shaping operation of the CLDLC does not necessarily have to be probabilistic. In addition it is important to note that the reduced dimensionality simply serves as a mean to efficiently decode CLDLC’s in the MIMO channel.

The proposed scheme attains better performance than the Alamouti scheme which uses 1 NDCU, by adding an extra $\frac{1}{3}$ NDCU. While our scheme is more complex than the Alamouti scheme, it still requires a low computational complexity which is significantly smaller than exhaustive search over the entire codebook. For instance, when transmitting 8 bits per channel use over three channel uses, the computational complexity required to decode the codebook consisting of $2^{24}$ words, is an order of 100 operations.

The outline of the chapter is as follows. In Section 4.2 basic definitions are given. In Section 4.3 we derive the max-product algorithm for point-to-point channels, present the max-product algorithm for the AWGN channel and its relation to the sum-product algorithm, and also extend the parametric decoder to approximate the max-product algorithm. In Section 4.4 we derive the max-product algorithm for general MIMO channels and show the connection between the passed messages and MMSE estimation, when the prior is a Gaussian. Section 4.5 presents the design of CLDLC for MIMO channel with two transmit and two receive antennas, and finally in Section 4.6 we present numerical results.

4.2 Basic Definitions

A lattice $\Lambda$ is a discrete set in the Euclidean space $\mathbb{R}^n$, closed under addition. $n$-dimensional lattice can be represented by a squared generating matrix $G$. In this case we can write $x = G \cdot b$, where $x \in \Lambda$ and $b \in \mathbb{Z}^n$. The Voronoi region of a lattice point $x \in \Lambda$ is the set of points in $\mathbb{R}^n$ that are closer to $x$ than to any other lattice point. The Voronoi regions are identical up to a translate. The Voronoi region volume equals $|\det(G)|$, i.e. the absolute value of the determinant.

Similarly to low-density parity-check codes (LDPC) [10], LDLC’s are lattices that have a sparse “parity-check matrix” $Q = G^{-1}$, i.e. the inverse of the lattice generator matrix. In this case we get for $x \in \Lambda$ that $Q \cdot x \in \mathbb{Z}^n$. A Latin square LDLC has the same non-zero values in each row and each column of its parity check matrix, $Q$, up to a permutation and sign. In this case we denote the absolute values of the non-zero entries of the rows and columns of $Q$ by $q = \{|q_1|, \cdots, |q_l|\}$, where $l$ is the LDLC degree, and we assume
\[|q_1| \geq \cdots \geq |q_l| > 0.\] Note that we can represent each LDLC by a bipartite graph, where the variable nodes represent the lattice symbols and the check nodes represent the parity check equations.

An \(n\)-complex dimensional complex lattice \(A_C\) is a discrete set in \(\mathbb{C}^n\) closed under addition. Each lattice point satisfies \(z = G \cdot b\), where \(b \in \mathbb{Z}[i]^n\) and \(G \in \mathbb{C}^{n \times n}\) is an invertible matrix, i.e. each lattice point equals to the multiplication of the lattice generator matrix \(G\), and a Gaussian integer vector. The Voronoi region of a lattice point \(x \in \Lambda\) is the set of points in \(\mathbb{C}^n\) closest to \(x\). The Voronoi regions are identical up to a translate and the volume of each Voronoi region equals \(\det(G^\dagger G)\). CLDLC [33] is a complex lattice with sparse parity check matrix \(Q = G^{-1}\). When referring to CLDLC in this chapter we assume that \(Q\) is a Latin square, i.e. its rows and columns are identical up to a permutation and a phase.

For the point-to-point AWGN channel we can write \(y = x + w\), where \(x \in \Lambda\) and \(w \sim \mathcal{C}\mathcal{N}(0, \sigma^2 \cdot I)\) is the \(n\)-dimensional AWGN with variance \(\sigma^2\). When considering regular lattice decoding, i.e. considering a channel with no power constraint, the classic definition of channel capacity is meaningless. This channel was analyzed in [20]. For lattices, this channel generalized capacity as presented in [20] gives \(\sigma^2 < \frac{\det(G)^{\frac{1}{2}}}{2\pi e}\).

For the quasi static flat-fading channel with \(M\) transmit and \(N\) receive antennas

\[Y = H \cdot X + W\tag{4.1}\]

where \(H \in \mathbb{C}^{N \times M}\) is an \(N \times M\) matrix, \(X \in \mathbb{C}^{M \times T}\) is the transmitted signal over \(M\) transmit antennas and \(T\) channel uses, and \(W \in \mathbb{C}^{N \times T}\) is the additive noise. The entries of \(W\) are independent with distribution \(\mathcal{C}\mathcal{N}(0, 2\sigma^2)\), where \(\mathcal{C}\mathcal{N}\) represents complex-normal, and \(2\sigma^2\) is the additive noise variance. \(Y \in \mathbb{C}^{N \times T}\) is the received signal, and the entries of \(H\), \(h_{i,j} \sim \mathcal{C}\mathcal{N}(0, 1)\), \(1 \leq i \leq N, 1 \leq j \leq M\), are also independent. \(H\) remains constant during the \(T\) channel uses and we assume perfect channel knowledge at the receiver and no channel knowledge at the transmitter. A codebook \(C\) with rate \(R\) bits per channel use consists of \(|C| = 2^{RT}\) codewords. In this setting the average transmitted power of this code is \(P = \frac{1}{T} \sum_{X \in C} \|X\|_F^2\), where \(\|\cdot\|_F\) is a matrix Frobenius norm, and the signal to noise ratio (SNR) in each receive antenna is \(\text{SNR} = \frac{P}{2\sigma^2}\).

Finally, the conv-sup operation between two functions is defined by

\[f(x) \circ g(x) = \max_{t} f(t) \cdot g(x - t).\tag{4.2}\]

### 4.3 Max-Product Algorithm for Point-to-Point Channels

#### 4.3.1 General Formulation

We would like to calculate the following marginal

\[\psi_j(x_j) = \max_{\underline{x} \in \Lambda, \sim x_j} f_{Y|X}(y|\underline{x}) \quad j = 1, \ldots, n\tag{4.3}\]

where \(\{\underline{x} \in \Lambda, \sim x_j\}\) means the set of lattice points in which the \(j\)'th component equals \(x_j\), and \(f_{Y|X}(y|\underline{x}) = \prod_{k=1}^{r} f_{Y_k|X_k}(y_k|x_k)\) is the probability of receiving the observation \(y\) given that \(\underline{x}\) was transmitted. Note that
if we take for each function in (4.3) the argument \( x_j \), \( j = 1, \ldots, n \) that maximizes it, we get exactly the ML lattice decoding decision. We can factorize the problem and rewrite it as:

\[
\psi_j(x_j) = \max_{x \in \mathbb{R}^n, x_j} \prod_{k=1}^n f_{Y_k|x_k}(y_k|x_k) \prod_{i=1}^n \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z})
\]  

where \( q_{ij} \) is the \( i \)’th row of \( Q \), and the indicator function \( \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}) \) equals 1 if \( q_{ij} \cdot x \) equals an integer and zero else. The multiplication \( \prod_{i=1}^n \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}) \) in (4.4) takes into account all of the lattice check equations, hence this product equals 1 if and only if \( x \in \mathbb{Z} \). We can translate the factorized function in (4.4) to a factor graph, for which the variable nodes represent the symbols \( x_j \), \( j = 1, \ldots, n \), and the check nodes represent the check equations indication functions \( \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}) \), \( i = 1, \ldots, n \). Edges in the factor graph are stretched from each check equation indication function to the variable nodes that take place in it, i.e. the variables corresponding to the non-zero entries in the relevant row in \( Q \). In addition, each variable node \( x_k \) is connected to the function that represents its channel observation \( f_{Y_k|x_k}(y_k|x_k) \), \( k = 1, \ldots, n \).

Under the tree assumption, the marginalization in (4.4) can be broken into similar independent marginalization subproblems corresponding to the subtrees. Without loss of generality let us observe the marginalization of \( x_1 \). In this case under the tree assumption we get that \( \psi_1(x_1) \) equals:

\[
\max_{x^{(i)}, \ldots, x^{(s)}} f(y_1|x_1) \prod_{i=1}^s c_i(x_1, x^{(i)}) \cdot \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z})
\]  

where we assumed that \( x_1 \) takes place in the first \( s \) check equations in \( Q \), and \( x^{(i)} \) are the variables that take place in the \( i \)’th check equation with \( x_1 \). \( c_i(x_1, x^{(i)}) \cdot \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}) \) is related to the subtree of the \( i \)’th check equation that \( x_1 \) takes place in. Note that due to the tree assumption the elements in \( x^{(i)} \) \( i = 1, \ldots, s \) are different. Hence we can rewrite (4.5) as:

\[
\psi_1(x_1) = f(y_1|x_1) \prod_{i=1}^s \max_{x^{(i)}} c_i(x_1, x^{(i)}) \cdot \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}).
\]

In order to calculate the marginalization in (4.6) we can divide the operation into two phases. The first phase takes place in the check nodes connected to \( x_1 \), and the second phase takes place in the variable node of \( x_1 \). For the \( i \)’th check equation let us denote the check node message by:

\[
c^{(i)}(x_1) = \max_{x^{(i)}} c_i(x_1, x^{(i)}) \cdot \mathbb{1}(q_{ij} \cdot x \in \mathbb{Z}).
\]

Assuming \( q_{ij} \) has \( m+1 \) non-zero values, we would like to break the calculation of \( c^{(i)}(x_1) \) into \( m \) maximization operations. Hence, let us assume that \( q_{i} \cdot x = \sum_{k=1}^{m} q_{ik} x_k + q_{im+1} x_1 \), where \( q_{ik} \), \( k = 1, \ldots, m+1 \) are the non-zero entries of \( q_{ij} \) and \( x_k \), \( k = 1, \ldots, m \) are their corresponding variable nodes. By assuming
\( t_1 = q_1^i \cdot x_1^{(i)} \), \( t_2 = q_2^2 x_2^{(i)} + t_1 \) and in general \( t_k = q_k^k x_k^{(i)} + t_{k-1}, k = 2, \ldots, m \), we write

\[
P(t_m) = \max_{t_1, \ldots, t_{m-1}} v_x^{(i)} \left( \frac{t_1}{q_i^i} \right) \cdot \prod_{k=2}^{m} v_x^{(i)} \left( \frac{t_k - t_{k-1}}{q_i^k} \right)
\]

(4.7)

where \( v_x^{(i)}(\cdot) \) is the message sent from variable \( x_k^{(i)} \), \( k = 1, \ldots, m \). Note that due to the tree assumption \( v_x^{(i)}(\cdot) \) is the marginalization of \( x_k^{(i)} \) over its subtree excluding its edge with \( x_1 \) i'th check equation, i.e. to calculate \( x_k^{(i)} \) message we need to take the same marginalization operation over its subtree. We can break the maximization in (4.7) as follows

\[
v_1(t_2) = \max_{t_1} v_x^{(i)} \left( \frac{t_1}{q_i^i} \right) \cdot v_x^{(i)} \left( \frac{t_2 - t_1}{q_i^2} \right)
\]

(4.8)

\[
v_{k-1}(t_k) = \max_{t_{k-1}} v_{x_k}(t_{k-1}) \cdot v_x^{(i)} \left( \frac{t_k - t_{k-1}}{q_i^k} \right)
\]

(4.9)

where \( k = 3, \ldots, m - 1 \).

\[
P(t_m) = \max_{t_{m-1}} v_{x_m}(t_{m-1}) \cdot v_x^{(i)} \left( \frac{t_m - t_{m-1}}{q_i^m} \right).
\]

(4.10)

Hence, by assigning \( t_m = b - q_i^{m+1} x_1 \), where \( b \in \mathbb{Z} \), we get that the check node message equals

\[
c^{(i)}(x_1) = \max_{b \in \mathbb{Z}} P(b - q_i^{m+1} \cdot x_1).
\]

(4.11)

In the second phase, that takes place in the variable node, we simply multiply the messages related to variable node \( x_1 \)

\[
\psi_1(x_1) = f_{Y_1|X_1}(y_1|x_1) \cdot \prod_{i=1}^{s} c^{(i)}(x_1).
\]

(4.12)

In general the tree assumption does not necessarily hold. In this case we take the following steps to get the max-product message passing algorithm.

- **Initialization**: Variable node \( x_j \) first message is initialized to its channel observation \( f_{Y_j|X_j}(y_j|x_j) \), \( j = 1, \ldots, n \).

Each iteration is divided into two phases.

- **Check node**: Consider a certain message sent to variable node \( x_j \) from a certain check equation. First we calculate the maximization over the messages sent from the other variable nodes that take place in this check equation, after expanding these messages. This maximization was described in (4.8)-(4.10). Then we calculate the maximization on the replications of \( P(t_m) \), after contracting it. This operation was described in (4.11).

- **Variable Node**: In the variable node we multiply the incoming check node messages with the variable node channel observation in a similar manner to (4.12). However, this time we exclude in the
multiplication the message from the check node for which we send the message.

- **Final Decision:** We multiply all the variable node incoming messages with its channel observation to get $\psi_j(x_j)$. Then we find $\hat{x}_j = \arg\max_{x_j} \psi_j(x_j)$ and take $\hat{b} = \lfloor Q \cdot \hat{x}_j \rfloor$

Note that in order to get the sum-product algorithm derived in [22], we only need to replace the maximization in (4.7) by an integral, i.e. we get convolution between the expanded messages (4.8)-(4.10), and we replace the maximization operation in (4.11) by summation.

### 4.3.2 The AWGN Channel

In this section we analyze the max-product algorithm for the AWGN channel. We prove that the maximization of the passed messages (4.7) still enables us to represent the passed messages with Gaussians, just like in the sum-product algorithm where the passed messages are Gaussian-mixtures. Moreover, we show that the Gaussians representing the passed messages in both algorithms are identical in each iteration. The only difference between the algorithms passed messages is how we process these Gaussians. In the sum-product algorithm we sum these Gaussians. In the max-product algorithm we take the maximal value in each point between these Gaussians.

We characterize each Gaussian by three parameters: amplitude $a \geq 0$, mean value $\mu$, and variance $v$.

We define the Gaussian function $N(x; \mu, v)$ as follows

$$a \cdot N(x; \mu, v) = \frac{a}{\sqrt{2\pi v}} \cdot e^{-\frac{(x-\mu)^2}{2v}}. \quad (4.13)$$

For the AWGN channel we get that $f_{Y_j|X_j}(y_j|x_j) = N(x_j; y_j, \sigma_j^2)$, $j = 1, \ldots, n$.

Next we prove several properties of Gaussians combined with multiplication and maximization operations. This properties enable us to prove that the passed messages can be represented via Gaussians, and also to prove the relation to the sum-product algorithm. The following lemma relates to the operations presented in (4.8)-(4.10).

**Lemma 4.1.** Assume $v_1 > 0$, $v_2 > 0$.

$$\sup_{x'} N(x'; \mu_1, v_1) \cdot N(x - x'; \mu_2, v_2)$$

$$= \sqrt{\frac{v_1 + v_2}{2\pi v_1 \cdot v_2}} N(x; \mu_1 + \mu_2, v_1 + v_2).$$

**Proof.** We need to minimize the exponent absolute value $\frac{(x - \mu_1)^2}{2v_1} + \frac{(x - x' - \mu_2)^2}{2v_2}$. Taking the first derivative according to $x'$ and finding $x'$ that zeros it gives us $x' = \frac{v_1 + v_2}{2v_1 + v_2} \cdot (\mu_1 - \mu_2)$. Assigning this value in the exponent gives us $\frac{(x - \mu_1 - \mu_2)^2}{2(v_1 + v_2)}$ that corresponds to $N(x; \mu_1 + \mu_2, v_1 + v_2)$. \qed

Interestingly, the expression received in Lemma 4.1 is identical to the convolution result between this Gaussians up to a coefficient that depends on the variance values. Now let us define an infinite set of
Gaussians \( \{ a_k \cdot N(x; \mu_k, v_k) \} \), \( k \geq 1 \) (for a finite set we can zero the irrelevant amplitudes). The maximal envelope of these Gaussians is defined as follows

\[
G_{Env}(x) \triangleq \sup_{k \geq 1} a_k \cdot N(x; \mu_k, v_k)
\]

(4.14)
i.e., for each point \( x \) we take the maximal value between the Gaussians. Next we prove a property of Gaussian-envelopes that relates to the multiplication in the variable node (4.12).

**Proposition 4.1.** Consider two Gaussian-envelopes \( \sup_{k \geq 1} a_k^{(1)} \cdot N(x; \mu_k^{(1)}, v_k^{(1)}) \) and \( \sup_{k' \geq 1} a_k^{(2)} \cdot N(x; \mu_k^{(2)}, v_k^{(2)}) \). In this case their multiplication yields

\[
\sup_{k \geq 1} a_k^{(1)} \cdot N(x; \mu_k^{(1)}, v_k^{(1)}) \cdot \sup_{k' \geq 1} a_k^{(2)} \cdot N(x; \mu_k^{(2)}, v_k^{(2)})
\]

\[
= \sup_{k \geq 1, k' \geq 1} a_k^{(1)} \cdot a_k^{(2)} \cdot N(x; \mu_k^{(1)}, v_k^{(1)}) \cdot N(x; \mu_k^{(2)}, v_k^{(2)}).
\]

which is also a Gaussian-envelope.

**Proof.** This proposition states that multiplying the Gaussians-envelopes is equivalent to first multiplying the Gaussians constituting these Gaussian-envelopes, and then taking the maximization over this multiplication. The proof is straight forward. The Gaussians and their coefficients are positive. Hence the maximal value must be the multiplication of the maximal values of each envelope. Considering the other multiplications does not affect the result. The result is a Gaussian-envelope since the multiplication of Gaussians yields a Gaussian [22]. Hence the maximization over the Gaussians multiplication yields a Gaussian-envelope.

Now we prove another proposition relating to (4.8)-(4.10).

**Proposition 4.2.** Consider two Gaussian-envelopes consisting of Gaussians with the same variance: \( G_{Env}^{(1)}(x) = \sup_{k \geq 1} a_k^{(1)} \cdot N(x; \mu_k^{(1)}, v^{(1)}) \) and \( G_{Env}^{(2)}(x) = \sup_{k' \geq 1} a_k^{(2)} \cdot N(x; \mu_k^{(2)}, v^{(2)}) \). Assume \( v^{(1)} \), \( v^{(2)} > 0 \), and that these functions are bounded. We get

\[
\sup_{x'} G_{Env}^{(1)}(x') \cdot G_{Env}^{(2)}(x - x') = \sqrt{\frac{v^{(1)} + v^{(2)}}{2\pi v^{(1)} \cdot v^{(2)}}}.
\]

\[
\sup_{k \geq 1, k' \geq 1} a_k^{(1)} \cdot a_k^{(2)} \cdot N(x; \mu_k^{(1)} + \mu_k^{(2)}, v_k^{(1)} + v_k^{(2)}).
\]

**Proof.** From Proposition 4.1 we can write the Gaussian-envelopes multiplication as \( \sup_{x'} \sup_{k \geq 1, k' \geq 1} a_k^{(1)} \cdot a_k^{(2)} \cdot N(x; \mu_k^{(1)}, v^{(1)}) \cdot N(x - x'; \mu_k^{(2)}, v^{(2)}) \). Since both functions are bounded we can reverse the maximization order, and then in the inner maximization we need to find for each \( k \geq 1 \) and \( k' \geq 1 \), the \( \sup_{x'} a_k^{(1)} \cdot a_k^{(2)} \cdot N(x; \mu_k^{(1)}, v^{(1)}) \cdot N(x - x'; \mu_k^{(2)}, v^{(2)}) \). From Lemma 4.1 we know that this maximization gives \( \sqrt{\frac{v^{(1)} + v^{(2)}}{2\pi v^{(1)} \cdot v^{(2)}}} \cdot a_k^{(1)} \cdot a_k^{(2)} \cdot N(x; \mu_k^{(1)} + \mu_k^{(2)}, v^{(1)} + v^{(2)}) \). The variance values do not depend on \( k, k' \), and so they can be taken out of the maximization.

68
Theorem 4.1. For each message in each iteration there exists a set of Gaussians (set of mean values, variance values and amplitudes) for which the maximal envelope of these Gaussians gives the max-product algorithm message, and the sum of these Gaussians gives the sum-product algorithm message.

Proof. We prove this theorem by induction. We begin with the initialization step. In this case the variable node messages consist of a Gaussian that equals to the channel observation, i.e. variable node $x_j$ sends $N(x_j; y_j, \sigma^2)$. This messages are bounded functions. In the check node we begin by analyzing the calculation of $P(t_m)$ presented in (4.8)-(4.10). We keep the notations used in these equations, and without loss of generality consider a check node message sent to $x_1$. In this case if we denote the mean values of the incoming messages to the check node by $\mu_k$, $k = 1, \ldots, m + 1$, and the variance values by $\sigma^2$, based on Lemma 4.1 we get that in the first iteration $P(t_m) \propto N(t_m; \sum_{k=1}^{m} q_k^k \cdot \mu_k, \sigma^2 \sum_{k=1}^{m} (q_k^k)^2)$. Note that since the incoming messages are bounded, $P(t_m)$ is also a bounded function. For the calculation of $c^{(i)}(x_1)$ in (4.11), first we take the set of Gaussians $P(b - q_i^{m+1}, x_1) \propto N(x_1; \frac{b - \sum_{k=1}^{m} q_i^k \cdot \mu_k}{q_i^{m+1}}, \frac{\sigma^2 \sum_{k=1}^{m} (q_i^k)^2}{(q_i^{m+1})^2})$, where $b \in \mathbb{Z}$. These Gaussians are all replications of the Gaussian $P(-q_i^{m+1}, x_1)$, and so they all have the same variance and the same coefficient. Hence we get that $c^{(i)}(x_1) \propto \sup_{b \in \mathbb{Z}} N(x_1; \frac{b - \sum_{k=1}^{m} q_i^k \cdot \mu_k}{q_i^{m+1}}, \frac{\sigma^2 \sum_{k=1}^{m} (q_i^k)^2}{(q_i^{m+1})^2})$ which is a Gaussian-envelope consisting of Gaussians with the same variance. Based on this operation we can see that $c^{(i)}(x_1)$ is also a bounded function. In the sum-product algorithm [22] the initialization is the same. This time in the first iteration in each check node we perform convolution between the incoming Gaussians, which gives the same result as $P(t_m)$ up to a factor. We perform the same replication of the Gaussian as in the max-product algorithm, only this time we sum the replications. Hence, the check node messages in the first iteration consist of the same Gaussians in both algorithms. Now we turn to analyze the max-product operations in the first iteration in the variable node. In this case we calculate $\psi_1(x_1)$ according to (4.12) by multiplying the incoming Gaussian-envelopes messages, with the channel observation. The incoming messages are Gaussian-envelopes consisting of Gaussians with the same variance. Based on Proposition 4.1, the multiplication is equivalent to first multiplying the Gaussians from the different messages, and then taking a Gaussian-envelope on the multiplication. Multiplying two Gaussians with variance values $v_a$ and $v_b$ gives a Gaussian with variance $(\frac{1}{v_a} + \frac{1}{v_b})^{-1}$ [22]. Hence, since the Gaussians in each message have the same variance, the Gaussians product also yields Gaussians that have the same variance. Since the incoming messages are bounded, their multiplication $\psi_1(x_1)$ is also bounded. For the sum-product algorithm this multiplication is performed between Gaussian-mixtures and yields the same Gaussians as in the max-product, only this time these Gaussians are summed.

Next, assume that in iterations $1, \ldots, I$ the check and variable nodes messages are Gaussian-envelopes, where the Gaussians in each message have the same variance. Also assume that these Gaussians are identical to the Gaussians in the sum-product messages, and that the messages are bounded. In the check node, since the incoming messages are bounded we can use Proposition 4.2. We get that $P(t_m)$ is a Gaussian-envelope consisting of Gaussians that equal to the convolution between the Gaussians of the incoming messages. Calculating $P(t_m)$ requires multiplications and maximization of bounded functions, hence $P(t_m)$ is bounded. In the sum-product algorithm, these Gaussians undergo convolution. Hence the Gaussians are identical in both algorithms. The calculation of $c^{(i)}(x_1)$ (4.11) is equivalent to replicating each Gaussian in $P(-q_i^{m+1}, x_1)$ and taking the Gaussian-envelope on these Gaussians. This operation keeps $c^{(i)}(x_1)$ bounded,

69
Figure 4.1: (a) This figure presents three Gaussians. The strongest Gaussian (dotted) mean value is 0. The other two Gaussians have mean value 1. (b) The sum-product algorithm message. This is a Gaussian mixture consisting of the three Gaussians sum. For this case the message maximal value is at 1. (c) The max-product algorithm message. This is the maximal Gaussian-envelope of the three Gaussians. In this case the message maximal value is at 0.

and also the replicated Gaussians variance values are identical. For the sum-product algorithm we take the same replications as in the max-product algorithm. Hence the Gaussians are identical between both algorithms. In the variable node the arguments are identical to the arguments given for the first iteration.

In [22] the authors formulated necessary conditions for convergence of the mean and variance values of the Gaussians in the passed messages. A partial convergence analysis of the amplitudes was also given. As the Gaussians in each iteration are identical to the sum-product Gaussians, the conditions and analysis also hold for the max-product algorithm.

The Gaussians in the passed messages are identical in both algorithms. Hence, the difference in each iteration comes from the processing each algorithm performs on these Gaussians to obtain the messages. In some cases, the different processing leads to different final decisions made by the algorithms. For instance it may occur when there is rather “tall” Gaussian concentrated around a certain point, and also several smaller Gaussians concentrated around another point, whose sum yields larger value than the tall Gaussian. In this case the decisions may be different. For illustration see Figure 4.1.

4.3.3 Extension of the Parametric Algorithm

In [32] a parametric algorithm for the sum-product case was presented. In this section we adapt this algorithm to the max-product case. We will briefly go over the parametric algorithm [32] and highlight the required changes to adapt it to the max-product case. The parametric approach uses the fact that the passed messages consist of Gaussians. In this approach the Gaussians in the passed messages are represented
by lists of their means, variance values and amplitudes. The operations in the check nodes and variable nodes can be done by calculating the Gaussians parameters. However, the number of Gaussians in each message is infinite. Hence, a key component of the parametric algorithm is to efficiently approximate the infinite parametric Gaussians list of each message, by a finite list of $J'$ Gaussians. The Gaussians in each message are consolidated in each iteration by first choosing the Gaussian with largest coefficient in the list, and consolidating it with Gaussians that fall within a certain range around its mean value. Assume $\{a_k, \mu_k, v_k, 1 \leq k \leq J\}$ are the Gaussians to be consolidated. In this case we approximate these Gaussians by a single Gaussian with mean $\hat{\mu}_k = \sum_{k=1}^J a'_k \mu_k$ and variance $\hat{v}_k = \sum_{k=1}^J a'_k v_k$, where $a'_k = \frac{a_k}{\sum_{k=1}^J a_k}$. In [32] the amplitude of this Gaussian equals $\hat{a} = \sum_{k=1}^J a_k$. In the max-product case we take $\hat{a} = \max_{1 \leq k \leq J} a_k$. This is the first difference between the algorithms. After consolidating these Gaussians, we erase them from the message, and find the message next Gaussian with largest coefficient. We repeat this process $J'$ times at most. The second difference between the algorithms is the amplitudes calculation in the check nodes. Lemma 4.1 proves that in the check node, both algorithms operations yield the same Gaussians up to a coefficient that depends on the variance. In the theoretical algorithm the variance values of the Gaussians in each message are identical. However the parametric approximation gives Gaussians with different variance values. Hence, when calculating the convolution between Gaussians pairs in the sum-product algorithm [32], the result Gaussian amplitude is $a_1 \cdot a_2$, where $a_1$ and $a_2$ are the Gaussians amplitudes, and for the max-product algorithm we take $\sqrt{\frac{v_1 + v_2}{v_1 \cdot v_2}} \cdot a_1 \cdot a_2$, where $v_1$ and $v_2$ are the Gaussians variance values. Besides that both algorithms are identical. The parametric algorithms complexity is the same. For typical lists length they yield an improvement of an order of magnitude in complexity and two orders of magnitudes in storage requirements, compared to straightforward implementation of the algorithms [32].

### 4.4 Max-Product Algorithm for MIMO Channels

In this section we derive an iterative max-product algorithm for CLDLC’s when considering a-prior probabilities in general MIMO channels. We analyze the case of Gaussian additive noise in the MIMO channel, together with Gaussian a-priori probabilities, and reveal an interesting connection between the passed messages and MMSE estimation. This algorithm is used to efficiently decode the CLDLC presented in Section 4.5, for the case $M = N = 2$.

#### 4.4.1 General Formulation

We would like to calculate the marginal

$$
\psi_j(x_j) = \max_{x \in \Lambda_C, \sim x_j} f_{Y|X}(y|x) \cdot f_X(x) \quad j = 1, \ldots, n
$$

(4.15)

where $\{x \in \Lambda_C, \sim x_j\}$ is the set of lattice points in which the $j$’th component equals $x_j$ and $f_{Y|X}(y|x)$ is the probability density function (PDF) of receiving the observation $y$ given that $x$ was transmitted. $f_X(x) = \prod_{k=1}^n f_X(x_k)$ is a PDF which is proportional to $\frac{\prod_{k=1}^n f_X(x_k)}{\sum_{x \in \Lambda_C} \prod_{k=1}^n f_X(x_k)}$, the a-priori probability of transmitting
a lattice point $x \in \Lambda_C$. Note that if we take for each function in (4.15) the argument $x_j, j = 1, \ldots, n$ that maximizes it we get the MAP decision for a lattice $\Lambda_C$.

Assume that the channel PDF can be broken into the multiplications
\[
f_{Y|X}(y|x) = \prod_{s=1}^{S} f_{Y_s|X_s}(y_s|x_s)
\]
where
\[
y_s = H_s \cdot x_s + w_s \quad 1 \leq s \leq S.
\]
$H_s \in \mathbb{C}^{N_s \times M_s}$ is an $N_s \times M_s$ matrix, where the number of columns equals to the number of elements in $x_s$. The vectors $x_s, 1 \leq s \leq S$, are disjoint subsets of $\underline{x}$ and their union equals $\underline{x}$. We can factorize the problem and rewrite (4.15) as
\[
\psi_j(x_j) = \max_{\underline{x} \in \mathbb{C}^n, \sim x_j} \prod_{s=1}^{S} f_{Y_s|X_s}(y_s|x_s) \cdot \prod_{k=1}^{n} \mathbb{1}(q_k \cdot \underline{x} \in \mathbb{Z}[i]) \cdot \prod_{k'=1}^{n} f_X(x_{k'})
\]
where $q_k$ is the $k$’th row of $Q$, and the indication function $\mathbb{1}(q_k \cdot \underline{x} \in \mathbb{Z}[i])$ equals 1 if $q_k \cdot \underline{x}$ equals a Gaussian integer and zero else. The multiplication $\prod_{k=1}^{n} \mathbb{1}(q_k \cdot \underline{x} \in \mathbb{Z}[i])$ in (4.18) takes into account all the lattice check equations. Since $Q \cdot \underline{x} \in \mathbb{Z}[i]^n$ if and only if $\underline{x} \in \Lambda_C$, also $\prod_{k=1}^{n} \mathbb{1}(q_k \cdot \underline{x} \in \mathbb{Z}[i])$ equals 1 if and only if $\underline{x} \in \Lambda_C$.

Without loss of generality we consider the marginalization of $x_1$, $\psi_1(x_1)$, and assume that $\underline{x}_1 = (x_1, x_2, \ldots, x_{K'})^T$, i.e. in the MIMO channel $x_1$ takes place with the first $K'$ symbols.

In order to derive the max-product algorithm for the MIMO channel we make an extended tree assumption. This assumption states that the tree resulting from the check equations, with a certain element from $\underline{x}_s$ as its root, does not have leaves from $\underline{x}_s$. For instance the tree resulting from the check equations with $x_1$ as its root, does not have leaves from $x_2, \ldots, x_{K'}$. An example for the extended tree assumption for $\underline{x}_1 = (x_1, x_2)^T$ is given in Figure 4.2.

Figure 4.2: The extended tree structure. The solid lines connect variables that take place in the same check equation. The dotted lines connect variables that take place in the same channel. For instance $x_1$ and $x_2$ are connected via a dotted line because they are connected via $H_1$. 

72
Under the extended tree assumption we can factorize (4.18) and write

$$\psi_1(x_1) = \max_{x_2,\ldots,x_{k'}} f_{x_1|x_2,\ldots,x_{k'}} f_{x_1} \cdot \prod_{k=1}^{K'} f_{X}(x_k) \cdot \tilde{c}_k(x_k)$$  \hspace{1cm} (4.19)$$

where

$$\tilde{c}_k(x_k) = \max_{z^{(1)},\ldots,z^{(i)}} \prod_{m=1}^{l} c_m(z^{(m)}) \cdot \mathbb{1}(q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]})$$  \hspace{1cm} (4.20)$$

and we assume that $x_k$ takes place in the first $l$ check equations of $Q$, where $x^{(m)}$ are the variables that take place in the $m$th check equation with $x_k$. Note that under the tree assumption $x_k$ appears in (4.20) only in $\mathbb{1}(q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]})$, $1 \leq m \leq l$. From the tree assumption we get that the elements in $x^{(m)}$, $1 \leq m \leq l$ are different and so we can write

$$\tilde{c}_k(x_k) = \prod_{m=1}^{l} \max_{z^{(m)}} c_m(z^{(m)}) \cdot \mathbb{1}(q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]}).$$  \hspace{1cm} (4.21)$$

Assuming $q_m$ has $l'+1$ non-zero values we can write $q_m \cdot z^{(m)} = \sum_{s=1}^{l'} q_m^{s} \cdot x^{(m),s} + q_m^{l'+1} \cdot x_k$, where $q_m^{s}$, $s = 1, \ldots, l' + 1$ are the non-zero entries of $q_m^s$, and $x^{(m),s}$, $s = 1, \ldots, l'$, are the symbols in $x^{(m)}$. In this case $c_m(x^{(m)})$ can be factorized to the following multiplication

$$c_m(x^{(m)}) = \prod_{s=1}^{l'} v_{x^{(m)},s}(t_s)$$  \hspace{1cm} (4.22)$$

where $v_{x^{(m)},s}(t_s)$ is the marginalization of $x^{(m),s}$ (4.19), done over its subtree excluding the edges with the other variables that take place in the $m$th check equation. The solution to the optimization problem in (4.21) $\max_{z^{(m)}} c_m(z^{(m)}) \cdot \mathbb{1}(q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]})$ is obtained when $q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]}$. Hence, by calculating

$$P(t') = v_{x^{(m),1}} \left( \frac{t_1}{q_m^{1}} \right) \odot \cdots \odot v_{x^{(m),l'}} \left( \frac{t_l'}{q_m^{l'}} \right)$$  \hspace{1cm} (4.23)$$

and assigning $t'_k = b - q_m^{l'+1} x_k$, where $b \in \mathbb{Z}^{[i]}$, we get that each element in the multiplication in (4.21) can be written as

$$\max_{z^{(m)}} c_m(z^{(m)}) \cdot \mathbb{1}(q_m \cdot z^{(m)} \in \mathbb{Z}^{[i]}) = \max_{b \in \mathbb{Z}^{[i]}} P(b - q_m^{l'+1} \cdot x_k)$$

where the operation $\odot$ is defined in (4.2).

In order to obtain the marginalization, the aforementioned operations can be performed iteratively over a bipartite graph. This graph is divided into three groups: check nodes, variable nodes and channel nodes. The variable nodes represent the $n$ symbols in $x$, each check node represents a parity check equation in $Q$, and each channel node represents a channel in (4.17). An edge from a check node to variable node is stretched if the corresponding entry of $Q$ does not equal zero. An edge from a variable node to the channel
node representing $H_s$ (4.17) is stretched in case the symbol is in $x_s$, $1 \leq s \leq S$. An example to such a bipartite graph is given in Figure 4.3.

The iterative decoding is divided into several phases. Without loss of generality we describe the messages to variable node $x_1$, and the messages from this variable node.

• **Initialization**: The initialization is performed on the variable to check node messages. The message of $x_1$ is initialized to

$$\max_{x_2,\ldots,x_{K'}} f_{Y_1|X_1} (y_1|x_1) \prod_{s=1}^{K'} f_X (x_s).$$

In each iteration the algorithm is divided into the following steps.

• **Check to variable node messages**: Let us consider the message sent to $x_1$. First a conv-sup is performed between the messages sent from the *other* variable nodes that take place in the check equation, after expanding and rotating this messages. This operation was described in (4.23). Then, we calculate

$$\max_{b \in \mathbb{Z}_q} P \left( b - q^{\ell+1} \cdot x_1 \right)$$

to obtain the message.

• **Variable to channel node message**: Multiply the incoming check node messages and send them to the channel node.

• **Channel to variable node message**: Consider the channel message to variable node $x_1$. The variable to channel node messages from the other variable nodes are $\tilde{c}_k (x_k)$, $k = 2, \ldots, K'$. Therefore, the channel to variable node message is

$$\max_{x_2,\ldots,x_{K'}} f_{Y_1|X_1} (y_1|x_1) f_X (x_1) \cdot \prod_{k=2}^{K'} f_X (x_k) \cdot \tilde{c}_k (x_k)$$

(4.24)

• **Variable to check node message**: Multiply the incoming channel node message with the incoming check node messages. However, we exclude in the multiplication the message from the check node for which we send the message.

• **Final Decision**: Multiply the incoming channel node message with *all* incoming check node messages to get $\psi_j (x_j)$. Then find $\hat{x}_j = \arg\max_{x_j} \psi_j (x_j)$ and take $\hat{b} = \lfloor Q \cdot \hat{x}_j \rfloor$. 

74
Note that the sum-product algorithm for CLDLC in MIMO channels is obtained by simply replacing the conv-sup in (4.23) with a convolution, replacing the maximization in (4.19) with an integral, and also replacing \( \max_{b \in \mathbb{Z}[i]} P \left( b - d_{m}^{l+1} \cdot x_{1} \right) \) with \( \sum_{b \in \mathbb{Z}[i]} P \left( b - d_{m}^{l+1} \cdot x_{1} \right) \).

### 4.4.2 Gaussian Prior Assumption

In this section we analyze the max-product algorithm for MIMO channels with additive Gaussian noise, i.e. in the model in (4.17) we assume that all entries in \( w_{s} \), \( s = 1, \ldots, S \) are independent with complex normal distribution \( \mathcal{CN} \left( 0, 2\sigma^{2} \right) \). We also assume that the a-priori probabilities of the symbols are independent with distribution \( \mathcal{CN} \left( 0, 2P_{0} \right) \), where \( P_{0} \) is the average power of the codebook, i.e.

\[
\begin{align*}
   f_{X} (x_{k}) &= \mathcal{CN} (x_{k}; 0, 2P_{0}) \quad k = 1, \ldots, n \tag{4.25}
\end{align*}
\]

where

\[
   a \cdot \mathcal{CN} (x; \mu, 2v) = \frac{a}{2\pi v} e^{-\frac{|x-\mu|^{2}}{2v}} \tag{4.26}
\]

is a complex Gaussian with mean value \( \mu \in \mathbb{C} \), variance \( 2v \) and amplitude \( a \).

Under these assumptions we show that the passed messages can be represented by Gaussians. In addition, we show an interesting connection between the channel node messages (4.24) and MMSE estimation. Keeping the notations in (4.24) we show that the channel node message to \( x_{1} \) can be represented by Gaussians, where the mean value of each Gaussian is the MMSE estimation of \( x_{1} \) given the observation \( y_{1} \), and the variance value is the MMSE of the symbol. Each Gaussian in the channel node message is created by Gaussians in the incoming messages \( f_{X} (x_{k}) \cdot \tilde{c}_{k} (x_{k}) \), where the a-priori probabilities for calculating the MMSE estimation correspond to these Gaussians.

We begin by presenting several properties of complex normal Gaussians when performing maximization and conv-sup operation (4.2). These properties were proved in Subsection 4.3.2 for real Gaussians. Since the extension to complex normal Gaussians is straightforward we only present these properties. The first property relates to conv-sup operation between two complex normal Gaussians \( \mathcal{CN} (x; \mu_{1}, 2v_{1}) \) and \( \mathcal{CN} (x; \mu_{2}, 2v_{2}) \), where both \( v_{1} > 0 \) and \( v_{2} > 0 \). In this case it was shown in Lemma 4.1 that

\[
   \mathcal{CN} (x; \mu_{1}, 2v_{1}) \odot \mathcal{CN} (x; \mu_{2}, 2v_{2}) = \frac{v_{1} + v_{2}}{2\pi \cdot v_{1} \cdot v_{2}} \mathcal{CN} (x; \mu_{1} + \mu_{2}, 2 \cdot (v_{1} + v_{2})) \tag{4.27}
\]

Note that (4.27) equals to the convolution between two complex normal Gaussians up to a coefficient that depends on the variance values. Now let us define a set of complex normal Gaussians \( \{a_{m} \cdot \mathcal{CN} (x; \mu_{m}, 2v_{m})\} \), where \( a_{m} \geq 0 \) and \( m \geq 1 \). The Gaussian envelope of this set is defined as

\[
   CG_{Env} (x) \triangleq \sup_{m \geq 1} a_{m} \cdot \mathcal{CN} (x; \mu_{m}, 2v_{m}) \tag{4.28}
\]

Note that replacing the supremum with a sum yields a Gaussian mixture. Let us define two Gaussian envelopes \( CG_{Env}^{(1)} (x) = \sup_{m \geq 1} a_{m}^{(1)} \cdot \mathcal{CN} (x; \mu_{m}^{(1)}, 2v^{(1)}) \) and \( CG_{Env}^{(2)} (x) = \sup_{m \geq 1} a_{m}^{(2)} \cdot \mathcal{CN} (x; \mu_{m}^{(2)}, 2v^{(2)}) \),
where each Gaussian envelope consists of Gaussians with the same variance value. In this case it was shown in Proposition 4.1 that

\[
CG^{(1)}_{Env}(x) \cdot CG^{(2)}_{Env}(x) = \sup_{m \geq 1, m' \geq 1} a^{(1)}_m \cdot a^{(2)}_{m'} \cdot \text{CN}(x; \mu^{(1)}_m, 2\nu^{(1)}) \cdot \text{CN}(x; \mu^{(2)}_{m'}, 2\nu^{(2)})
\]  

(4.29)

which is also a Gaussian envelope. Basically (4.29) states that multiplying Gaussian envelopes is equivalent to first multiplying the Gaussians constituting these Gaussian envelopes, where multiplication of Gaussians yields a Gaussian \[22\], and then taking the maximization over the new set of Gaussians. The third property that was proven in Proposition 4.2 states that when \(CG^{(1)}_{Env}(x), CG^{(2)}_{Env}(x)\) are bounded, and \(\nu^{(1)} > 0, \nu^{(2)} > 0\) we get

\[
CG^{(1)}_{Env}(x) \odot CG^{(2)}_{Env}(x) = \nu^{(1)} + \nu^{(2)} - \frac{1}{2\pi^{(1)} \cdot \nu^{(2)}} \cdot a^{(1)}_m \cdot a^{(2)}_{m'} \cdot \text{CN}(x; \mu^{(1)}_m + \mu^{(2)}_{m'}, 2\left(\nu^{(1)} + \nu^{(2)}\right))
\]  

(4.30)

This means that conv-sup between bounded Gaussian envelopes is equivalent to first calculating the conv-sup between all Gaussian pairs of the two Gaussian envelopes, and then taking the maximization over the new set of Gaussians.

In order to show the connection between the channel node message and the MMSE estimation we first prove a lemma that relates to the maximization in (4.24). For simplicity we keep the notations in (4.24), and assume that for \(H_1\) in (4.17) the number of rows \(N_1\) is greater or equal to the number of columns \(M_1\).

**Lemma 4.2.** For \(\nu_k > 0, k = 1, \ldots, K'\), we get

\[
\max_{x_2, \ldots, x_{K'}} f_{Y_1|X_1} \left( y_1 | x_1 \right) \prod_{k=1}^{K'} \text{CN}(x_k; \mu_k, 2\nu_k) = \frac{\nu_{x_1|y_1}}{(2\pi)^{K' - 1} \cdot |C_{x_1|y_1}|} \cdot f_{Y_1} \left( y_1 \right) \cdot \text{CN}(x_1; \mu_{x_1|y_1}, \nu_{x_1|y_1})
\]

where \(\mu_{x_1|y_1}\) is the MMSE estimation of the symbol \(x_1\) given the observation \(y_1\), \(\nu_{x_1|y_1}\) is the MMSE of this estimation, and \(C_{x_1|y_1}\) is the covariance matrix of the vector \(x_1\) given the observation, assuming the elements in \(x_1\) are independent and that \(x_k \sim \text{CN}(\mu_k, 2\nu_k), k = 1, \ldots, K'\).

**Proof.** We can rewrite the maximization problem as

\[
f_{Y_1} \left( y_1 \right) \cdot \max_{x_2, \ldots, x_{K'}} f_{X_1|X_1} \left( x_1 | y_1 \right)
\]

(4.31)

assuming \(f_{X_1} \left( x_1 \right) = \prod_{k=1}^{K'} \text{CN}(x_k; \mu_k, 2\nu_k)\). Since \(x_1, y_1\) are joint Gaussians, by “completing the square”
we can write
\[
\frac{f_{X_1 | \mathbf{Y}_n} (\mathbf{x}_1 | \mathbf{y}_1)}{\prod_{k=1}^{K'} \mathbb{C}^{(k)} (x_k)} = \mathbb{C} \left( x_1; \mu_{x_1 | y_1}, v_{x_1 | y_1} \right),
\]
\[
\frac{v_{x_1 | y_1}}{(2\pi)^{K' - 1} | C_{x_1 | y_1}|} e^{-\frac{1}{2}(\mathbf{x}_1 - \mathbf{\mu}_{x_1 | y_1})^\top A(\mathbf{x}_1 - \mathbf{\mu}_{x_1 | y_1})}
\]
(4.32)
where \( \mathbf{x}_1 = (x_2, \ldots, x_{K'})^\top \) and the mean value \( \mathbf{\mu}_{x_1 | y_1} = (\mathbf{\mu}_2 \cdot x_1 + c_2, \ldots, \mathbf{\mu}_{K'} \cdot x_1 + c_{K'})^\top \), i.e. linear combinations of \( x_1 \). In this case we get for any value of \( x_1 \)
\[
\max_{x_2, \ldots, x_{K'}} e^{-\frac{1}{2}(\mathbf{x} - \mathbf{\mu}_{x_1 | y_1})^\top A(\mathbf{x} - \mathbf{\mu}_{x_1 | y_1})} = 1
\]
and by assigning (4.33), (4.32) in (4.31) we get the proof. 

Note that by replacing the maximization with an integral we get the same result, only without the factor
of \( (2\pi)^{K' - 1} | C_{x_1 | y_1}| \).

As we will show the passed messages in the iterative decoding algorithm are Gaussian envelopes. Hence, we prove the following lemma on the channel node messages in (4.24) when the incoming messages are Gaussian envelopes.

**Lemma 4.3.** Consider \( K' \) bounded Gaussian envelopes consisting of Gaussians with the same variance values: \( \mathbb{C}^{(k)} (x_k) = \sup_{m_k \geq 1} \mathbb{C}^{(k)} (x_k; \mu_{x_1 | y_1}^{(k)}, 2 \cdot v_1^{(k)}) \), \( k = 1, \ldots, K' \). In this case

\[
\max_{x_2, \ldots, x_{K'}} f_{X_1 | \mathbf{Y}_n} (\mathbf{x}_1 | \mathbf{y}_1) = \prod_{k=1}^{K'} \mathbb{C}^{(k)} (x_k)
\]
\[
\cdot \mathbb{C} \left( x_1; \mu_{x_1 | y_1}^{(m)}, v_{x_1 | y_1}^{(m)} \right) \cdot \prod_{k=1}^{K'} \mathbb{C}^{(k)} (x_k)
\]
\[
\cdot \frac{v_{x_1 | y_1}}{(2\pi)^{K' - 1} | C_{x_1 | y_1}|} e^{-\frac{1}{2}(\mathbf{x}_1 - \mathbf{\mu}_{x_1 | y_1}^{(m)})^\top C_{x_1 | y_1}^{-1} (\mathbf{x}_1 - \mathbf{\mu}_{x_1 | y_1}^{(m)})}
\]
which is also a Gaussian envelope. \( \mathbf{m} = (m_1, \ldots, m_{K'}) \), \( \mu_{x_1 | y_1}^{(m)} \) is the MMSE estimation of the symbol \( x_1 \) given the observation \( y_1 \), assuming that the symbols in \( \mathbf{e}_1 \) are independent where \( x_k \sim \mathbb{C} \left( \mu_{x_1 | y_1}^{(k)}, 2 \cdot v_1^{(k)} \right) \),

and \( f_{X_1 | Y_1}^{(m)} (y_1) = \frac{1}{(2\pi)^{K' | C_{y_1}} | C_{y_1}|} e^{-\frac{1}{2}(y_1 - H \mu^{(m)})^\top C_{y_1}^{-1} (y_1 - H \mu^{(m)})} \), where \( \mu^{(m)} = \left( \mu_{m_1}, \ldots, \mu_{m_{K'}} \right)^\top \) and \( C_{y_1} \) is the covariance matrix of \( y_1 \).

**Proof.** Since the Gaussian envelopes are bounded we can switch the order of the maximization and the supremum. Then we get for each \( m \geq 1 \)
\[
\max_{x_2, \ldots, x_{K'}} f_{X_1 | \mathbf{Y}_n} (\mathbf{x}_1 | \mathbf{y}_1) \prod_{k=1}^{K'} \mathbb{C} \left( x_k; \mu_{x_1 | y_1}^{(k)}, 2 \cdot v_1^{(k)} \right).
\]
From Lemma 4.2 we know that the solution to this maximization problem is

\[
\frac{v_{x_1|y_1}}{(2\pi)^{K'-1}|C_{x_1|y_1}|} \prod_{k=1}^{K'} a^{(k)}_{m_k} \cdot f^{(m)}_{y_1}(x_1) \cdot CN\left(x_1; \mu^{(m)}_{x_1|y_1}, v_{x_1|y_1}\right)
\]

and so by taking the supremum over all Gaussians in the case \(m \geq 1\) we prove the lemma.

From Lemma 4.3 we get that for each value of \(m\), the amplitude of the resultant Gaussian is affected by the term \(f^{(m)}_{y_1}(x_1)\) that depends on the mean values of the incoming Gaussians. On the other hand, since the variance values of the Gaussians in each Gaussian envelope are identical, we get that the terms \(v_{x_1|y_1}\) and \(\frac{v_{x_1|y_1}}{(2\pi)^{K'-1}|C_{x_1|y_1}|}\) are independent of \(m\).

Now we can describe the iterative decoding algorithm. First, as in Subsection 4.3.2 the passed messages are bounded Gaussian envelopes. Each variable to check node message is initialized to a bounded Gaussian envelope. In the check nodes the conv-sup (4.23) between bounded Gaussian envelopes yields a bounded Gaussian envelope (4.30), and so does the operation

\[
\max_{b \in \mathbb{Z}[i]} P \left( b - \frac{q^{(m)}_{m+1} \cdot x_k}{v_{x_1|y_1}} \right).
\]

In the variable to channel node and the variable to check node messages the multiplication of bounded Gaussian envelopes also yields a bounded Gaussian envelope (4.29). Finally, in the channel to variable node messages the maximization (4.24) on the incoming bounded Gaussian envelopes also yields a bounded Gaussian envelope as shown in Lemma 4.3.

Each Gaussian envelope in the passed messages can be represented by a set of Gaussians. Each Gaussian is obtained by performing the following operations on a certain set of Gaussians from different incoming messages, \(a_k \cdot CN\left(x_k; \mu_k, 2v_k\right), k = 1, \ldots, K'.\)

- **Check to variable node messages:** Consider the aforementioned \(K'\) Gaussians from \(K'\) incoming messages. Also assume that the message is sent to symbol with coefficient \(q_{K'+1}\) in the check equation. In this case the Gaussian in the check to variable node message, corresponding to these Gaussians, has variance value \(v = \frac{\sum_{k=1}^{K'} |q_k|^2 v_k}{|q_{K'+1}|^2} \), mean value \(\mu = \frac{b - \sum_{k=1}^{K'} q_k \mu_k}{q_{K'+1}}, b \in \mathbb{Z}[i]\), and amplitude \(a = \prod_{k=1}^{K'} a_k\).

- **Variable to channel and variable to check node message:** The multiplication of \(K'\) incoming Gaussians, yields a Gaussian with variance value \(v = \left(\sum_{k=1}^{K'} \frac{1}{v_k}\right)^{-1}\), mean value \(\mu = \frac{\sum_{k=1}^{K'} \mu_k}{\sum_{k=1}^{K'} v_k} \), and amplitude \(a = \prod_{k=1}^{K'} a_k \cdot e^{-\sum_{k=1}^{K'} \frac{|\mu_k - \mu|^2}{2v_k}}\).

- **Channel to check node message:** Keeping the notations from (4.24) we get that \(K'\) Gaussians from \(K'\) incoming messages, yield in the message sent to \(x_1\) a Gaussian with variance \(v_{x_1|y_1}\), mean value \(\mu_{x_1|y_1}\), and amplitude \(a = \prod_{k=1}^{K'} a_k \cdot e^{-\frac{1}{2}(z_1 - H_1\mu)^T C_{x_1}^{-1} (z_1 - H_1\mu)}\), where \(\mu = (\mu_1, \ldots, \mu_{K'})^T\). Note
that from (4.24) we get that the message sent to \( x_1 \) always takes as a-priori probability \( CN(x;0,2P_0) \), i.e. \( \mu_1 = 0, v_1 = P_0 \) and \( a_1 = 1 \).

In the sum-product algorithm the passed messages are Gaussian mixtures. In addition, for the sum-product algorithm an integral is performed in the channel node instead of maximization, and convolution is performed in the check node instead of conv-sup. The similarity between conv-sup and convolution, and also between maximization and integration in the channel node, leads us to the conclusion that just like in Subsection 4.3.2, the messages in the max-product algorithm in each iteration can be represented by the same set of Gaussians required to represent the messages in the sum-product algorithm in each iteration. The only difference is that the passed messages in the max-product algorithm are obtained by performing maximization over these Gaussians, where the messages in the sum-product algorithm are obtained by summing these Gaussians.

4.5 CLDLC for the 2 \( \times \) 2 MIMO Fading Channel

In this section we present CLDLC for MIMO fading channel with \( M = N = 2 \). We begin by presenting a construction of CLDLC with a very sparse parity check matrix. Then we present a transmission scheme for the MIMO fading channel that enables to decode the CLDLC efficiently and attain good performance.

4.5.1 Construction of the CLDLC

We construct an \( n \)-complex dimensional CLDLC with parity check matrix which is a Latin square with degree two, i.e. each row and column of \( Q \) consists of the same two non-zero elements up to permutation and phase. However, we put another constraint on the position of these non-zero values. Assuming \( n \) is even and \( |q_1| > |q_2| > 0 \) we state that \( Q = \begin{pmatrix} I (|q_1|) & \Pi_1 (|q_2|) \\ \Pi_2 (|q_2|) & I (|q_1|) \end{pmatrix} \), where \( I (|q_1|) \) is an \( \frac{n}{2} \times \frac{n}{2} \) diagonal matrix where each diagonal element equals \( |q_1| \), and \( \Pi_1 (|q_2|) \), \( \Pi_2 (|q_2|) \) are permutation matrices, where the absolute value of each non-zero element in these matrices equals \( |q_2| \). An example for the case \( n = 4 \) is

\[
Q = \begin{pmatrix}
1 & 0 & |q_2|e^{j\frac{\pi}{4}} & 0 \\
0 & 1 & 0 & |q_2|e^{-j\frac{\pi}{4}} \\
0 & |q_2|e^{j\frac{3\pi}{4}} & 1 & 0 \\
|q_2|e^{-j\frac{3\pi}{4}} & 0 & 0 & 1
\end{pmatrix},
\]

(4.34)

4.5.2 The Transmission Scheme

Based on the constraint \( Q \cdot x = \mathbb{Z}[i]^n \) if and only if \( x \in \Lambda_C \), the sparseness of \( Q \) can be exploited to attain an efficient decoding algorithm for CLDLC. Note that the \( i \)'th column of \( Q \) corresponds to \( x_1 \). Let us consider as an illustrative example the case of \( n = 4 \), and \( Q \) presented in (4.34). In this case if we know \( x_3 \) and do not know \( x_1 \), based on the first equation of \( Q \) we get \( x_1 = b - |q_2|e^{j\frac{\pi}{4}} \cdot x_3 \), where \( b \in \mathbb{Z}[i] \), i.e. we know \( x_1 \) up to a Gaussian integer. The same thing applies to the fourth equation of \( Q \) in (4.34), when we know
The conjunction of these equations assists in finding \( x_1 \). In more practical scenarios, we may have very good observations of \( x_3 \) and \( x_4 \), which leads in each equation to a good knowledge of \( x_1 \) up to a Gaussian integer. The conjunction of the information on \( x_1 \) from the two equations, together with the observation of \( x_1 \), enables to decode this symbol.

In order to be able to find \( x_1 \) when its observation is not good, via the lattice check equations, we would like to create a certain “irregularity”. Therefore, we would like to use a transmission scheme that protects some of the symbols more than the others, where the protected symbols always take place in the check equations with less protected symbols. The symbols are more protected in the sense that the probability that their observation is not good, is smaller than the probability that the observation of the unprotected symbols is not good. For instance, in our example the irregularity needs to ensure that the observations of \( x_3 \) and \( x_4 \) are better (with larger probability) than the observations of \( x_1 \) and \( x_2 \). In general, we would like to ensure that the observations of symbols \( x_{\frac{n}{2}+1}, \ldots, x_n \), are more probable to be better than the observations of symbols \( x_1, \ldots, x_{\frac{n}{2}} \).

In this subsection we reduce the dimensionality of the transmission scheme to \( \frac{4}{3} \) average number of dimensions per channel use, in order to increase the probability that the observations of symbols \( x_{\frac{n}{2}+1}, \ldots, x_n \) are good, i.e. half of the transmitted symbols are more protected than the other half. Then we use the sparse structure of \( Q \) in order to decode the codewords very efficiently based on these observations, via the parity check equations. For the example of \( n = 4 \) and the parity check matrix in (4.34), the proposed transmission scheme is

\[
\begin{pmatrix}
    x_1 & x_3 & -\frac{x_1}{\sqrt{2}} \\
    x_2 & x_4 & \frac{x_2}{\sqrt{2}} \\
\end{pmatrix}
\]

which leads at the receiver to effective channel of \( H \) induced on symbols \( x_1, x_2 \), and effective channel of \( \sqrt{\sum_{k=1}^{n/2} \sum_{j=1}^{n/2} |h_{k,j}|^2} \) for symbols \( x_3, x_4 \), i.e. after performing at the receiver the same manipulations as in [1], on symbols \( x_3, x_4 \), we get

\[
\begin{pmatrix}
    y_1 \\
    y_2
\end{pmatrix} = H \cdot \begin{pmatrix}
    x_1 \\
    x_2
\end{pmatrix} + \begin{pmatrix}
    w_1 \\
    w_2
\end{pmatrix}
\]

and

\[
\begin{pmatrix}
    \tilde{y}_3 \\
    \tilde{y}_4
\end{pmatrix} = \sqrt{\sum_{k=1}^{n/2} \sum_{j=1}^{n/2} |h_{k,j}|^2 / 2} \cdot \begin{pmatrix}
    x_3 \\
    x_4
\end{pmatrix} + \begin{pmatrix}
    \tilde{w}_3 \\
    \tilde{w}_4
\end{pmatrix}
\]

where \( w_1, w_2, \tilde{w}_3, \tilde{w}_4 \) are independent with distribution \( \mathcal{CN} (0, 2\sigma^2) \). In general when \( n \) is a multiple of four, the transmission scheme for tuples of three channel uses is

\[
\begin{pmatrix}
    x_{2j-1} & \frac{x_{2j-1} + x_{2j-1}}{\sqrt{2}} & -\frac{x_{2j-1}}{\sqrt{2}} \\
    x_{2j} & \frac{x_{2j} + x_{2j}}{\sqrt{2}} & \frac{x_{2j}}{\sqrt{2}}
\end{pmatrix},
\]

\( j = 1, \ldots, \frac{n}{4} \).

While the Alamouti transmission scheme [1] uses 1 average number of dimensions per channel use, the transmission scheme we propose uses \( \frac{4}{3} \) average number of dimensions per channel use, i.e. uses in average \( \frac{1}{3} \) dimensions more than the Alamouti scheme. As we will show, using CLDLC with this transmissions
scheme enables to outperform the performance attained by the Alamouti scheme, while maintaining very low computational complexity.

### 4.5.3 Efficient Parametric Decoder

In this section we present a very efficient algorithm for decoding CLDLC’s in a MIMO channel when \( M = N = 2 \). We consider CLDLC with degree two, using the transmission scheme presented in Subsection 4.5.2. In addition, we use parametric implementation of the algorithm presented in Section 4.4, using only one iteration.

In the parametric algorithm we represent each passed message by a list of Gaussians. Each Gaussian in the list can be represented by three parameters: mean value, variance and amplitude, as defined in (4.26). The parameters of each Gaussian in the list are calculated according to the description at the end of Subsection 4.4.2. However, while each passed message consists of infinite number of Gaussians, the parametric algorithm represents each message by a finite list of Gaussians. Since the CLDLC degree is two, the parametric algorithm can be implemented more efficiently than the parametric algorithms presented in [33], [32] and in Subsection 4.3.3.

In accordance with the transmission scheme in Subsection 4.5.2 we can partition the symbols into two groups. The protected symbols \( x_{\frac{n}{2}+1}, \ldots, x_n \), and the less protected symbols \( x_1, \ldots, x_{\frac{n}{2}} \). We describe the parametric algorithm for each of these groups separately, since they have different effective channels and therefore different number of edges in the channel nodes.

We begin by describing the parametric algorithm for symbols \( x_{\frac{n}{2}+1}, \ldots, x_n \). In this case the effective channel of each symbol is\[
\sqrt{\sum_{k=1}^{\frac{n}{2}} \sum_{j=1}^{\frac{n}{2}} |h_{j,k}|^2} x_{\frac{n}{2}+m} + \bar{w}_{\frac{n}{2}+m}, m = 1, \ldots, \frac{n}{2}.
\]

The messages from the variable nodes of these symbols are \( \text{CN} (x_{\frac{n}{2}+m}, \mu_{x_{\frac{n}{2}}+m} | y_{\frac{n}{2}+m}, v_{x_{\frac{n}{2}}+m}) \), where the a-priori PDF for the MMSE estimation is \( \text{CN} (x_{\frac{n}{2}+m}; 0, 2P_0) \). Since these symbols are already protected, in the final decision we estimate these symbols by \( \hat{x}_{\frac{n}{2}+m} = \mu_{x_{\frac{n}{2}}+m} | y_{\frac{n}{2}+m} \) and we do not use information from symbols \( x_1, \ldots, x_{\frac{n}{2}} \).

For symbols \( x_1, \ldots, x_{\frac{n}{2}} \) we get at the receiver \( H \cdot \left( \begin{array}{c} x_{2j-1} \\ x_{2j} \end{array} \right) + \left( \begin{array}{c} u_{2j-1} \\ u_{2j} \end{array} \right), j = 1, \ldots, \frac{n}{4} \). Hence, in this case each channel node has two edges. For simplicity let us consider the messages sent from symbol \( x_1 \), and the messages sent to this symbol.

- **Initialization:** Since the degree is only two and the protected symbols do not use information from the less protected symbols there is no need for initialization.

- **Check to variable node message:** Since the degree is two and we perform only one iteration there is no need to perform conv-sup. The initialized Gaussian from the protected symbol that takes place in the check equation with \( x_1 \), is simply sent to \( x_1 \) after streching/expanding according to the corresponding check equation coefficient. This does not require any significant operation.

- **Variable to channel node message:** Since the algorithm performs only a single iteration each incoming message consists of a single Gaussian. Based on \( \max_{b \in \mathbb{Z}[i]} P \left( b - q_{m+1}' \cdot x_1 \right) \) we take replications...
of this Gaussian. However, in the parametric algorithm we take only a finite number of replications. For simplicity let us consider the replications in messages sent from the variable node of $x_1$. When $v_{x_1 \mid y_1} < 1$ we take the replications within a ball with a radius of length $2.5$, centered at $\mu_{x_1 \mid y_1}$, where the MMSE estimation is done with a-priori probabilities $\text{CN}(x_1; 0, 2P_0)$, $\text{CN}(x_2; 0, 2P_0)$. In case $v_{x_1 \mid y_1} \geq 1$ we take the replications within a ball with a radius $\lceil \sqrt{n \cdot P_0} \rceil$, centered at zero. We take replications for each incoming message to the check node. Since we consider only one iteration and the degree is two, the Gaussians in the list have the same amplitude. Hence, in order to calculate the variable to channel node message we first need to find the $J''$ Gaussian pairs from the two incoming messages (after replication) that have the closest mean values. Then, we multiply these $J''$ pairs of Gaussians. This operation requires $O\left(J''\right)$ multiplications. Finally, we multiply these $J''$ Gaussians with the a-priori probability $\text{CN}(x_1; 0, 2P_0)$ in order to obtain the variable to channel node message that approximates $f_X(x_1) \cdot \tilde{c}_1(x_1)$. This operation also requires $O\left(J''\right)$ multiplications. Hence, to calculate the variable to channel node message we need to take $O\left(J''\right)$ multiplications.

- **Channel to variable node message:** Keeping the notations from (4.24) let us consider the message from the channel node to variable node $x_1$. In this case the incoming message from $x_2$ consists of $J''$ Gaussians. We calculate $J''$ MMSE estimations, where each MMSE estimation takes as a-priori probabilities a different Gaussian from the approximation of $f_X(x_2) \cdot \tilde{c}_2(x_2)$, together with $\text{CN}(x_1; 0, 2P_0)$. This requires $O\left(J''\right)$ multiplications. Therefore, the channel to variable node message also consists of $J''$ Gaussians.

- **Final decision:** Since there is only one iteration, in this phase we calculate the estimated marginal. We multiply the $J''$ Gaussians from the channel to variable node message with the $J''$ Gaussians from the multiplication of the check node messages after replication. This operation requires $O\left(\left(J''\right)^2\right)$ multiplications.

In the final decision phase of the protected symbols we can decide together on symbols connected via the same channel node. This is due to the fact that each symbol marginal consists of Gaussians list, and the lists of amplitudes are identical between symbols connected to the same channel node. Hence, in the final decision phase we can calculate the amplitudes only once for all symbols connected via a channel node, instead of calculating it for each symbol separately. For instance in order to estimate $\hat{x}_1$, we can calculate the amplitudes only once for symbols $x_1, x_2$. In this case the final decision is made by finding the maximal amplitude, and taking the mean value of the corresponding Gaussian in the symbols’ marginal. Therefore, symbols $x_1, \ldots, x_N$ require complexity in the order of $O\left(\frac{n}{2} \cdot \left(J''\right)^2\right)$. In order to decide on the estimated Gaussian integer vector we take $\hat{b} = \lfloor Q \cdot \hat{x} \rfloor$.

Altogether, the decoding of $n$-complex dimensional CLDLC with degree two, using a single iteration of the max-product algorithm, requires complexity in the order of $O\left(\frac{n}{2} \cdot \left(J''\right)^2\right)$ operations.
4.6 Numerical Results

4.6.1 The AWGN Channel

We compare the performance of both algorithms. In all cases we took Gaussian lists of length $J' = 10$. Also, we normalize the Voronoi region volume to one. We begin by comparing the performance for Latin square LDLC of dimension $n = 8$, degree $l = 3$, and generating sequence $|q| = \{\frac{1}{2.31}, \frac{1}{3.17}, \frac{1}{5.11}\}$. We normalize the WER by a factor of $\frac{2}{n}$. For normalized WER (NWER) of $2 \cdot 10^{-5}$ the max-product algorithm improves the performance by 0.2 dB compared to the sum-product algorithm. Next, we compared the algorithms performance for Latin square LDLC of dimension $n = 16$, $l = 3$, and the same generating sequence as for $n = 8$. In this case for NWER of $2 \cdot 10^{-5}$ we can see that the max-product algorithm is 0.15 dB closer to the channel capacity than the sum-product algorithm. In [12], the smallest gap from channel capacity for a certain dimension and a certain error probability was presented. For dimensions $n = 8, 16$, we also compared the performance at NWER $2 \cdot 10^{-5}$ to the smallest gap from channel capacity. In accordance with [12] we choose $\epsilon_1 = 10^{-5}$ which gives NWER $\frac{2}{n}(1 - \epsilon_1)^n \approx 2 \cdot 10^{-5}$ for $n = 8, 16$. We can see that for $n = 8$ the max-product algorithm is 1.5 dB from the smallest gap from channel capacity. For $n = 16$ the max-product algorithm has gap of 1.35 dB from the smallest gap from channel capacity. Finally, we take Latin square LDLC of dimension $n = 100$, $l = 5$ and $|q| = \{\frac{1}{2.31}, \frac{1}{3.17}, \frac{1}{5.11}, \frac{1}{7.33}, \frac{1}{11.71}\}$. In this case the performance of both algorithms is essentially the same. Indeed we can see that for rather small dimensions, and at moderate gap from channel capacity, the max-product algorithm improves the performance. As the gap from channel capacity increases the improvement decreases. Also, for large dimensions both algorithms attain essentially the same performance.

![Figure 4.4: Normalized word error rate for different block lengths for the AWGN channel](image)

Figure 4.4: Normalized word error rate for different block lengths for the AWGN channel
4.6.2 The $2 \times 2$ MIMO Fading Channel

In this section we present the performance of CLDLC for the $2 \times 2$ MIMO fading channel. We consider a 4-complex dimensional CLDLC with the parity check matrix presented in (4.34), where we take $|q_2| = \sqrt{\frac{1}{17}}$, and normalize the Voronoi region volume to one, i.e. normalize Q such that $|Q| = 1$. We transmit the CLDLC using the transmission scheme presented in (4.35), and for each channel realization we decode using the parametric decoder presented in Section 4.5.3. We present numerical results for transmission at rates $R_1 = 8$ and $R_2 = 16$ bits per channel use. Since the transmission scheme (4.35) spreads over three channel uses, we need to transmit 24 bits to attain rate $R_1$ and 48 bits to attain rate $R_2$. Therefore, to attain rate $R_1$ we take a subset of the CLDLC consisting of $2^{24}$ words, and to attain rate $R_2$ we take a subset of the CLDLC consisting of $2^{48}$ words.

In order to find the lattice points with minimal power for transmission we use a variant of the M-algorithm for nested lattices [22]. The average transmitted power per channel use when considering the transmission scheme in (4.35) is $P_c = \frac{1}{|C|} \sum_{x \in C} \frac{1}{3} \sum_{j=1}^4 |x_j|^2$, where $C$ is the aforementioned codebook which is a subset of the CLDLC, with rate of either $R_1$ or $R_2$, and $|C|$ is the number of codewords. Hence, in accordance with the definitions in 4.2 we get $\text{SNR} = \frac{P_c}{\sigma^2}$. For $R_1$ we get $\text{SNR}_{dB} = 7.33 - \sigma_{dB}^2$, and for $R_2$ we get $\text{SNR}_{dB} = 26.13 - \sigma_{dB}^2$.

The simulation results are presented in Figure 4.5. We compare the performance of CLDLC’s decoded using the parametric algorithm, to the performance of the Alamouti scheme, and also to tilted QAM and golden code, both decoded using maximum-likelihood (ML) decoder. Note that although we compared the performance of CLDLC to golden code with the same rate, still the CLDLC

![Figure 4.5: Performance of CLDLC for rates $R_1 = 8$, $R_2 = 16$ bits per channel use, compared to the performance of the Alamouti scheme and golden code for rate $R_1$, and also to tilted QAM and the Alamouti scheme for rate $R_2$. The tilted QAM and golden code are decoded using ML decoder. The CLDLC is decoded using the parametric algorithm.](image-url)
transmission scheme spreads over three channel uses, where golden code only spreads over two. For instance for rate $R_1$ it means that the CLDLC codebook consists of $2^{24}$ words, where the golden code consists of $2^{16}$ words. In this case normalizing the WER according to the methods presented in [12], would mean comparing the performance of three concatenated golden codes to the performance of two concatenated CLDLC’s such that both codes have the same rate and the same number of words. We compare the normalized WER of these schemes for rates $R_1$ and $R_2$. Another normalization that leads to the same results for small WER, is normalizing the WER of each scheme by its number of channel uses.

For rate $R_1$ the parametric algorithm average lists length is $J'' = 14$, where for rate $R_2$ the average lists length of a message is $J'' = 25$. The a-priori probabilities in the parametric algorithm are $CN \left(x_j; 0, 2P_0\right)$, $j = 1, \ldots, 4$, where $P_0 = \frac{2^{\frac{3}{2}} R_1}{2\pi e}$, $\frac{2^{\frac{3}{2}} R_2}{2\pi e}$ for rates $R_1$ and $R_2$ respectively. For further improvement of the performance for rate $R_1$, we take the parametric algorithm decision, and check whether there are more likely lattice codewords of the transmitted lattice $Q$, within a ball with radius of length $1.6$, around this decision. Since there is a small amount of lattice points within this radius, this search has a very small effect on the complexity. For both rates we compare the performance for normalized WER of $10^{-4}$. For rate $R_1$ we can see that CLDLC outperforms the Alamouti scheme by $3.5\ dB$ and has a gap of $1.7\ dB$ from the golden code. For rate $R_2$ the CLDLC outperforms the Alamouti scheme by $9\ dB$ and has a gap of $4\ dB$ from the tilted QAM.
Chapter 5

Summary and Conclusions

In this final chapter we summarize the thesis, outline several directions for further research and provide concluding remarks.

5.1 Summary

In this thesis we considered the problem of reliable communication over fading channels using multiple-antennas. In this problem it is desired to decrease the error probability as much as possible, at a certain transmission rate. This problem is of practical interest since the fading channel models the wireless medium which is used by a great number of devices and applications.

In the first part of this thesis we analyzed the performance of IC’s in MIMO fading channels. We considered both point-to-point and MAC channels and presented a new tradeoff between the NDCU and the best DMT that may be attained by any IC (or more precisely by any sequence of IC’s). First, based on this tradeoff we showed that IC’s attain the optimal DMT of the point-to-point MIMO Rayleigh fading channel. Then, we showed for the MAC channel that when the number of users is greater than\( \max \left( 1, \frac{N-M+1}{M} \right) \) IC’s can not attain the optimal DMT of finite constellations. On the other hand in a user limited regime that is bounded by\( \max \left( 1, \frac{N-M+1}{M} \right) \) we showed that IC’s in general and lattices using regular lattice decoding in particular attain the optimal DMT.

In the second part we showed a practical implementation of lattice codes in SISO and MIMO fading channels. We derived a max-product algorithm for LDLC’s for those channels, and for the Gaussian case showed an interesting connection between the max-product and sum-product algorithms, and also between the passed messages and MMSE estimation. For a MIMO fading channel with two transmit and two receive antennas we designed CLDLC combined with a transmission scheme that can be decoded very efficiently by a parametric algorithm that approximates the max-product algorithm. Numerical results show that this transmission scheme attains performance comparable to state of the art coding schemes in this channel.
5.2 Suggestions for Further Research

5.2.1 IC’s in Point-to-Point MIMO Fading Channels

In Chapter 2 we showed that there exist sequences of IC’s in general and lattices using regular lattice decoding in particular that attain the optimal DMT. However, the existence of such sequences was proved by using probabilistic methods. It would be interesting to find an explicit construction that attains the optimal DMT, and also to show that each segment of the optimal DMT is attained by a single (scaled) lattice rather than a sequence of lattices.

5.2.2 IC’s in MAC MIMO Fading Channels

The optimality of IC’s in MAC channels was studied in Chapter 3. For the case \( N \geq (K + 1) M - 1 \) an analytical upper bound on the optimal DMT of IC’s, and also a transmission scheme that attains it were presented. However, for the case \( N < (K + 1) M - 1 \) we only proved the sub-optimality of IC’s compared to the optimal DMT of finite constellations. In this case an explicit analytical expression for the upper bound on the optimal DMT of IC’s was given only for the symmetric case, while for the general case the upper bound was presented in the form of an optimization problem. Getting an analytical expression for the upper bound for the the non-symmetric case when \( N < (K + 1) M - 1 \), and also finding a transmission scheme that attains it, are still open problems.

5.2.3 LDLC’s for MIMO Fading Channels

The max-product algorithm for CLDLC’s was derived for general MIMO fading channels. We used an approximation of the algorithm in a channel with two transmit and two receive antennas. In this case reducing the CLDLC dimensionality enabled to attain good performance using only a single iteration. It would be interesting to see whether applying lattice reduction methods, as presented in [13], can improve the performance of both the transmission scheme presented in Chapter 4 using a single iteration, and full dimensional CLDLC with no short loops using larger number of iterations. Also, we used CLDLC’s in Chapter 4 as a modulation (due to the CLDLC small dimension). Extending the CLDLC dimension and using it as a coded modulation is yet to be done. Finally, the design of CLDLC’s for larger antenna arrays and the use of the max-product algorithm in these cases may also be a subject for future research.

5.3 Concluding Remarks

This work has both theoretical and practical significance. From a practical point of view the communication engineer benefits from knowing the assumptions he is allowed to make in the receiver design when transmitting lattice constellations. In addition, this thesis presents practical implementation of lattice codes for the MIMO fading channel, along with an efficient decoding algorithm for LDLC’s. From a theoretical point of view this thesis gives a geometrical interpretation to the optimal DMT, by designing lattice constellations.
that “match” the MIMO fading channel. Also, from a DMT perspective, it shows for which settings IC’s are optimal, and when IC’s are inherently suboptimal.

On a larger perspective, this work provides a better understanding of the MIMO fading channel, and also of the allowed assumptions in the receiver design. We hope that this work inspires future research in this field, leading to practical coding schemes and deep theoretical results.
Appendix A

The Optimal DMT of IC’s in Point-to-Point Channels: Proofs

A.1 Proof of Theorem 2.1

We prove the result for any IC with density $\gamma_{rc}$. The proof outline is as follows. We prove the theorem by contradiction. First, for a given IC with receiver density $\gamma_{rc}$, we assume an average decoding error probability that equals to the lower bound we wish to prove. Then, we derive a “regular” IC from the given IC with the same density $\gamma_{rc}$ and the same average decoding error probability. Regularizing the IC allows us to find a lower bound on the IC maximal error probability that depends on its density. We expurgate half of the codewords with the largest error probability and get another regular IC with density $\gamma_{rc}/2$. Based on the average decoding error probability, we upper bound the expurgated IC maximal error probability, and based on its density we lower bound the same maximal error probability, and get a contradiction.

Let us consider a $D \cdot T$-complex dimensional IC at the receiver, $S'_{D,T}(\rho)$, with receiver density $\gamma_{rc}$ and average decoding error probability

$$P_e(H, \rho) = (1 - \epsilon)^{D \cdot T} e^{-\mu_{rc} \cdot \overline{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})}$$ (A.1)

where $\overline{A}(D \cdot T) = \left(\frac{1}{D \cdot T - 1}\right)^{D \cdot T - 1} e^{\Gamma(D \cdot T + 1) D \cdot T}$, $C(D \cdot T) = \left(\frac{1}{D \cdot T - 1}\right)^{D \cdot T - 1} e^{\frac{3}{2} \Gamma(D \cdot T + 1) D \cdot T}$, and $0 < \epsilon_1, \epsilon_2 < 1$.

Next we construct a regularized IC, $S''_{D,T}(\rho)$, from $S'_{D,T}(\rho)$, whose Voronoi regions are bounded and have finite volumes, i.e. there exists a finite radius $r$ such that $V(x) \subset Ball(x, r), \forall x \in S''_{D,T}(\rho)$, where $Ball(x, r)$ is a $D \cdot T$-complex dimensional ball centered around $x$. We construct $S''_{D,T}(\rho)$ in the following manner. Let us define $C_0(\rho, H) = \{S'_{D,T}(\rho) \cap (H \cdot cube_{D,T}(b))\}$, i.e. a finite constellation derived from $S'_{D,T}(\rho)$. We turn this finite constellation into an IC by tiling $C_0(\rho, H)$ in the following manner

$$S''_{D,T}(\rho) = C_0(\rho, H) + (b + b') \hat{H}_ex \mathbb{Z}^{2D \cdot T}$$ (A.2)
where for simplicity we assumed that $\text{cube}_{D,T}(b) \subset \mathbb{C}^{D,T}$, i.e. contained within the first $D \cdot T$ complex dimensions. Correspondingly, under this assumption, $\bar{H}_{ex}$ equals the first $D \cdot T$ complex columns of $H_{ex}$. In this case, the tiling of $C_0(\rho, H)$ is done according to the complex integer combinations of $\bar{H}_{ex}$ columns. In general, $\text{cube}_{D,T}(b)$ may be a rotated cube within $\mathbb{C}^{MT}$. In this case the tiling is done according to some $D \cdot T$ complex linearly independent vectors, consisting of linear combinations of $H_{ex}$ columns. An alternative way to construct $S''_{D,T}(\rho)$ is by considering the transmitter IC $S_{D,T}(\rho)$. In this case we can construct another IC at the transmitter

$$\mathcal{S}_{D,T}(\rho) = \{S_{D,T}(\rho) \cap \text{cube}_{D,T}(b)\} + (b + b')\mathbb{Z}^{2D-T} \quad (A.3)$$

where without loss of generality we assumed again that $\text{cube}_{D,T}(b) \subset \mathbb{C}^{D,T}$. In this case $S''_{D,T}(\rho) = \{H_{ex} \cdot \mathcal{S}_{D,T}(\rho)\}$.

Next we would like to set $b$ and $b'$ to be large enough such that $S''_{D,T}(\rho)$ has average decoding error probability smaller or equal to $\frac{\mathbb{E}(D,T)}{2}e^{-\mu_{ec}A(D,T)+(D\cdot T-1)\ln(\mu_{ec})}$ and density larger or equal to $\gamma_{irrc}$. Due to the symmetry that results from the tiling (A.2), it is sufficient to upper bound the average decoding error probability of the points $x \in C_0(\rho, H) \subset S''_{D,T}(\rho)$ denoted by $P_e^{S''_{D,T}}(C_0)$ in order to upper bound the average decoding error probability of the entire IC $S''_{D,T}(\rho)$. Hence $P_e^{S''_{D,T}}(C_0)$ is also the average decoding error probability for the IC $S''_{D,T}(\rho)$. We can upper bound the error probability in the following manner

$$P_e^{S''_{D,T}}(C_0) \leq P_e(C_0) + P_e(S_{D,T} \setminus C_0) \quad (A.4)$$

where $P_e(C_0)$ is the average decoding error probability of the finite constellation $C_0(\rho, H)$ and $P_e(S_{D,T} \setminus C_0)$ is the average decoding error probability to points in the set $\{S_{D,T} \setminus C_0(\rho, h)\}$, i.e. the error probability inflicted by the replicated codewords outside the set $C_0(\rho, H)$.

We begin by upper bounding $P_e(S_{D,T} \setminus C_0)$ by choosing $b'$ to be large enough. By the tiling at the transmitter (A.3) and the fact that we have finite complex dimension $D \cdot T$, for a certain channel realization $H_{ex}$ we get that there exists $\delta(H_{ex})$ such that any pair of points $x_1, x_2 \in C_0(\rho, H), x_2 \in \{S_{D,T} \setminus C_0(\rho, h)\}$ fulfills $\|x_1 - x_2\| \geq 2b' \cdot \delta(H_{ex})$. The term $\delta(H_{ex})$ is a factor that defines the minimal distance between these 2 sets for a given channel realization. Note that also for the case $M > N$, there must exist such $\delta(H_{ex})$, as we assumed that $S''_{D,T}(\rho)$ is $D \cdot T$-complex dimensional IC, i.e. the projected IC $S''_{D,T}(\rho) = H_{ex} \mathcal{S}_{D,T}(\rho)$ is also $D \cdot T$-complex dimensional. Hence, we get that

$$P_e(S_{D,T} \setminus C_0) \leq Pr(\|\tilde{n}_{ex}\| \geq b' \delta(H_{ex}))$$

where $\tilde{n}_{ex}$ is the effective noise in the $D \cdot T$-complex dimensional hyperplane where $S''_{D,T}(\rho)$ resides. By using the upper bounds from [20], we get that for $\frac{(b' \delta(H_{ex}))^2}{2D \cdot T} > \sigma^2$

$$Pr(\|\tilde{n}_{ex}\| \geq b' \delta(H_{ex})) \leq e^{-\frac{(b' \delta(H_{ex}))^2}{2\sigma^2}}(\frac{(b' \delta(H_{ex}))^2}{2D \cdot T \sigma^2})^{D \cdot T}.$$
Hence, for $b'$ large enough we get that

$$P_e(S''_{D,T} \setminus C_0) \leq (1 - \epsilon^*) \frac{\mathcal{C}(D \cdot T)}{4} e^{-\mu_{rc} \mathcal{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})}.$$  

Now we would like to upper bound the error probability, $P_e(C_0)$, of the finite constellation $C_0(\rho, H)$. According to the definition of the average decoding error probability in (2.10), the definition of $C_0(\rho, H)$ and the assumption in (A.1), we get that

$$P_e(C_0) \leq \frac{(1 - \epsilon^*)(1 + \epsilon(b))}{4} \mathcal{C}(D \cdot T) e^{-\mu_{rc} \mathcal{A}(D \cdot T)} \cdot \mu_{rc}^{(D \cdot T - 1)}$$

where $\lim_{b \to \infty} \epsilon(b) = 0$. It results from the fact that in (2.10) we take the limit supremum, and so for $b$ large enough the average decoding error probability of the IC must be upper bounded by the aforementioned term. Also, for any $b$ the average decoding error probability of the finite constellation $C_0(\rho, H)$ is smaller or equal to the error probability, defined in (2.10), of decoding over the entire IC. Based on the upper bound from (A.4) we get the following upper bound on the error probability of $S''_{D,T}(\rho)$

$$P_e(S''_{D,T}(C_0) \leq \frac{(1 - \epsilon^*)(1 + \epsilon(b))}{2} \mathcal{C}(D \cdot T) e^{-\mu_{rc} \mathcal{A}(D \cdot T)} \cdot \mu_{rc}^{(D \cdot T - 1)}.$$ (A.5)

According to the definition of $\gamma_{rc}$ and due to the fact that we are taking limit supremum: for any $0 < \epsilon_1 < 1$ there exists $b$ large enough such that

$$\frac{|C_0(\rho, H)|}{\text{vol}(H_{ex} \cdot \text{cube}_{D,T}(b))} \geq (1 - \epsilon_1)\gamma_{rc}.$$ (A.6)

where $|C_0(\rho, H)|$ is the number of points in $C_0(\rho, H)$. In fact there exists large enough $b$ that fulfils both (A.5) and (A.6).

In (A.2) we tiled by $b + b'$. If we had tiled $C_0(\rho, H)$ only by $b$, then for large enough $b$ we would have got IC with density larger or equal to $(1 - \epsilon_1)\gamma_{rc}$. However, as we tile by $b + b'$, we get for $b$ large enough that $S''_{D,T}(\rho)$ has density greater or equal to $(1 - \epsilon_1)(1 - \epsilon_2)\gamma_{rc}$. Hence, for any $0 < \epsilon_2 < 1$ there exists $b$ large enough such that

$$\gamma''_{rc} \geq (1 - \epsilon_1)(1 - \epsilon_2)\gamma_{rc}.$$ (A.7)

where $\gamma''_{rc}$ is the density of $S''_{D,T}(\rho)$. Again, there also must exist large enough $b$ that fulfils (A.5) and (A.7) simultaneously. Hence, for large enough $b$ we can derive from $S'_{D,T}(\rho)$ an IC $S''_{D,T}(\rho)$ with density

$$\gamma''_{rc} \geq (1 - \epsilon_1)(1 - \epsilon_2)\gamma_{rc}$$

and average decoding error probability smaller or equal to

$$\frac{(1 - \epsilon^*)(1 + \epsilon(b))}{2} \mathcal{C}(D \cdot T) e^{-\mu_{rc} \mathcal{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})}.$$  

By averaging arguments we know that expurgating the worst half of the codewords in $S''_{D,T}(\rho)$, yields
an IC $S''_{D,T}(\rho)$ with density
\[ \gamma_{rc}' \geq (1 - \epsilon_1)(1 - \epsilon_2) \frac{\tilde{\gamma}_{rc}}{2} = \frac{\gamma_{rc}}{2} \]  
(A.8)
and maximal decoding error probability
\[ \sup_{x \in S''_{D,T}} P_e^{S''_{D,T}}(x) \leq (1 - \epsilon^*)(1 + \epsilon(b)) \overline{C}(D \cdot T) e^{-\mu_{rc} \overline{A}(D \cdot T)} \mu_{rc}^{-D \cdot T - 1} \]  
(A.9)
where $P_e^{S''_{D,T}}(x)$ is the error probability of $x \in S''_{D,T}(\rho)$.

From the construction method of $S''_{D,T}(\rho)$, defined in (A.2), it can be easily shown that tiling $C_0(\rho, H)$ yields bounded and finite volume Voronoi regions, i.e. there exists a finite radius $r$ such that $V(x) \subset Ball(x, r), \forall x \in S''_{D,T}(\rho)$. Due to the symmetry that results from $S''_{D,T}(\rho)$ construction (A.2), it also applies to $S''_{D,T}(\rho)$. Hence, there must exist a point $x_0 \in S''_{D,T}(\rho)$ that satisfies $|V(x_0)| \leq \frac{r_{rc}}{r} \leq \frac{1}{r_{rc}}$.

According to the definition of the effective radius in (2.1), we get that $r_{eff}(x_0) \leq r_{eff}(\frac{\gamma_{rc}}{2})$. Hence, we get
\[ \sup_{x \in S''_{D,T}} P_e^{S''_{D,T}}(x) > P_e^{S''_{D,T}}(x_0) > Pr(\|\tilde{u}_{ex}\| \geq r_{eff}(x_0)) \geq Pr(\|\tilde{u}_{ex}\| \geq r_{eff}(\frac{\gamma_{rc}}{2})) \]  
(A.10)
where the lower bound $P_e^{S''_{D,T}}(x_0) > Pr(\|\tilde{u}_{ex}\| \geq r_{eff}(x_0))$ was proven in [20]. We calculate the following lower bound
\[ Pr(\|\tilde{u}_{ex}\| \geq r_{eff}(\frac{\gamma_{rc}}{2})) > \int_{r_{eff}^2}^{r_{eff}^2 + \sigma^2} \frac{r^{D \cdot T - 1} e^{-\frac{r^2}{2\sigma^2}}}{\sigma^{2D \cdot T - 2} \Gamma(D \cdot T)} dr \geq \frac{r_{eff}^{2D \cdot T - 2} e^{-\frac{r_{eff}^2}{2\sigma^2}}}{\sigma^{2D \cdot T - 2} \Gamma(D \cdot T) \sqrt{\pi}} \]  
(A.11)
By assigning $r_{eff}^2 = (\frac{2 \Gamma(D \cdot T + 1)}{\gamma_{rc}D \cdot T})^{\frac{1}{2D \cdot T}}$ we get
\[ \sup_{x \in S''_{D,T}} P_e^{S''_{D,T}}(x) > \overline{C}(D \cdot T) \cdot e^{-\frac{\gamma_{rc} D \cdot T}{2\epsilon_{rc}}} \overline{A}(D \cdot T) + (D \cdot T - 1) \ln(\frac{\gamma_{rc}}{2\epsilon_{rc}}). \]  
(A.12)
Hence, for certain $\epsilon_1$ and $\epsilon_2$ we get
\[ \sup_{x \in S''_{D,T}} P_e^{S''_{D,T}}(x) > \overline{C}(D \cdot T) \cdot e^{-\mu_{rc} \overline{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})} \]  
(A.13)
where $\mu_{rc} = \frac{\gamma_{rc}}{2\epsilon_{rc}}$. For $b$ large enough we get $(1 - \epsilon^*)(1 + \epsilon(b)) < 1$, and so (A.13) contradicts (A.9).
As a result we get contradiction of the initial assumption in (A.1). This contradiction also holds for any $P_e^c(H, \rho) < \frac{(1 - \epsilon^*)\overline{C}(D \cdot T)}{4} e^{-\mu_{rc} \overline{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})}$. Hence, we get that
\[ P_e^c(H, \rho) > \frac{\overline{C}(D \cdot T)}{4} e^{-\mu_{rc} \overline{A}(D \cdot T) + (D \cdot T - 1) \ln(\mu_{rc})}. \]  
(A.14)
Note that the lower bound holds for any $0 < \epsilon_1, \epsilon_2, \epsilon^* < 1$ and also that the expressions in (A.1), (A.14) are continuous. As a result we can also set $\epsilon_1 = \epsilon_2 = \epsilon^* = 0$ and get the desired lower bound. Finally, note that we are interested in a lower bound on the error probability of any IC for a given channel realization. Hence,
we are free to choose different values for \( b \) and \( b' \) for each channel realization.

### A.2 Proof of the Optimization Problem in Theorem 2.2

We would like to solve the optimization problem in (2.17) for any value of \( D = B + \beta \leq L \), where \( B \in \mathbb{N} \) and \( 0 < \beta \leq 1 \). First we consider the case of \( 0 < D \leq 1 \), i.e. the case where \( B = 0 \). In this case the constraint boils down to \( \alpha_1 = 1 - \frac{r}{\beta} \). By assigning \( \alpha_1 = \cdots = \alpha_L = 1 - \frac{r}{\beta} \) we get that \( d_{D,T}(r) \leq MN(1 - \frac{r}{\beta}) \). Next we analyze the case where \( D > 1 \). Due to the constraint, the minimal value must satisfy \( \alpha_1 = \cdots = \alpha_{L-B} \). From the constraint we also know that \( \alpha_L = D - r - \sum_{i=1}^{B-1} \alpha_{L-i} - \beta \alpha_{L-B} \). By assigning in (2.17) we get

\[
\min_{\alpha > 0} (D - r)(N + M - 1) + ((M - B)(N - B) - \beta(N + M - 1)) \alpha_{L-B} - \sum_{i=1}^{B-1} 2i \cdot \alpha_{L-i} \quad (A.15)
\]

where \( \alpha > 0 \) signifies \( \alpha_1 \geq \cdots \geq \alpha_L \geq 0 \). We would like to consider two cases. The case where \( ((M - B)(N - B) - \beta(N + M - 1)) > \sum_{i=1}^{B-1} 2i \) and the case where \( ((M - B)(N - B) - \beta(N + M - 1)) \leq \sum_{i=1}^{B-1} 2i \). The first case, where \( ((M - B)(N - B) - \beta(N + M - 1)) > B(B - 1) \), is achieved for \( D < \frac{MN}{N + M - 1} \). In this case we use the following Lemma in order to find the optimal solution

**Lemma A.1.** Consider the optimization problem

\[
\min_{\mathcal{L}} B_1c_1 - \sum_{i=2}^{E} B_ic_i
\]

where: (1) \( c_1 \geq \cdots \geq c_E \geq 0 \); (2) \( B_1 \geq \sum_{i=2}^{E} B_i \) and \( B_2 > \cdots > B_E > 0 \); (3) \( \beta c_1 + \sum_{i=2}^{E} c_i = \delta > 0 \), where \( 0 < \beta \leq 1 \). The minimal value is achieved for \( c_1 = \cdots = c_E = \frac{\delta}{E - 1 + \beta} \).

**Proof.** We prove by induction. First let us consider the case where \( E = 2 \). In this case we would like to find

\[
\min_{\mathcal{L}} B_1c_1 - B_2c_2 \quad (A.16)
\]

where \( c_1 \geq c_2 \geq 0, \beta c_1 + c_2 = \delta > 0, B_1 > B_2 > 0 \) and \( 0 < \beta \leq 1 \). It is easy to see that for this case the minimum is achieved for \( c_1 = c_2 \), as increasing \( c_1 \) while decreasing \( c_2 \) to satisfy \( \beta c_1 + c_2 = \delta \) will only increase (A.16).

Now let assume that for \( E \) elements, the minimum is achieved for \( c_1 = \cdots = c_E = \frac{\delta}{E - 1 + \beta} \). Let us consider \( E + 1 \) elements with constraint \( \beta c_1 + \sum_{i=2}^{E+1} c_i = \delta \). If we take \( c_1 = \cdots = c_{E+1} = \frac{\delta}{E + \beta} \) we get

\[
(B_1 - \sum_{i=2}^{E+1} B_i) \frac{\delta}{E + \beta}. \quad (A.17)
\]

We would like to show that this is the minimal possible value for this problem. Take \( c_{E+1}' = \frac{\delta}{E + \beta} - \epsilon \geq 0 \). In this case \( \beta c_1' + \sum_{i=2}^{E} c_i' = \frac{(E - 1 + \beta) \delta + (E + \beta) \epsilon}{E + \beta} \) in order to satisfy \( \beta c_1' + \sum_{i=2}^{E+1} c_i' = \delta \). According to our
assumption $B_1c_1' - \sum_{i=2}^{E} B_ic_i'$ is minimal for $c_1' = \cdots = c_E' = \frac{\delta}{E+\beta} + \frac{\epsilon}{E-1+\beta}$. By assigning these values we get
\[
(B_1 - \sum_{i=2}^{E+1} B_i) \frac{\delta}{E+\beta} + (B_1 - \sum_{i=2}^{E} B_i) \frac{\epsilon}{E-1+\beta} + B_{E+1}\epsilon
\]
which is greater than (A.17). This concludes the proof.

For the case $((M-B)(N-B) - \beta(N+M-1)) > B(B-1)$, the optimization problem coincides with Lemma A.1 as it fulfils the condition $B_1 > \sum_{i=2}^{E} B_i$ in the lemma. Hence, the optimization problem solution for $D < \frac{MN}{N+M-1}$ is $\alpha_1 = \cdots = \alpha_{L-1} = \frac{D-r-\alpha_l}{N+M-1} = \alpha$. The minimum is achieved when $\alpha_L = \alpha$, i.e. the maximal value $\alpha_L$ can receive under the constraint $\alpha_1 \geq \cdots \geq \alpha_L \geq 0$. We get that $\alpha = 1 - \frac{r}{D}$, and so the optimization problem solution to (2.17) for the case $D < \frac{MN}{N+M-1}$ is $d_{D,T}(r) \leq MN(1 - \frac{r}{D})$.

For the case $((M-B)(N-B) - \beta(N+M-1)) \leq B(B-1)$, or equivalently $D \geq \frac{MN}{N+M-1}$, we would like to show that the optimal solution must fulfil $\alpha_L = 0$. It results from the fact that for the optimal solution, the term $((M-B)(N-B) - \beta(N+M-1))\alpha_{L-B} - \sum_{i=1}^{B-1} 2i \cdot \alpha_{L-i}$ in (A.15) must be negative. This is due to the fact that taking $\alpha_1 = \cdots = \alpha_{L-1}$ gives negative value. Hence, for the optimal solution we would like to maximize $\sum_{i=1}^{B-1} \alpha_{L-i} - \beta \alpha_{L-B} = D - r - \alpha_L$. By taking $\alpha_L = 0$ the sum is maximized. Hence, the optimal solution for $D \geq \frac{MN}{N+M-1}$ must have $\alpha_L = 0$.

Now consider the general case. Assume that for $D \geq \frac{(M-l+1)(N-l+1)}{N+M-1-2(l-1)} + l - 1$ the optimal solution must have $\alpha_{L-l} = \cdots = \alpha_{L-l+1} = 0$. First consider the case where $1 \leq l \leq B - 1$. For this case the constraint is $\sum_{i=l}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} = D - r$, i.e. the constraint contains at least two singular values. We can rewrite (2.17) as follows
\[
\min_{D>0} (D-r)(N+M-1-2l) + ((M-B)(N-B) - \beta(N+M-1-2l))\alpha_{L-B} - \sum_{i=l+1}^{B-1} 2(i-l)\alpha_{L-i}. \quad (A.18)
\]

For the case $((M-B)(N-B) - \beta(N+M-1-2l)) > (B-1-l)(B-l)$ we get that $D < \frac{(M-l)(N-l)}{N+M-1-2(l-1)} + l - 1$. For this case we can use Lemma A.1 and get that the optimization problem solution is $\alpha_{L-l} = \cdots = \alpha_{L-B} = \frac{D-r-\alpha_l}{D-l} = \alpha$. The minimum is achieved for $\alpha_L = \cdots = \alpha_{L-l+1} = 0$ and $\alpha_1 = \cdots = \alpha_{L-l} = \frac{D-r}{D-l}$. Hence, for the case $D \geq \frac{(M-l)(N-l)}{N+M-1-2l} + l - 1 \leq D < \frac{(M-l)(N-l)}{N+M-1-2l} + l$ the solution is $d_{D,T}(r) \leq (N-l)(M-l)\frac{D-r}{D-l}$.

For the case $((M-B)(N-B) - \beta(N+M-1-2l)) \leq (B-1-l)(B-l)$, or equivalently $D \geq \frac{(M-l)(N-l)}{N+M-1-2l} + l$, the term $((M-B)(N-B) - \beta(N+M-1-2l))\alpha_{L-B} - \sum_{i=l+1}^{B-1} 2(i-l)\alpha_{L-i}$ in (A.18) must be negative for the optimal solution. This is due to the fact that by taking $\alpha_1 = \cdots = \alpha_{L-l} = 0$ we get a negative value. Hence we would like to maximize the sum $\sum_{i=l+1}^{B-1} \alpha_{L-i} + \beta \alpha_{L-B} = D - r - \alpha_{L-l}$. The sum is maximized by taking $\alpha_{L-l} = 0$. Hence the optimal solution for the case $D \geq \frac{(M-l)(N-l)}{N+M-1-2l} + l$ must have $\alpha_{L-l} = \cdots = \alpha_L = 0$. Note that for the case $l = B - 1$ we have only two terms in the constraint $\alpha_{L-B+1} + \beta \alpha_{L-B} = D - r$. However, the solution remains the same.

For the case $D \geq \frac{(M-l)(N-l)}{N+M-1-2(l-1)} + l - 1$ and $l = B$ the constraint is of the form $\alpha_{L-B} = \frac{D-r}{D_l}$. Again we assume that $\alpha_{L-B+1} = \cdots = \alpha_L = 0$. In this case the solution is $\alpha_1 = \cdots = \alpha_{L-l} = \frac{D-r}{D_l}$ and so $d_{D,T}(r) \leq (M-l)(N-l)\frac{D-r}{D_l}$. This concludes the proof.

96
A.3 Proof of Lemma 2.1

We begin by proving the case \( N \geq M \). From the construction of \( G_l \) it can be seen that a set of columns \( \{h_j, \ldots, h_i\} \) may occur in \( N - i + j \) blocks at most. It results from the fact that we can only subtract \( M - i \) columns to the right of \( h_j \) (2.20), and \( j - 1 \) columns to the left of \( h_j \) (2.21), and still get a block that contains \( \{h_j, \ldots, h_i\} \) (or even more specifically a block that contains \( \{h_j, h_i\} \)). In addition, columns \( \{h_j, \ldots, h_i\} \) must occur in the first \( N + M + 1 \) blocks, as these blocks equal to \( H \) (2.19). Hence, we can upper bound the number of occurrences by \( N - M + 1 + j - 1 + M - i = N - i + j \).

Next we prove the case \( M > N \). When \( 0 \leq i - j < N \), the set of columns \( \{h_j, \ldots, h_i\} \) may occur in \( N - i + j \) blocks at most. We divide the proof into four cases.

1. \( i \leq N \) and \( j \geq M - N + 1 \). In this case the set of columns \( \{h_j, \ldots, h_i\} \) occurs in the first \( M - N + 1 \) blocks (2.22). As for the additional \( N - 1 - l \) pairs of columns, the set of columns belongs both to the set \( \{h_1, \ldots, h_N\} \) and \( \{h_{M - N + 1}, \ldots, h_M\} \). Hence, in the additional column pairs we can subtract \( N - i \) columns to the right of \( h_j \) (2.23) and \( j - M + N - 1 \) columns to the left of \( h_j \) (2.24). Added together we observe that the number of occurrences can not exceed \( N - i + j \).

2. \( i \leq N \) and \( j < M - N + 1 \). In this case the set of columns can have only \( j \) occurrences in the first \( M - N + 1 \) blocks. In this case the set \( \{h_j, \ldots, h_i\} \) occurs within \( \{h_1, \ldots, h_N\} \) but does not occur within \( \{h_{M - N + 1}, \ldots, h_M\} \). Hence, the transmission scheme only subtracts columns to the right of \( h_j \) (2.23). In this case we can have \( N - i \) subtractions and together we get \( N - i + j \) occurrences at most.

3. \( i > N \) and \( j \geq M - N + 1 \). We have here \( M - i + 1 \) occurrences in the first \( M - N + 1 \) blocks. In this case the set \( \{h_j, \ldots, h_i\} \) occurs within \( \{h_{M - N + 1}, \ldots, h_M\} \) but does not occur within \( \{h_1, \ldots, h_N\} \). Hence we can subtract up to \( j - M + N - 1 \) columns to the left of \( h_j \) (2.24). Together there are \( N - i + j \) occurrences at most.

4. Last case, \( i > N \) and \( j < M - N + 1 \). Here the set of columns can only occur in the first \( M - N + 1 \) blocks. In this case there are exactly \( N - i + j \) occurrences in the first \( M - N + 1 \) blocks.

In case \( i - j \geq N \), the set of columns does not occur in any block as each column of \( G_l \) does not have more than \( N \) non-zero entries.

A.4 Proof of Theorem 2.3

Based on [20] we have the following upper bound on the maximum-likelihood (ML) decoding error probability of each \( D_l \cdot T_l \)-complex dimensional IC point \( \bar{x}' \in S_{D_l \cdot T_l} \)

\[
P_e(\bar{x}') \leq Pr(\|\bar{n}_{x'}\| \geq R) + \sum_{l \in \text{Ball}(\bar{x}', 2R) \cap S_{D_l \cdot T_l}, \bar{x}' \neq \bar{x}'} Pr(\|\bar{L} - \bar{x}' - \bar{n}_{x'}\| < \|\bar{n}_{x'}\|) \tag{A.19}
\]
where $Ball(x', 2R)$ is a $D_l \cdot T_l$-complex dimensional ball of radius $2R$ centered around $x'$, and $\vec{n}_{ex}$ is the effective noise in the $D_l \cdot T_l$-complex dimensional hyperplane where the IC’s resides. Note that the second term in (A.19) represents the pairwise error probability to points within $Ball(x', 2R)$, i.e. the decision region is at distance $R$ at most.

Next we upper bound the average decoding error probability of an ensemble of constellations drawn uniformly within $cube_{D_l \cdot T_l}(b)$. Each code-book contains $|\gamma_{tr}b^{2D_l \cdot T_l}|$ points, where each point is drawn uniformly within $cube_{D_l \cdot T_l}(b)$. At the receiver, the random ensemble is uniformly distributed within $\{H_{eff}^{(l)} \cdot cube_{D_l \cdot T_l}(b)\}$. Let us consider a certain point, $x' \in \{H_{eff}^{(l)} \cdot cube_{D_l \cdot T_l}(b)\}$, from the random ensemble at the receiver. We denote the ring around $x'$ by $Ring(x', i\Delta) = Ball(x', i\Delta) \setminus Ball(x', (i-1)\Delta)$. The average number of points within $Ring(x', i\Delta)$ of the random ensemble is

$$Av(x', i\Delta) = \gamma_{rc}|H_{eff}^{(l)} \cdot cube_{D_l \cdot T_l}(b) \cap Ring(x', i\Delta)|$$

$$\leq \gamma_{rc}|Ring(x', i\Delta)| \leq \frac{\gamma_{rc} \pi_{D_l \cdot T_l} 2D_l \cdot T_l (i\Delta)^{2D_l \cdot T_l - 1}}{T(D_l \cdot T_l + 1)} (i\Delta)^{2D_l \cdot T_l - 1}\Delta$$

(A.20)

where $\gamma_{rc} = \rho^{T_l + \sum_i D_i T_i} \eta_i$. By using the upper bounds on the error probability (A.19), and the average number of points within the rings (A.20), we get for a certain channel realization the following upper bound on the average decoding error probability of the finite constellations ensemble, at point $x'$

$$P_{e FC}(x', \rho, \eta) \leq Pr(\|\vec{n}_{ex}\| \geq R) + \gamma_{rc}Q(D_l \cdot T_l) \sum_{i=1}^{\frac{2R}{\Delta}} Pr(\vec{n}_{ex, 1} > \frac{(i-1)\Delta}{2}) \cdot (i\Delta)^{2D_l \cdot T_l - 1}$$

where $Q(D_l \cdot T_l) = \frac{\pi_{D_l \cdot T_l} 2D_l \cdot T_l}{T(D_l \cdot T_l + 1)}$, and $\vec{n}_{ex, 1}$ is the first component of $\vec{n}_{ex}$ (the pairwise error probability has scalar decision region). By taking $\Delta \to 0$ we get

$$P_{e FC}(x', \rho, \eta) \leq Pr(\|\vec{n}_{ex}\| \geq R) + \gamma_{rc}Q(D_l \cdot T_l) \int_0^{\frac{2R}{\Delta}} Pr(\vec{n}_{ex, 1} > \frac{x}{2}) x^{2D_l \cdot T_l - 1} dx.$$  

(A.22)

Note that this upper bound applies to any value of $R \geq 0$ and $b$, and does not depend on $x'$, meaning

$$P_{e FC}(x', \rho, \eta) = P_{e FC}(\rho, \eta).$$

Now we divide the channel realization into two subsets: $A = \{\eta \mid \sum_{i=1}^{D_l \cdot T_l} \eta_i \leq T_l (D_l - r), \eta_i \geq 0\}$, where $\eta = (\eta_1, \ldots, \eta_{D_l \cdot T_l})$ and $A = \{\eta \mid \sum_{i=1}^{D_l \cdot T_l} \eta_i > T_l (D_l - r), \eta_i \geq 0\}$. For each set we upper bound the error probability. We begin with the case $\eta \in A$. For this case we upper bound the terms in (A.22) and find an upper bound on the error probability as a function of the receiver VNR, $\mu_{rc} = \rho^{1 - \frac{\sum_i ^{D_l \cdot T_l} \eta_i}{T_l \cdot T_l}}$. We begin by upper bounding the integral of the second term in (A.22). Note that

$$Pr(\vec{n}_{ex, 1} \geq \frac{x}{2}) \leq e^{-\frac{x^2}{8\sigma^2}}.$$
Hence, the integral in the second term in (A.22) can be upper bounded by
\[ \sigma^{2D_1T_1}\Gamma(D_1, T_1)^{2D_1T_1 - 2}\int_0^{2R} \frac{e^{-\frac{x^2}{2\sigma^{2D_1T_1}}}x^{2D_1T_1 - 1}}{\sigma^{2D_1T_1}\Gamma(D_1, T_1)^{2D_1T_1 - 2}}dx \]
where \( \int_0^{2R} \frac{e^{-\frac{x^2}{2\sigma^{2D_1T_1}}}x^{2D_1T_1 - 1}}{\sigma^{2D_1T_1}\Gamma(D_1, T_1)^{2D_1T_1 - 2}}dx = Pr(\|\tilde{n}_{ex}\| \leq 2R) \leq 1 \). As a result we get the following upper bound
\[ \int_0^{2R} Pr(\tilde{n}_{ex,1} > \frac{x}{2})x^{2D_1T_1 - 1}dx \leq \sigma^{2D_1T_1}\Gamma(D_1, T_1)^{2D_1T_1 - 2}. \]
By assigning this upper bound in the second term of (A.22) we get
\[ \gamma_{rc}Q(D_1, T_1) \int_0^{2R} Pr(\tilde{n}_{ex,1} > \frac{x}{2})x^{2D_1T_1 - 1}dx \leq \gamma_{rc}\sqrt{\pi}^{2D_1T_1}2^{2D_1T_1-1}\Gamma(D_1, T_1)^{2D_1T_1 - 2} = \rho^{T_1(D_1-r) + \sum_{i=1}^{D_1} \eta_i} \frac{4^{D_1T_1}}{2e^{D_1T_1}}. \]

Next we upper bound \( Pr(\|\tilde{n}_{ex}\| \geq R) \), the first term in (A.22). We choose
\[ R^2 = R^2_{eff} = \frac{2D_1T_1}{\pi} \gamma_{rc} \frac{1}{\gamma_{rc}} = \frac{2D_1T_1}{\pi} \rho^{T_1(D_1-r) + \sum_{i=1}^{D_1} \eta_i} \frac{4^{D_1T_1}}{2e^{D_1T_1}}. \]
For \( \eta \in \mathcal{A} \) we get that
\[ \frac{R^2_{eff}}{2D_1T_1 \sigma^2} = \rho^{1 - \sum_{i=1}^{D_1} \eta_i} \frac{\eta_i}{\sigma^{2D_1T_1}} \geq 1. \]
By using the upper bounds from [20], we know that for the case \( \frac{R^2_{eff}}{2D_1T_1 \sigma^2} \geq 1 \), \( Pr(\|\tilde{n}_{ex}\| \geq R_{eff}) \leq e^{-\frac{R^2_{eff}}{2\sigma^2}(\frac{R^2_{eff}}{2D_1T_1 \sigma^2})}^{D_1T_1} \). Hence we get
\[ Pr(\|\tilde{n}_{ex}\| \geq R_{eff}) \leq e^{-D_1T_1\rho^{1 - \sum \eta_i} \frac{\eta_i}{\sigma^{2D_1T_1}} \cdot \rho^{T_1(D_1-r) + \sum_{i=1}^{D_1} \eta_i} \cdot e^{D_1T_1}. \]
The fact that \( \eta \in \mathcal{A} \) has two significant consequences: the VNR is greater or equal to 1, and as \( \rho \) increases the maximal VNR in the set also increases. For very large VNR at the receiver, the upper bound of the first term, (A.25), is negligible compared to the upper bound on the second term, (A.24). On the other hand, the set of rather small VNR values is fixed for increasing \( \rho \) (the VNR is greater or equal to 1). Hence there must exist a coefficient \( F'(D_1, T_1) \) that gives us
\[ \overline{P_e^{FC}}(\rho, \eta) \leq F'(D_1, T_1)\rho^{-T_1(D_1-r) + \sum_{i=1}^{D_1} \eta_i}. \]
for any \( \rho \) and \( \eta \in \mathcal{A} \), where \( \overline{P_e^{FC}}(\rho, \eta) \) is the average decoding error probability of the ensemble of constellations, for a certain channel realizations.

Note that we could also take \( R \geq R_{eff} \), as the upper bound in (A.24) does not depend on \( R \) and the upper bound in (A.25) would only decrease in this case. It results from the fact that we are interested in the
exponential behavior of the error probability, and we consider a fixed VNR (as a function of $\rho$) as an outage event. This allows us to take cruder bounds than [20] in (A.24), that do not depend on $R$.

For the case $\eta \in \mathcal{A}$, we get
\[
\rho^{-T_i(D_i-r)} + \sum_{i=1}^{D_i} \eta_i \geq 1.
\]
Hence, we can upper bound the error probability for $\eta \in \mathcal{A}$ by 1. We can also upper bound the error probability for this case by the upper bound from equation (A.26), as long as we state that $F'(D_i T_i) \geq 1$. Hence, the upper bound from (A.26) applies to $\eta_i \geq 0, 1 \leq i \leq D_i \cdot T_i$.

So far we upper bounded the average decoding error probability of the ensemble of finite constellations. We extend now these finite constellations into an ensemble of IC’s with density $\gamma_{tr}$, and show that the upper bound on the average decoding error probability does not change. Let us consider a certain finite constellation, $C_0(\rho, b) \subset \text{cube}_{D_i \cdot T_i}(b)$, from the random ensemble. We extend it into IC

\[
IC(\rho, D_i \cdot T_i) = C_0(\rho, b) + (b + b') \cdot \mathbb{Z}^{2D_i \cdot T_i}
\]

where without loss of generality we assumed that $\text{cube}_{D_i \cdot T_i}(b) \in \mathbb{C}^{D_i \cdot T_i}$. At the receiver we have

\[
IC(\rho, D_i \cdot T_i, H_{\text{eff}}(l_i)) = H_{\text{eff}}(l_i) \cdot C_0(\rho, b) + (b + b') H_{\text{eff}}(l_i) \cdot \mathbb{Z}^{2D_i \cdot T_i}.
\]

By extending each finite constellation in the ensemble into an IC according to the method presented in (A.27), we get a new ensemble of IC’s. We would like to set $b$ and $b'$ to be large enough such that the IC’s ensemble average decoding error probability has the same upper bound as in (A.26), and a density that equals $\gamma_{rc}$ up to a coefficient. First we would like to set a value for $b'$. Increasing $b'$ decreases the error probability inflicted by the codewords outside the set $\{H_{\text{eff}}(l_i) \cdot C_0(\rho, b)\}$. Without loss of generality, we upper bound the error probability of the points $x \in \{H_{\text{eff}}(l_i) \cdot C_0(\rho, b)\} \subset IC(\rho, D_i \cdot T_i, H_{\text{eff}}(l_i))$, denoted by $P_e IC(H_{\text{eff}}(l_i) \cdot C_0)$. Due to the tiling symmetry, $P_e IC(H_{\text{eff}}(l_i) \cdot C_0)$ is also the average decoding error probability of the entire IC. We begin with $\eta \in \mathcal{A}$. For this case, we upper bound the IC error probability in the following manner

\[
P_e IC(H_{\text{eff}}(l_i) \cdot C_0) \leq P_e FC(H_{\text{eff}}(l_i) \cdot C_0) + P_e (H_{\text{eff}}(l_i) \cdot (IC \setminus C_0))
\]

where $P_e FC(H_{\text{eff}}(l_i) \cdot C_0)$ is the error probability of the finite constellation $\{H_{\text{eff}}(l_i) \cdot C_0\}$, and $P_e (H_{\text{eff}}(l_i) \cdot (IC \setminus C_0))$ is the average decoding error probability to points in the set $\{H_{\text{eff}}(l_i) \cdot (IC \setminus C_0)\}$. For the case $\eta \in \mathcal{A}$, we know that $0 \leq \eta_i \leq T_i(D_i - r)$. Hence, the constriction caused by the channel in each dimension can not be smaller than $\rho^{-\frac{T_i}{2}(D_i-r)}$. As a result, for any $x_1 \in \{H_{\text{eff}}(l_i) \cdot C_0\}$ and $x_2 \in \{H_{\text{eff}}(l_i) \cdot (IC \setminus C_0)\}$ we get $\|x_1 - x_2\| \geq 2b' \cdot \rho^{-\frac{T_i}{2}(D_i-r)}$. By choosing $b' = \sqrt{\frac{T_i}{\pi \epsilon}} \rho^{\frac{T_i}{2}(D_i-r)+\epsilon}$, we get for $\eta \in \mathcal{A}$ that $\|x_1 - x_2\| \geq 2\sqrt{\frac{T_i \cdot \pi \epsilon}{\pi \epsilon}} \rho^\epsilon$. Hence we get

\[
P_e (H_{\text{eff}}(l_i) \cdot (IC \setminus C_0)) \leq Pr(\|\tilde{I}_{\text{ex}}\| \geq \sqrt{\frac{T_i \cdot \pi \epsilon}{\pi \epsilon}} \rho^\epsilon).
\]

100
For \( \rho \geq 1 \) we get according to the bounds in [20] that

\[
Pr(\|\tilde{n}_{ex}\| \geq \sqrt{D_l \cdot T_l \rho^\epsilon}) \leq e^{-D_l \cdot T_l \rho^{1+\epsilon}} \rho^{D_l \cdot T_l (1+\epsilon)} e^{D_l \cdot T_l}.
\]

As a result, there exists a coefficient \( F''(D_l \cdot T_l) \) such that

\[
P_e(H^{(l)}_{eff} : (IC \setminus C_0)) \leq F''(D_l \cdot T_l) \rho^{T_l(D_l-\rho)+\sum_{i=1}^{D_l \cdot T_l} \eta_i}
\]

for \( \eta \in A \) and \( \rho \geq 1 \). This bound applies to any IC in the ensemble. From (A.26) we can state that

\[
P_e(\rho, \eta) = E_{C_0} (P_{IC} (H^{(l)}_{eff} : C_0)) \leq F'(D_l \cdot T_l) \rho^{-T_l(D_l-\rho)+\sum_{i=1}^{D_l \cdot T_l} \eta_i}.
\]

Hence

\[
P_e(\rho, \eta) \leq F(D_l \cdot T_l) \rho^{-T_l(D_l-\rho)+\sum_{i=1}^{D_l \cdot T_l} \eta_i} \quad (A.29)
\]

where \( F(D_l \cdot T_l) = E_{C_0} (P_{IC} (H^{(l)}_{eff} : C_0)) \) is the average decoding error probability of the ensemble of IC’s defined in (A.28), and \( F = 2 \max(F', F'') > 1 \).

Next, we set the value of \( b \) to be large enough such that each IC density from the ensemble in (A.28), \( \gamma'_{rc} \), equals \( \gamma_{rc} \) up to a factor of 2. By choosing \( b = b' \cdot \rho' \) we get

\[
\gamma'_{rc} = \gamma_{rc}\left(\frac{b}{b+b'}\right)^{2D_l T_l} = \gamma_{rc}\left(\frac{1}{1+\rho^{-\epsilon}}\right).
\]

For each value \( \rho \geq 1 \), we get \( \frac{1}{2} \gamma_{rc} \leq \gamma'_{rc} \leq \gamma_{rc} \). As a result we have

\[
\mu_{rc} \leq \mu'_{rc} = \frac{(\gamma'_{rc})^{-\frac{1}{2}}}{2\pi e \sigma^2} \leq 2\mu_{rc}.
\]

Note that in our proof we referred to a matrix of dimension \( NT_l \times D_l \cdot T_l \). However, these results apply to any full rank matrix with number of rows which is greater or equal to the number of columns.

### A.5 Proof of Theorem 2.4

Specifically, we first lower bound the contribution of \( h_j \) to the determinant (2.33), by upper bounding \( \sum_{k=0}^{\min(j,L)-1} b_j(k) a(k+i) \). Based on Lemma 2.1, and the fact that when two columns of \( H \) occur together in a block of \( H^{(l)}_{eff} \), all the columns of \( H \) between them must also occur in the same block, we get

\[
\sum_{s=k}^{\min(j,L)-1} b_j(s) \leq N - k \quad 0 \leq k \leq \min(j,L) - 1.
\]

where \( \sum_{s=k}^{\min(j,L)-1} b_j(s) \) is the number of occurrences of \( \{h_j, \ldots, h_{j-k}\} \) in the blocks of \( H^{(l)}_{eff} \). Hence, we can state that

\[
\sum_{s=0}^{\min(j,L)-1} b_j(s) \leq N
\]
by assigning \( k = 0 \) in (A.30). Also note that for \( l = 0 \), the sum \( \sum_{s=0}^{\min(j,L)} b_j(s)a(s,\xi_j) \) is larger than for any other \( 1 \leq l \leq L - 1 \). From the inequalities in (2.31), and the fact that for \( l = 0 \) we get \( b_j(k) > 0 \) for any \( 1 \leq k \leq \min(j,L) - 1 \), we can state that

\[
\min(j,L) - 1 \sum_{s=0}^{\min(j,L)} b_j(s)a(s,\xi_j) \leq \sum_{s=0}^{\min(j,L) - 2} a(s,\xi_j) + (N - \min(j,L) + 1)a(\min(j,L) - 1,\xi_j) = c(j). \tag{A.31}
\]

Using (2.33) and (A.31) we can state that for a vector \( \xi_j \), whose PDF is proportional to \( \rho^{-\sum_{i=1}^{N} \xi_{i,j}} \), we can lower bound the contribution of \( h_j \) to \( |H_{eff}^{(l)} H_{eff}^{(l)^\dagger}| \) by

\[
\|h_j\|^{2b_j(0)} \prod_{k=1}^{\min(j,L) - 1} \|h_{j\perp j-1,...,j-k}\|^{2b_j(k)} \geq \rho^{-c(j)}. \tag{A.32}
\]

By taking into account the contribution of each column \( h_j \) to the determinant we get that

\[
|H_{eff}^{(l)} H_{eff}^{(l)^\dagger}| = \prod_{j=1}^{M} \|h_j\|^{2b_j(0)} \prod_{k=1}^{\min(j,L) - 1} \|h_{j\perp j-1,...,j-k}\|^{2b_j(k)}. \tag{A.33}
\]

By considering the set of vectors \( \xi_1, \ldots, \xi_M \), whose PDF is proportional to \( \rho^{-\sum_{j=1}^{M} \sum_{i=1}^{N} \xi_{i,j}} \), and by using the lower bound from (A.32) we get

\[
|H_{eff}^{(l)} H_{eff}^{(l)^\dagger}| \geq \rho^{-\sum_{j=1}^{M} c(j)} \tag{A.34}
\]

The upper bound on the error probability presented in Theorem 2.3 is proportional to

\[
\rho^{-T_l(D_l - r)} |H_{eff}^{(l)} H_{eff}^{(l)^\dagger}|^{-1} = \rho^{-T_l(D_l - r) + \sum_{i=1}^{D_l} \eta_i} \tag{A.35}
\]

for \( \eta_i \geq 0 \) and \( 1 \leq i \leq D_l \cdot T_l \), where \( \rho^{-\eta_i} \) are the singular values of \( H_{eff}^{(l)} \). Hence, in order to use the upper bound from Theorem 2.3 in our analysis, we need to show that by taking \( \xi_{i,j} \geq 0 \), \( 1 \leq i \leq N \), \( 1 \leq j \leq M \) we also get that \( \eta_i \geq 0 \), \( 1 \leq i \leq D_l \cdot T_l \). Note that the entries of \( H_{eff}^{(l)} \) are elements of the channel matrix \( H \). Also, all the columns of \( H \) must appear in \( H_{eff}^{(l)} \). Hence, from trace considerations we get

\[
\rho^{-\min_{i,j}(\xi_{i,j})} \leq \rho^{-\min_{s}(\eta_s)} \leq N \cdot D_l \cdot T_l^2 \rho^{-\min_{i,j}(\xi_{i,j})}.
\]

As a result \( \min_{i,j}(\xi_{i,j}) \geq 0 \) if and only if \( \min_{s}(\eta_s) \geq 0 \), and so \( \eta_s \geq 0 \) for every \( 1 \leq s \leq D_l \cdot T_l \). As the upper bound on the error probability in (A.35) applies to \( \eta_i \geq 0 \), \( 1 \leq i \leq D_l \cdot T_l \), this upper bound also applies whenever \( \xi_{i,j} \geq 0 \), \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \). In equation (A.34) we found a lower bound on the determinant. We use this lower bound to upper bound the determinant of the matrix inverse \( |H_{eff}^{(l)} H_{eff}^{(l)^\dagger}|^{-1} \)

\[
|H_{eff}^{(l)} H_{eff}^{(l)^\dagger}|^{-1} \leq \rho^{-\sum_{j=1}^{M} c(j)} \tag{A.36}
\]
and as a consequence we can upper bound the error probability.

We can express the average decoding error probability over the ensemble of IC’s for large \( \rho \) as follows

\[
\mathbb{P}_e(\rho) = \int_{H} P_e(\rho, H) dH = \int_{\xi_{\mathbb{L}} \geq 0} P_e(\rho, \xi_{\mathbb{L}}) f(\xi_{\mathbb{L}}) d\xi_{\mathbb{L}} \tag{A.37}
\]

where \( P_e(\rho, H) = P_e(\rho, \xi_{\mathbb{L}}) \) is the ensemble average decoding error probability per channel realization, and \( \xi_{\mathbb{L}} \geq 0 \) means \( \xi_{i,j} \geq 0 \) for \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \). We divide the integration range into two sets:

\[ A = \{ \xi_{\mathbb{L}} \mid \sum_{i=1}^{N} \sum_{j=1}^{M} \xi_{i,j} \leq T_l(D_l - r); \xi_{\mathbb{L}} \geq 0 \} \]

\[ A = \{ \xi_{\mathbb{L}} \mid \sum_{i=1}^{N} \sum_{j=1}^{M} \xi_{i,j} > T_l(D_l - r); \xi_{\mathbb{L}} \geq 0 \}. \]

Hence, we can write the average decoding error probability as follows

\[
\mathbb{P}_e(\rho) = \int_{\xi_{\mathbb{L}} \in A} P_e(\rho, \xi_{\mathbb{L}}) f(\xi_{\mathbb{L}}) d\xi_{\mathbb{L}} + \int_{\xi_{\mathbb{L}} \in A} P_e(\rho, \xi_{\mathbb{L}}) f(\xi_{\mathbb{L}}) d\xi_{\mathbb{L}} \tag{A.38}
\]

We begin by upper bounding the first term of the error probability in (A.38). Based on Theorem 2.3, the average decoding error probability per channel realization is upper bounded by

\[
P_e(\rho, H) \leq \rho^{\frac{-T_l(D_l - r) + \sum_{i=1}^{N} \xi_{i,j}}{\eta}} \]

Using the upper bound on the determinant (A.36) and the fact that \( |H_{\text{eff}}^{(l)} H_{\text{eff}}^{(l)}|^\dagger = \rho^{\sum_{i=1}^{N} \xi_{i,j}} \eta \), we get that the first term of the error probability (A.38) is upper bounded by

\[
\int_{\xi_{\mathbb{L}} \in A} \rho^{\frac{-T_l(D_l - r) + \sum_{j=1}^{M} (c(j) - \sum_{i=1}^{N} \xi_{i,j})}{\eta}} d\xi_{\mathbb{L}}. \tag{A.39}
\]

Now we prove a Lemma that shows that the exponent of the integrand in the upper bound from (A.39) is negative for \( \xi_{\mathbb{L}} \geq 0 \).

**Lemma A.2.** consider \( \xi_{i,j} \geq 0 \) for \( 1 \leq i \leq N \) and \( 1 \leq j \leq M \). The sum

\[
c(j) - \sum_{i=1}^{N} \xi_{i,j} \leq 0
\]

for every \( 1 \leq j \leq M \).

**Proof.** See appendix A.6. \( \square \)

In a similar manner to [35], [6], for a very large \( \rho \) and a finite integration range, we can approximate the integral by finding the most dominant exponential term in (A.39). Based on Lemma A.2 we know that the exponent of the integrand is always negative. Hence, we can approximate the upper bound by finding

\[
\min_{\xi_{\mathbb{L}} \in A} T_l(D_l - r) + \sum_{j=1}^{M} (\sum_{i=1}^{N} \xi_{i,j} - c(j)).
\]

As \( \sum_{i=1}^{N} \xi_{i,j} - c(j) \geq 0 \) the minimum is achieved when \( \sum_{i=1}^{N} \xi_{i,j} - c(j) = 0 \) for \( 1 \leq j \leq M \). This can be achieved for instance by taking \( \xi_{i,j} = 0 \) for \( 1 \leq i \leq N, 1 \leq j \leq M \). In this case we get that the diversity order equals \( T_l(D_l - r) \) which is the best diversity order possible for IC’s of complex dimension \( D_l \cdot T_l \).
Next we upper bound the second term of the error probability from (A.38). For \( \xi_{i,j} \in \overline{\mathbb{A}} \) we upper bound the average decoding error probability per channel realization by 1. In this case we get
\[
\int_{\xi_{i,j} \in \overline{\mathbb{A}}} \rho^{-\sum_{j=1}^{M} \sum_{i=1}^{N} \xi_{i,j}} d\xi_{i,j}.
\]
Again we approximate this integral by calculating the most dominant exponential term, i.e. by calculating
\[
\min_{\xi_{i,j} \in \overline{\mathbb{A}}} \sum_{i=1}^{N} \sum_{j=1}^{M} \xi_{i,j}.
\]
The minimal value for this case is also \( T_l(D_l - r) \). Hence, we get a diversity order \( T_l(D_l - r) \) for the second term. As a result we can state that for both terms in (A.38) we get the same diversity order, and the transmission scheme diversity order is upper bounded by \( T_l(D_l - r) \). The proof is concluded.

### A.6 Proof of Lemma A.2

We know that
\[
c(j) = \sum_{s=0}^{\min(j,L)-2} a(s, \xi_j) + (N - \min(j,L) + 1)a(\min(j,L) - 1, \xi_j)
\]
where
\[
a(k, \xi_j) = \min_{s \in \{k+1, \ldots, N\}} \xi_{s,j} \quad 0 \leq k \leq \min(j,L) - 1
\]
and by definition
\[
a(\min(j,L) - 1, \xi_j) \geq \cdots \geq a(0, \xi_j) \geq 0.
\]
In order to prove the Lemma we begin with \( a(\min(j,L) - 1, \xi_j) \). We know that
\[
\sum_{s=\min(j,L)}^{N} \xi_{s,j} \geq (N - \min(j,L) + 1) \cdot \min_{s} \xi_{s,j} \quad (A.40)
\]
where \( s \in \{ \min(j,L), \ldots, N \} \). We can also see that
\[
\xi_{k+1,j} \geq \min_{s \in \{k+1, \ldots, N\}} \xi_{s,j} \quad (A.41)
\]
for \( 0 \leq k \leq \min(j,L) - 2 \). Hence we get
\[
c(j) - \sum_{i=1}^{N} \xi_{i,j} \leq 0.
\]
This concludes the proof.
A.7 Proof of Theorem 2.5

We prove that there exists a sequence of $2D_l \cdot T_l$-real dimensional lattices (as a function of $\rho$) that attains the same diversity order as in Theorem 2.4. By using the Minkowski-Hlawaka-Siegel Theorem [20],[11], we upper bound the error probability of the ensemble of lattices, for each channel realization. This upper bound equals to the upper bound derived in Theorem 2.3. Then we average the upper bound over all channel realizations, and receive the desired diversity order.

We consider a $2D_l \cdot T_l$-real dimensional ensemble of lattices, transmitted using the transmission scheme defined in Subsection 2.4.1. We spread the first $D_l \cdot T_l$ dimensions of the lattice on the real part of the non-zero entries of $G_l$, and the other $D_l \cdot T_l$ dimensions of the lattice on the imaginary part of the non-zero entries of $G_l$. Each lattice in the ensemble has transmitter density $\gamma_{tr} = \rho^{rT_l}$, i.e. multiplexing gain $r$. We begin by analyzing the performance of the ensemble of lattices at the receiver, for each channel realization. We assume a certain channel realization that induces a receiver VNR $\mu_{rc} = \rho^{1 - \frac{r}{2R} - \sum_{i=1}^{T_l} \frac{m_i}{m_{i+1}}} \cdot \sigma^2$, where $\rho \geq 0$. For each lattice in the ensemble we get that the channel realization induces a new lattice at the receiver, $H_{eff}^{(l)} \cdot \bar{x}$, with density $\gamma_{rc}$ in accordance with (2.5) and Subsection 2.4.2. For lattices with regular lattice decoding, the error probability is equal among all codewords. Hence, it is sufficient to analyze the lattice’s zero codeword error probability. We define the indication function

$$I_{Ball}(0,2R) (x) = \begin{cases} 
1, & \|x\| \leq 2R \\
0, & \text{else}
\end{cases}.$$ 

In a similar manner to (A.19) we can state that for each lattice induced at the receiver, $\Lambda_{rc}$, the lattice zero codeword error probability is upper bounded by

$$\sum_{x \in \Lambda_{rc}, x \neq 0} I_{Ball}(0,2R_{eff}) (x) \cdot Pr(\|\bar{n}_{ex}\| > \|x - \bar{n}_{ex}\|) + Pr(\|\bar{n}_{ex}\| \geq R_{eff})$$

(A.42)

where $\frac{R_{eff}^2}{2D_l \cdot T_l \sigma^2} = \mu_{rc}$, and $\bar{n}_{ex}$ is the effective noise in the $D_l \cdot T_l$-complex hyperplane where $\Lambda_{rc}$ resides in. By defining $f_{rc}(x) = I_{Ball}(0,2R_{eff}) (x) \cdot Pr(\|\bar{n}_{ex}\| > \|x - \bar{n}_{ex}\|)$, we can rewrite the upper bound on the error probability from (A.42)

$$\sum_{x \in \Lambda_{rc}, x \neq 0} f_{rc}(x) + Pr(\|\bar{n}_{ex}\| \geq R_{eff}).$$

(A.43)

Note that

$$\gamma_{rc} \int_{\mathbb{R}^{2D_l \cdot T_l}} f_{rc}(x) dx + Pr(\|\bar{n}_{ex}\| \geq R_{eff})$$

(A.44)

is equal to the expression in (A.22), where $\gamma_{rc}$ is the density of the lattice induced at the receiver $\Lambda_{rc}$, as defined above.

We need to show that there exists a single probability measure for all channel realizations, that gives an average decoding error probability over the ensemble, which is upper bounded by (A.44). Hence, we consider the ensemble of lattices at the transmitter which is fixed for each channel realization. For this
by the upper bound from Theorem 2.3 \((A.29)\). As a result, we can upper bound the ensemble average decoding error probability for each channel realization \(\gamma_{tr} = \rho^{T_i}\). Now we define the following indication function

\[
I_{\text{ellipse}}(H, 2R) (x) = \begin{cases} 
1, & \|H \cdot x\| \leq 2R \\
0, & \text{else}
\end{cases}
\]
that is the function is one if \(x\) is within the ellipse and zero otherwise. Let us denote the error probability of a lattice in the ensemble for certain channel realization \(\eta\) by \(P_e^{(\nu)}(\eta, \rho)\), where \(\nu\) is a random variable that represents a certain lattice in the ensemble. Using regular lattice decoding, we get the following upper bound on the error probability for each lattice codeword

\[
P_e^{(\nu)}(\eta, \rho) \leq \sum_{x \in \Lambda_{tr}} I_{\text{ellipse}}(H_{\text{eff}}) (x) \cdot P_r(\|A \cdot \hat{n}_{\text{eff}} - \|x - \hat{n}_{\text{eff}}\|) + P_r(\|A \cdot \hat{n}_{\text{eff}}\| \geq R_{\text{eff}}) \quad (A.46)
\]

where \(A\) is a \(D_l \cdot T_i \times D_l \cdot T_i\) matrix that satisfies \(A^\dagger A = H_{\text{eff}}^\dagger H_{\text{eff}}, \Lambda_{tr}\) is the lattice from the ensemble that corresponds to \(\nu\) and \(\hat{n}_{\text{eff}} \sim \mathcal{CN}(0,(H_{\text{eff}}^\dagger H_{\text{eff}})^{-1})\). Note that (A.46) is equal to (A.43), and the corresponding terms in the expressions are also equal.

Let us define \(g_{rc}(x) = I_{\text{ellipse}}(H_{\text{eff}}) (x) \cdot P_r(\|A \hat{n}_{\text{eff}}\| > \|A (x - \hat{n}_{\text{eff}})\|)\). We get that

\[
\gamma_{tr} \int_{\mathbb{R}^{2D_l \cdot T_i}} g_{rc}(x) d\bar{x} = \gamma_{rc} \int_{\mathbb{R}^{2D_l \cdot T_i}} f_{rc}(x) d\bar{x} \quad (A.47)
\]

Next we show that by averaging the upper bound in (A.46) over the ensemble of lattices at the transmitter, with the correct probability measure, we get

\[
E_\nu \{P_e^{(\nu)}(\eta, \rho)\} \leq \gamma_{rc} \int_{\mathbb{R}^{2D_l \cdot T_i}} f_{rc}(x) d\bar{x} + Pr(\|\hat{n}_{\text{eff}}\| \geq R_{\text{eff}}). \quad (A.48)
\]

We prove (A.48) by using the Minkowski-Hlawaka-Siegel theorem [20]:

**Theorem A.1.** (Minkowski-Hlawaka-Siegel Theorem) In the set of all the lattices of density \(\gamma\) in \(\mathbb{R}^{2D_l \cdot T_i}\), there exists a probability measure \(\nu\) such that for any Riemann integrable function \(f(x)\) which vanishes outside some bounded region we have

\[
E_\nu \{\sum_{x \in \Lambda} g(x)\} = \gamma \int_{\mathbb{R}^{2D_l \cdot T_i}} g(x) d\bar{x} \quad (A.49)
\]
where \(E_\nu \{\cdot\}\) represents the expectation with respect to the measure \(\nu\).

Note that considering a \(2D_l \cdot T_i\)-real dimensional lattices enables us to use this theorem. Hence, by choosing \(\gamma = \gamma_{tr}\), \(g(x) = g_{rc}(x)\), and considering (A.46), (A.47) we get the desired upper bound (A.48). As a result, we can upper bound the ensemble average decoding error probability for each channel realization by the upper bound from Theorem 2.3 (A.29).
Now we are ready to lower bound the diversity order. According to Theorem A.1 there exists a single probability measure that satisfies (A.49), for any Riemann integrable function that vanishes outside some bounded region. Based on (A.34) and Lemma A.2, we get for the set \( \{ \xi_{i,j} \mid \sum_{i=1}^{N} \sum_{j=1}^{M} \xi_{i,j} \leq T_i(D_l - r); \xi_{i,j} \geq 0 \} \) a set of functions, \( g_{\text{loc}}(\xi) \), which are bounded. As a result we can upper bound the ensemble average decoding error probability for this set by the expression from (A.29). For the set of events \( \{ \xi_{i,j} \mid \sum_{i=1}^{N} \sum_{j=1}^{M} \xi_{i,j} > T_i(D_l - r); \xi_{i,j} \geq 0 \} \) we upper bound the ensemble average decoding error probability by 1. This bounds are the exact same bounds we used in order to average over the channel realizations in Theorem 2.4. Hence, by averaging over the channel realizations we get for the ensemble the same lower bound on the diversity order as in Theorem 2.4. This concludes the proof.

A.8 Proof of Corollary 2.3

Let \( P_e(S(\rho), r) \) denote the average decoding error probability of the IC \( S(\rho) \) with density \( \gamma_{tr} = \rho^T \). Since \( S_{D,T}(\rho) \) has density \( \gamma_{tr} = 1 \) for every \( \rho \), this IC’s sequence has multiplexing gain \( r = 0 \). Hence, in accordance with our definitions, we denote \( S_{D,T}(\rho) \) average decoding error probability by \( P_e(S_{D,T}(\rho), 0) \).

Assume

\[
P_e(S_{D,T}(\rho), 0) = A'(\rho)\rho^{-d}
\]

where \( -\lim_{\rho \to \infty} \log \rho P_e(S_{D,T}(\rho), 0) = d \), i.e. \( S_{D,T}(\rho) \) has diversity order \( d \). By scaling the sequence of IC’s such that

\[
\overline{S}_{D,T}(\rho) = S_{D,T}(\rho) \cdot \rho^{-\frac{r}{D}} \quad 0 \leq r \leq D,
\]

i.e., scaling \( S_{D,T}(\rho) \) by a factor of \( \rho^{-\frac{r}{D}} \), we get that \( \overline{S}_{D,T}(\rho) \) has density \( \gamma_{tr} = \rho^{rT} \), multiplexing gain \( r \) and so its error probability

\[
P_e(\overline{S}_{D,T}(\rho), r) = P_e(S_{D,T}(\rho^{1-rT}), 0) = A'(\rho^{1-rT})\rho^{-d(1-rT)}.
\]

As a result we get \( -\lim_{\rho \to \infty} \log \rho P_e(\overline{S}_{D,T}(\rho), r) = d(1 - \frac{r}{D}) \), i.e. \( \overline{S}_{D,T}(\rho) \) has diversity order \( d(1 - \frac{r}{D}) \).

A.9 Proof of Corollary 2.5

The proof of this corollary relies heavily on Theorem 2.3. We begin by describing the \( L \) ensembles of IC’s and how they are transmitted. Then we use averaging arguments in order to show that there exists a single sequence of IC’s that attains the optimal DMT.

We begin by considering a sequence of \( D_0 \cdot T_0 \)-complex dimensional IC’s with multiplexing gain \( r = 0 \), i.e. the transmitter density \( \gamma_{tr} = 1 \) for any \( \rho \). In a similar manner to Theorem 2.3, we first consider an ensemble of finite constellations drawn uniformly within \( \text{cube}_{D_0 \cdot T_0}(b) \subset \mathbb{C}^{D_0 \cdot T_0} \). Each code-book contains \( \lceil \gamma_{tr} b^{2D_0 \cdot T_0} \rceil \) points, where each point is drawn uniformly within \( \text{cube}_{D_0 \cdot T_0}(b) \). Let us denote a certain finite constellation in the ensemble by \( C_{FC}(\rho, D_0 \cdot T_0, b) \subset \text{cube}_{D_0 \cdot T_0}(b) \). We extend each finite
ensemble in the aforementioned manner, and then reducing its dimensionality by dropping the
where \( l \) gives us a \( D \times d \) of IC’s. By transmitting this ensemble of IC’s on the transmission matrix
where \( \eta \) has multiplexing gain \( r = 0 \). For a certain channel realization \( \eta \geq 0 \) we get in accordance with Theorem
Next we derive from the \( D_0 \times T_0 \)-complex dimensional ensemble of IC’s, another \( D_l \times T_l \)-complex dimensional ensemble of IC’s, where \( l = 1, \ldots, L - 1 \). For each IC, \( IC(\rho, D_0 \cdot T_0) \), in the ensemble we take the first \( \lfloor b^{2D_l \cdot T_l} \rfloor \) points in \( C_{FC}(\rho, D_0 \cdot T_0, b) \). We take the components of these points inside \( cube_{D_l \cdot T_l}(b) \), and denote this new finite constellation as \( C_{FC}(\rho, D_1 \cdot T_1, b) \). Then we replicate these points in a similar
manner to (A.50). In this case we get a new \( D_l \cdot T_l \)-complex dimensional IC
By doing it to each IC in the ensemble, we get a new \( D_l \cdot T_l \)-complex dimensional ensemble of IC’s. This new ensemble is equivalent to ensemble of IC’s generated by drawing uniformly \( \lfloor b^{2D_l \cdot T_l} \rfloor \) points inside \( cube_{D_l \cdot T_l}(b) \), and then replicate these points according to \( (b+b')Z^{2D_l \cdot T_l} \). Each IC sequence in this ensemble has multiplexing gain \( r = 0 \). Since \( b > \sqrt{\frac{D_l \cdot T_l}{\pi e}} \rho \frac{D_l \cdot T_l}{2} +2 \varepsilon \) and \( b' > \sqrt{\frac{D_l \cdot T_l}{\pi e}} \rho \frac{D_l \cdot T_l}{2} +\varepsilon \), we get in accordance with Theorem 2.3 that for a certain channel realization \( \eta \geq 0 \)
where \( \mathcal{P}_e(\rho, \eta, D_l \cdot T_l) \) is the average decoding error probability of the \( D_l \cdot T_l \)-complex dimensional ensemble of IC’s. By transmitting this ensemble of IC’s on the transmission matrix \( G_l \), and averaging over the channel realizations, we get diversity order \( d_{D_l} = (M-l)(N-l) + l(N + M - 2 \cdot l - 1) \). Transmitting over \( G_l \)
gives us a \( D_l \cdot T_l \)-complex dimensional ensemble of IC’s within \( \mathbb{C}^{MT_l} \).
From the sequential structure of the transmission scheme we get that omitting the \( 2 \cdot l \) rightmost columns of \( G_0 \) yields \( G_l \). Hence we can derive from the \( D_0 \times T_0 \)-complex dimensional ensemble of IC’s, that attains diversity order \( d_{D_0} \), another \( D_l \times T_l \)-complex dimensional ensemble of IC’s that attains diversity order \( d_{D_l} \), where \( l = 1, \ldots, L - 1 \). We attain it by diluting the points of each \( D_0 \times T_0 \)-complex dimensional IC in the ensemble in the aforementioned manner, and then reducing its dimensionality by dropping the \( 2 \cdot l \) rightmost columns of \( G_0 \).
So far we have shown the connection between the ensembles. Now we would like to show that there exists a certain sequence of \( D_0 \times T_0 \)-complex dimensional IC’s, that gives us the desired diversity orders
by diluting its points and adapting its dimensionality. We denote the average decoding error probability of the $D_l \cdot T_l$-complex dimensional ensemble of IC’s by $A_l(\rho)\rho^{-d_{D_l}}$, where $\lim_{\rho \to \infty} \frac{\log(A_l(\rho))}{\log(\rho)} = 0$. We also define $I_{l,\rho}$ as the event where a $D_l \cdot T_l$-complex dimensional IC in the ensemble has average decoding error probability which is smaller or equal to $(L + 1)A_l(\rho)\rho^{-d_{D_l}}$, where $l = 0, \ldots, L - 1$. From averaging arguments we know that $\Pr(I_{l,\rho}) \geq \frac{1}{L+1}$. We wish to show that the probability of the event $\{I_{0,\rho} \cap I_{1,\rho} \cap \cdots \cap I_{L-1,\rho}\}$ is bounded away from zero. From averaging arguments we know that

$$\Pr(I_{0,\rho} \cap I_{1,\rho} \cap \cdots \cap I_{L-1,\rho}) \geq 1 - \sum_{i=0}^{L-1} \Pr(I_{i,\rho}) \geq \frac{1}{L+1}.$$ 

Hence there must exist a sequence of $D_0 \cdot T_0$-complex dimensional IC’s that attains diversity order $d_{D_0}$ and has multiplexing gain $r = 0$, from which we can derive for each $l = 1, \ldots, L - 1$, a sequence of $D_l \cdot T_l$-complex dimensional IC’s with multiplexing gain $r = 0$ and diversity order $d_{D_l}$.

Next we show that these $L$ sequences attain the optimal DMT. Consider a sequence of $D_l \cdot T_l$-complex dimensional IC’s, that has multiplexing gain $r = 0$ and attains diversity order $d_{D_l}$. From Corollary 2.3 we know that scaling this sequence by a scalar $\rho^{-\frac{M-1}{2D_l}}$ yields a new sequence of IC’s with multiplexing gain $r$ and diversity order

$$d_{D_l}(r) = (M - l)(N - l) - (r - l)(N + M - 2\cdot l - 1)$$

where $0 \leq r \leq D_l$ and $l = 0, \ldots, L - 1$. Each of the $L$ straight lines $d_{D_l}(r)$, $l = 0, \ldots, L - 1$, coincides with a different segment out of the $L$ segments of the optimal DMT. This concludes the proof.
Appendix B

On the DMT of IC’s in MAC Channels: Proofs

B.1 Proof of Lemma 3.2

The proof outline is as follows. First we show that for finite constellations, the single user DMT is smaller than the contracted optimal DMT of any number of users (up to $K$) pulled together. Then we use this relation, together with the anchor points presented in Corollary 3.1 for the upper bound on IC’s DMT, in order to prove the lemma.

Since $K > 1$ and $M$ are positive integers, we get for $N \geq (K + 1)M - 1$ that $M \leq \frac{N}{K}$, where $1 \leq i \leq K$. Hence for any $d^{*,D}_{i,M,N}(i \cdot r)$, the range of NDCU per user is $0 \leq D \leq \min(M, \frac{N}{K}) = M$, where $1 \leq i \leq K$.

We begin by showing that $d^{*(FC)}_{M,N}(r)$ is smaller or equal to $d^{*(FC)}_{i,M,N}(i \cdot r)$ for $2 \leq i \leq K$, where $d^{*(FC)}_{i,M,N}(i \cdot r)$ is the optimal DMT of finite constellations contracted by $i$, in a point-to-point channel with $i \cdot M$ transmit and $N$ receive antennas. For the case $N > (K + 1)M - 1$ we get that $\frac{N}{K+1} \geq M$. Hence we also get that $\frac{N}{i+1} \geq M$ for $1 \leq i \leq K$. Hence from Theorem 3.3 we can see that

$$d^{*(FC)}_{M,N}(r) \leq d^{*(FC)}_{i,M,N}(i \cdot r) \quad 2 \leq i \leq K$$

(B.1)

by replacing $K$ with $i$.

For $N = (K + 1)M - 1$ we still get that $\frac{N}{i+1} \geq M$ for $1 \leq i \leq K - 1$, and again based on Theorem 3.3

$$d^{*(FC)}_{M,N}(r) \leq d^{*(FC)}_{i,M,N}(i \cdot r) \quad 2 \leq i \leq K - 1.$$  (B.2)

There remains a case of $i = K$. We can see that for $N = (K + 1)M - 1$ we get $M - \frac{1}{K} \leq \frac{N}{K+1} \leq M$. Hence we get from Theorem 3.3

$$d^{*(FC)}_{M,N}(r) \leq d^{*(FC)}_{K,M,N}(K \cdot r) \quad 0 \leq r \leq M - \frac{1}{K}.$$  (B.3)
For $M - \frac{1}{K} \leq r \leq M$ both $d_{M,N}^{\ast, (FC)}(r)$ and $d_{K,M,N}^{\ast, (FC)}(K \cdot r)$ are on the last straight line of the piecewise linear functions. By simply assigning $N = (K + 1) M - 1$ we get for $M - \frac{1}{K} \leq r \leq M$

$$d_{M,N}^{\ast, (FC)}(r) = d_{K,M,N}^{\ast, (FC)}(K \cdot r) = KM (M - r). \quad (B.4)$$

From (B.1)-(B.4) we get for $N \geq (K + 1) M - 1$ and $0 \leq r \leq M$ that

$$d_{M,N}^{\ast, (FC)}(r) \leq d_{i,M,N}^{\ast, (FC)}(i \cdot r) \quad 2 \leq i \leq K. \quad (B.5)$$

So far we have proved the relation between the contracted optimal DMT of finite constellations with different number of users pulled together. Next we wish to use it in order to prove the relation between $d_{i,M,N}^{\ast, i-D}(i \cdot r)$ for $1 \leq i \leq K$. In Corollary 2.2 it was shown that for $0 < D \leq \min (M, N)$

$$d_{M,N}^{\ast, D}(r) \leq d_{M,N}^{\ast, (FC)}(r) \quad 0 \leq r \leq D. \quad (B.6)$$

On the other hand from Corollary 3.1 we can see that

$$d_{i,M,N}^{\ast, i-D}(l) = d_{i,M,N}^{\ast, (FC)}(l) = (i \cdot M - l) (N - l) \quad 1 \leq i \leq K \quad (B.7)$$

for $l = 0$ when $0 \leq i \cdot D \leq i \cdot \frac{MN}{i + (M + N - 1)}$, and also for $l = 1, \ldots, i \cdot M - 1$ when $i \cdot \frac{MN - l(l-1)}{i + (M + N - 1 - 2l)} \leq i \cdot D \leq i \cdot \frac{MN - (l+1)(l)}{i + (M + N - 1 - 2l)}$. Hence based on (B.5)-(B.7), and the fact that $d_{i,M,N}^{\ast, i-D}(i \cdot r)$ is a contraction of $d_{i,M,N}^{\ast, (FC)}(r)$ for $2 \leq i \leq K$ we get

$$d_{i,M,N}^{\ast, i-D}(0) \geq d_{i,M,N}^{\ast, D}(0) \quad 2 \leq i \leq K \quad (B.8)$$

for $0 \leq D \leq \frac{MN}{i + (M + N - 1)}$, and

$$d_{i,M,N}^{\ast, i-D}(l) \geq d_{i,M,N}^{\ast, D}(l) \quad 2 \leq i \leq K \quad (B.9)$$

for $l = 1, \ldots, i \cdot M - 1$ and $\frac{MN - l(l-1)}{i + (M + N - 1 - 2l)} \leq D \leq \frac{MN - (l+1)(l)}{i + (M + N - 1 - 2l)}$. Since $d_{i,M,N}^{\ast, i-D}(i \cdot r), 1 \leq i \leq K$, are straight lines as a function of $r$, and also all of these straight lines are equal zero for $r = D$, i.e. $d_{i,M,N}^{\ast, i-D}(i \cdot D) = 0$ for $1 \leq i \leq K$, the inequalities in (B.8), (B.9) leads to

$$d_{i,M,N}^{\ast, D}(r) \leq d_{i,M,N}^{\ast, i-D}(i \cdot r) \quad 2 \leq i \leq K$$

for any $0 \leq D \leq M$ and $0 \leq r \leq D$. This concludes the proof.

**B.2 Proof of Lemma 3.3**

First note that $\frac{N}{i+1} \geq \frac{L}{K}$ for $1 \leq i \leq K - 1$. Hence from Theorem 3.3 we get that

$$d_{M,N}^{\ast, (FC)}(r) \leq d_{i,M,N}^{\ast, (FC)}(i \cdot r) \quad 2 \leq i \leq K - 1 \quad (B.10)$$

112
for $0 \leq r \leq \frac{L}{K}$. Based on (B.6), (B.7), (B.10) and Corollary 3.1 we get that

$$d^*_{i,M,N} (0) \geq D_{M,N} (0) \quad 2 \leq i \leq K - 1$$

for $0 \leq D \leq \frac{MN}{1+M+N-1}$, and

$$d^*_{i,M,N} (l) \geq D_{M,N} \left( \frac{l}{i} \right) \quad 2 \leq i \leq K - 1$$

for $l = 1, \ldots, i \cdot M - 1$ and $\frac{MN-l(l-1)}{1+M+N-1} \leq D \leq \frac{MN-l(l+1)}{1+M+N-1}$. Again, since $d^*_{i,M,N} (i \cdot r), 1 \leq i \leq K$, are straight lines as a function of $r$, and also all of these straight lines are equal to zero for $r = D$, the inequalities in (B.11), (B.12) lead to

$$d^*_{M,N} (r) \leq d^*_{i,M,N} (i \cdot r) \quad 2 \leq i \leq K - 1$$

for any $0 \leq D \leq \frac{L}{K}$ and $0 \leq r \leq D$.

### B.3 Proof of Lemma 3.4

Since $M \geq 1$ we get for $N < (K - 1) M + 1$ that $L = \frac{N}{K}$. Hence we can consider the range $0 \leq r \leq \frac{N}{K}$. We begin the proof by showing that for $N < (K - 1) M + 1$, $d^*_{M,N} (r)$ is inferior compared to $d^*_{K,M,N} (K \cdot r)$, for any $0 \leq D \leq \frac{N}{K}$. Then we show that the maximization over $d^*_{M,N} (r)$ yields $M \cdot N = M \cdot K \cdot r$.

We begin by showing that

$$d^*_{M,N} (r) \leq d^*_{K,M,N} (K \cdot r) \quad 0 \leq D \leq \frac{N}{K}$$

for $0 \leq r \leq D$. By assigning $D = \frac{N}{K}$ in $d^*_{K,M,N} (K \cdot r)$ we get

$$d^*_{K,M,N} (K \cdot r) = (K \cdot M - N + 1) \cdot (N - Kr).$$

Since $N < (K - 1) M + 1$ we get

$$d^*_{K,M,N} (0) = (K \cdot M - N + 1) \cdot N > M \cdot N. \quad (B.13)$$

From Corollary 3.1 we get that

$$d^*_{K,M,N} (0) \leq d^*_{K,M,N} (0) \quad 0 \leq D \leq \frac{N}{K} \quad (B.14)$$

and also

$$d^*_{M,N} (0) \leq M \cdot N \quad 0 \leq D \leq \frac{N}{K} \quad (B.15)$$

Since $d^*_{i,M,N} (i \cdot r), 1 \leq i \leq K$ are straight lines as a function of $r$, that equal to zero for $r = D$, and also
based on (B.13), (B.14), (B.15) and Lemma 3.3 we get
\[
d^{*,D}_{M,N} (r) \leq d^{*,D}_{i-M,N} (i \cdot r) \quad 1 \leq i \leq K
\] (B.16)
for any \(0 \leq D \leq \frac{N}{K}\) and \(0 \leq r \leq D\). Hence the optimization problem takes the following form
\[
\max_D \min_{1 \leq i \leq K} d^{*,D}_{i-M,N} (i \cdot r) = \max_D d^{*,D}_{M,N} (r) \quad 0 \leq r \leq \frac{N}{K}.
\] (B.17)

For \(N < (K-1)M + 1\) we get that \(\frac{N}{K} < \frac{MN}{N+M-1}\). Also, from Corollary 3.1 we get that \(d^{*,D}_{M,N} (0) = M \cdot N\) for \(0 \leq D \leq \frac{MN}{N+M-1}\). Hence, in the range \(0 \leq D \leq \frac{N}{K}\) we get a set of straight lines as a function of \(r\), \(d^{*,D}_{M,N} (r)\), where \(d^{*,D}_{M,N} (0) = MN\) and \(d^{*,D}_{M,N} (D) = 0\). As a result the maximal value for each \(r\) is attained for \(D = \frac{N}{K}\), and equals
\[
\max_D d^{*,D}_{M,N} (r) = d^{*,\frac{N}{K}}_{M,N} (r) = MN - KMr \quad 0 \leq r \leq \frac{N}{K}.
\] (B.18)

### B.4 Proof of Lemma 3.5

The outline of the proof is as follows. We begin by finding the straight line that equals \(d^{*,(FC)}_{M,N} (\lfloor \frac{b}{2} \rfloor + 1)\) when \(r = \lfloor \frac{b}{2} \rfloor + 1\), and also equals \(d^{*,(FC)}_{K,M,N} ((K-1)M + \lfloor \frac{l+1}{2} \rfloor)\) for \(r = \frac{(K-1)M + \lfloor \frac{l+1}{2} \rfloor}{K}\). Then we show that the NDCU per user, \(D_i\), corresponding to this straight line fulfills Corollary 3.1, i.e. for \(d^{*,D}_{M,N} (r)\), \(D_i\) is in the range of NDCU that rotate around the anchor point \(d^{*,(FC)}_{M,N} (\lfloor \frac{b}{2} \rfloor + 1)\), and also for \(d^{*,K-D}_{K,M,N} (K \cdot r)\), \(D_i\) is in the range of NDCU that rotate around the anchor point \(d^{*,(FC)}_{K,M,N} (K \cdot \frac{(K-1)M + \lfloor \frac{l+1}{2} \rfloor}{K})\). By showing that the straight line fulfills Corollary 3.1 for both cases, we get that the straight line equals \(d^{*,D}_{M,N} (r)\) and also \(d^{*,K-D}_{K,M,N} (K \cdot r)\).

Let us denote the straight line by
\[
d^{*} (r) = MN - \lfloor \frac{b}{2} \rfloor \cdot (\lfloor \frac{l}{2} \rfloor + 1) - 2 \cdot (\lfloor \frac{l}{2} \rfloor + 1) \cdot \left( \frac{l}{2} - \lfloor \frac{l}{2} \rfloor \right) - (N + M - 1 - l) r.
\]

First we wish to show that
\[
d^{*} (\lfloor \frac{l}{2} \rfloor + 1) = d^{*,(FC)}_{M,N} (\lfloor \frac{l}{2} \rfloor + 1), \quad \text{and also that} \quad d^{*} \left( \frac{(K-1)M + \lfloor \frac{l+1}{2} \rfloor}{K} \right) = d^{*,(FC)}_{K,M,N} ((K-1)M + \lfloor \frac{l+1}{2} \rfloor).
\]

By simply assigning \(r = \lfloor \frac{b}{2} \rfloor + 1\) we get
\[
d^{*} \left( \lfloor \frac{l}{2} \rfloor + 1 \right) = \left( N - \lfloor \frac{l}{2} \rfloor - 1 \right) \cdot \left( M - \lfloor \frac{l}{2} \rfloor - 1 \right) = d^{*,(FC)}_{M,N} \left( \lfloor \frac{l}{2} \rfloor + 1 \right). \quad \text{(B.19)}
\]

For \(r = \frac{(K-1)M + \lfloor \frac{l+1}{2} \rfloor}{K}\) we consider two cases. In the first case assume \(l = 2b\), i.e. \(l\) is even. Under this assumption \(\lfloor \frac{l+1}{2} \rfloor = \lfloor \frac{b}{2} \rfloor = b\), and so \(r = \frac{(K-1)M + b}{K}\). By assigning \(KM = N + M - 1 - 2b\) in \(d^{*} (r)\) we
get

\[ d^* \left( \frac{(K - 1) M + b}{K} \right) = MN - b(b + M + 1) - (K - 1) M^2 \]
\[ = (N - (K - 1) M - b) \cdot (M - b) = d_{K,M,N}^{*(\text{FC})} ((K - 1) M + b). \]

In the second case \( l = 2b + 1 \), i.e. \( l \) is odd. In this case we get \( \left\lfloor \frac{l+1}{2} \right\rfloor = b + 1, \left\lfloor \frac{l}{2} \right\rfloor = b \) and \( r = \frac{(K - 1) M + b + 1}{K} \).

By assigning \( K M = N + M - 2 - 2b \) in \( d^* (r) \) we get

\[ d^* \left( \frac{(K - 1) M + b + 1}{K} \right) = MN - (b + 1)(b + M + 1) - (K - 1) M^2 = d_{K,M,N}^{*(\text{FC})} ((K - 1) M + b + 1). \]

Hence from both cases we get

\[ d^* \left( \frac{(K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K} \right) = d_{K,M,N}^{*(\text{FC})} \left( (K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor \right). \quad \text{(B.20)} \]

Now we wish to show that \( d^* (r) = d_{M,N}^{D_l^*} (r) = d_{K,M,N}^{D_l^* (\text{FC})} (K \cdot r) \). We begin by showing that \( d^* (r) = d_{M,N}^{D_l^*} (r) \). First note that

\[ d^* (D_l) = d_{M,N}^{D_l} (D_l) = d_{K,M,N}^{D_l (\text{FC})} (K \cdot D_l) = 0. \quad \text{(B.21)} \]

Now let us denote \( D_{\left\lfloor \frac{l}{2} \right\rfloor}^* = \frac{M - N - \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 2 \right)}{N + M - 1 - 2 \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right)} \) and \( D_{\left\lfloor \frac{l}{2} \right\rfloor + 1}^* = \frac{M - N - \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 2 \right)}{N + M - 1 - 2 \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right)} \). We wish to show that

\[ d_{M,N}^{D_{\left\lfloor \frac{l}{2} \right\rfloor}^*} (0) = M \cdot N - \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 2 \right) < d^* (0) \leq M \cdot N - \left\lfloor \frac{l}{2} \right\rfloor \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d_{M,N}^{D_{\left\lfloor \frac{l}{2} \right\rfloor}^*} (0). \quad \text{(B.22)} \]

In the first case we take \( l = 2b \). In this case

\[ d^* (0) = M \cdot N - b(b + 1). \]

On the other hand we also get

\[ M \cdot N - \left\lfloor \frac{l}{2} \right\rfloor \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = M \cdot N - b(b + 1) = d^* (0) \]

which proves (B.22) for the first case. In the second case we consider \( l = 2b + 1 \). In this case

\[ d^* (0) = M \cdot N - (b + 1)^2. \]

For this case we also get \( M \cdot N - \left\lfloor \frac{l}{2} \right\rfloor \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = M \cdot N - b(b + 1) \) and \( M \cdot N - \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) \cdot \left( \left\lfloor \frac{l}{2} \right\rfloor + 2 \right) = M \cdot N - (b + 1) \cdot (b + 2) \). It can be easily shown that for \( b \geq 0 \)

\[ M \cdot N - (b + 1) \cdot (b + 2) < d^* (0) = M \cdot N - (b + 1)^2 \leq M \cdot N - b(b + 1) \]
which proves (B.22) for the second case. From Corollary 3.1 and (B.19) we know that
\[
d^* \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d^{*,D^*_N}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d^{*,D^*_N}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d^{*,(FC)}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right).
\] (B.23)

Since \(d^* (r), d^{*,D^*_N}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) (r) and d^{*,D^*_N}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) (r)\) are all straight lines that fulfil (B.22), (B.23) we get
\[
D^{*,D^*_N}_{\left\lfloor \frac{l}{2} \right\rfloor} \leq D_{\left\lfloor \frac{l}{2} \right\rfloor + 1}.
\] (B.24)

As a result, from Corollary 3.1 and (B.24) we get
\[
d^{*,D^*_l}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) = d^{*,(FC)}_{M,N} \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right).
\] (B.25)

Since \(d^* (r) and d^{*,D^*_l}_{M,N} (r)\) are straight lines and based on the equalities in (B.19), (B.21) and (B.25) we get
\[
d^* (r) = d^{*,D^*_l}_{M,N} (r).
\] (B.26)

Next we prove \(d^* (r) = d^{*,K=D^*_l}_{K-M,N} (K \cdot r)\). Assume \(r_l = \frac{(K-1)M+\left\lfloor \frac{r_1}{2} \right\rfloor}{K} \text{ and } D^*_l = \frac{MN-(K \cdot r_l-1)l}{K \cdot M+N-1-2(K \cdot r_l-1)}\). We wish to show
\[
\frac{d^{*,K=D^*_l}_{K-M,N} r_l}{K} (0) \leq d^* (0) < \frac{d^{*,K=D^*_l}_{K-M,N} r_l}{K} (0).
\] (B.27)

We consider two cases. For the first case we take \(l = 2 \cdot b\). In this case we get \(r_{2b} = \frac{(K-1)M+b}{K}, d^* (0) = M \cdot N - b (b + 1) \text{ and } N = (K - 1) M + 1 + 2b\). Hence we get
\[
d^{*,K=D^*_l}_{K-M,N} r_{2b} \left( \frac{r_l}{K} \right) (0) = KM - ((K - 1) M + b) (N - b) = MN - b (N - (K - 1) M) + b^2.
\] (B.28)

Since \(N - (K - 1) M = 1 + 2b\) we get
\[
MN - b (N - (K - 1) M) + b^2 = MN - b (b + 1) + b^2 = MN - b (b + 1).
\] (B.29)

From (B.28) and (B.29) we get \(d^* (0) = d^{*,K=D^*_l}_{K-M,N} r_{2b} \left( \frac{r_l}{K} \right) (0)\), which proves (B.27) for the first case. For the second case we take \(l = 2b + 1\). In this case \(r_{2b+1} = \frac{(K-1)M+b+1}{K}, d^* (0) = MN - (b + 1)^2 \text{ and } N = (K - 1) M + 2b + 2\). For this case we get
\[
d^{*,K=D^*_l}_{K-M,N} r_{2b+1} (0) = KM - ((K - 1) M + b) (N - b - 1) = MN + (b + 1) (K - 1) M - bN + b (b + 1).
\] (B.30)

Hence according to (B.27) we need to show
\[
MN + (b + 1) (K - 1) M - bN + b (b + 1) > MN - (b + 1)^2.
\] (B.31)

By assigning \((K - 1) M = N - 2b - 2\) we get from (B.31) \(N > b + 1\). Since \(0 \leq l = 2b + 1 \leq 2M - 3\),
the maximal value of $b$ is $b = M - 2$, which gives for $N = (K - 1) M + 2b + l$

$$N > M > M - 1 \geq b + 1.$$  

Hence we get

$$d^* (0) < d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} (0) = d^{*,K \cdot D_{r_l}^{*}}_{K \cdot M,N} (0). \quad (B.32)$$

On the other hand we get

$$d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} \left( \frac{1}{K} \right) (0) = KMN - ((K - 1) M + b) (N - b). \quad (B.33)$$

Hence according to (B.27), (B.33) we need to show that

$$MN + b (K - 1) M - N (b + 1) + b(b + 1) \leq MN - (b + 1)^2 \quad (B.34)$$

which again leads to $N > b + 1$. Hence we get

$$d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} \left( \frac{1}{K} \right) (0) = d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} (0) \leq d^* (0). \quad (B.35)$$

From (B.32) and (B.35) we get (B.27) for the second case. Hence we have proved (B.27). From Corollary 3.1 and (B.20) we know that

$$d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} \left( \frac{1}{K} \right) (0) = d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} (0) = d^{*,(FC)}_{K \cdot M,N} \left( (K - 1) M + \left\lceil \frac{l + 1}{2} \right\rceil \right). \quad (B.36)$$

Since $d^* (r), d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} (K \cdot r)$ and $d^{*,K \cdot D_{r_l}^{*}b+1}_{K \cdot M,N} (K \cdot r)$ are all straight lines that fulfill (B.27), (B.36) we get

$$D_{r_l}^{*} < D_{r_l}^{*} \leq D_{r_l}^{*} \frac{1}{K}. \quad (B.37)$$

As a result, from Corollary 3.1 and (B.37) we get

$$d^{*,K \cdot D_{r_l}^{*}}_{K \cdot M,N} \left( (K - 1) M + \left\lceil \frac{l + 1}{2} \right\rceil \right) = d^{*,(FC)}_{K \cdot M,N} \left( (K - 1) M + \left\lceil \frac{l + 1}{2} \right\rceil \right). \quad (B.38)$$

Since $d^* (r)$ and $d^{*,K \cdot D_{r_l}^{*}}_{K \cdot M,N} (K \cdot r)$ are all straight lines, and based on the equalities in (B.20), (B.21) and (B.38) we get

$$d^* (r) = d^{*,K \cdot D_{r_l}^{*}}_{K \cdot M,N} (K \cdot r). \quad (B.39)$$

From (B.26), (B.39) we get the first part of the Lemma, where from (B.25), (B.38) we get the second part of the Lemma.
B.5 Proof of Theorem 3.4

We begin by showing that \( d_{K,M,N}^{*,(IC)} (r) \) is the solution to the optimization problem in (3.9), i.e. the case where all users have the same NDCU, \( D \). Then we show that this is also the solution for (3.8).

First we find \( \max_D \min_{1 \leq i \leq K} \left( d_{i,M,N}^{*,D} (i \cdot r) \right) \), where \( 0 \leq r \leq \frac{L}{K} \). For the case \( N = (K + 1) M - 1 \), we can see from Lemma 3.2 that

\[
\max_D \min_{1 \leq i \leq K} \left( d_{i,M,N}^{*,D} (i \cdot r) \right) = \max_D d_{M,N}^{*,D} (r) = d_{M,N}^{*,(FC)} (r).
\]

For the case \( N < (K - 1) M + 1 \) it was shown in Lemma 3.4 that \( d_{K,M,N}^{*,(IC)} (r) \) is the optimization problem solution. For \( N = (K - 1) M + 1 + l \), where \( l = 0, \ldots, 2M - 3 \) we know from Lemma 3.3 that \( d_{M,N}^{*,D} (r) \) is smaller than \( d_{i,M,N}^{*,D} (i \cdot r) \) for \( 2 \leq i \leq K - 1 \) and any \( 0 \leq D \leq \frac{L}{K}, 0 \leq r \leq D \). Hence the optimization problem for this case boils down to

\[
\max_D \min \left\{ d_{M,N}^{*,D} (r), d_{K,M,N}^{*,D} (K \cdot r) \right\}
\]

for \( 0 \leq D \leq \frac{L}{K} \) and \( 0 \leq r \leq D \). From Lemma 3.5 we know that \( d_{M,N}^{*,D_l} (\lfloor \frac{l}{2} \rfloor + 1) = d_{M,N}^{*,(FC)} (\lfloor \frac{l}{2} \rfloor + 1) \). As a result, based on Corollary 3.1 we get that for \( 0 < D \leq D_l \)

\[
d_{M,N}^{*,D} (\lfloor \frac{l}{2} \rfloor + 1) \leq d_{M,N}^{*,(FC)} (\lfloor \frac{l}{2} \rfloor + 1) = d_{M,N}^{*,D_l} (\lfloor \frac{l}{2} \rfloor + 1)
\]

and also

\[
d_{M,N}^{*,D} (r) = 0 \leq d_{M,N}^{*,D_l} (r) \quad r \geq D_l.
\]

Hence we get for \( 0 < D \leq D_l \)

\[
d_{M,N}^{*,D} (r) \leq d_{M,N}^{*,D_l} (r) \quad \lfloor \frac{l}{2} \rfloor + 1 \leq r \leq \frac{L}{K}.
\]  

In a similar manner we also know from Lemma 3.5 that

\[
d_{K,M,N}^{*,D_l} \left( (K - 1) M + \lfloor \frac{l+1}{2} \rfloor \right) = d_{K,M,N}^{*,(FC)} \left( (K - 1) M + \lfloor \frac{l+1}{2} \rfloor \right)
\]

As a result, based on Corollary 3.1 we get that for \( D_l \leq D \leq \frac{L}{K} \)

\[
d_{K,M,N}^{*,D} \left( (K - 1) M + \lfloor \frac{l+1}{2} \rfloor \right) \leq d_{K,M,N}^{*,(FC)} \left( (K - 1) M + \lfloor \frac{l+1}{2} \rfloor \right) = d_{K,M,N}^{*,D_l} \left( (K - 1) M + \lfloor \frac{l+1}{2} \rfloor \right)
\]

and also

\[
d_{K,M,N}^{*,D_l} (K \cdot r) = 0 \leq d_{K,M,N}^{*,D} (K \cdot r) \quad r \geq D_l.
\]
Since $D_l \geq \frac{(K-1)M + \left\lfloor \frac{l-1}{2} \right\rfloor}{K}$ and these are straight lines, we also get for $D_l \leq D \leq \frac{L}{K}$

$$d^{*,K\cdot D}_{K,M,N}(K\cdot r) \leq d^{*,K\cdot D}_{K,M,N}(K\cdot r) \quad 0 \leq r \leq \frac{(K-1)M + \left\lfloor \frac{l-1}{2} \right\rfloor}{K}. \quad \text{(B.42)}$$

Hence, based on (B.41), (B.42) and the fact that $d^{*,D_l}_{M,N}(r) = d^{*,K\cdot D_l}_{K,M,N}(K\cdot r) = d^*(r)$ (Lemma 3.5), we get that

$$\max_D \min \left\{ d^{*,D}_{M,N}(r), d^{*,K\cdot D}_{K,M,N}(K\cdot r) \right\} = d^*(r) = d^{*,(FC)}_{K,M,N}(r) \quad \left\lfloor \frac{l}{2} \right\rfloor + 1 \leq r \leq \frac{(K-1)M + \left\lfloor \frac{l-1}{2} \right\rfloor}{K}. \quad \text{(B.43)}$$

Next we find the solution for $0 \leq r \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$. Our starting point is $D = D_l$ for which $d^{*,D_l}_{M,N}(r) = d^{*,K\cdot D_l}_{K,M,N}(K\cdot r)$. Since $d^*(\left\lfloor \frac{l}{2} \right\rfloor + 1) = d^{*,(FC)}_{M,N}(\left\lfloor \frac{l}{2} \right\rfloor + 1)$ we get from Corollary 3.1 and (B.24) that

$$\frac{MN - \left\lfloor \frac{l}{2} \right\rfloor (\left\lfloor \frac{l}{2} \right\rfloor + 1)}{M + N - 1 - 2\left\lfloor \frac{l}{2} \right\rfloor} \leq D_l \leq \frac{MN - (\left\lfloor \frac{l}{2} \right\rfloor + 1)(\left\lfloor \frac{l}{2} \right\rfloor + 2)}{M + N - 1 - 2(\left\lfloor \frac{l}{2} \right\rfloor + 1)}. \quad \text{(B.44)}$$

We also get from Corollary 3.1 that for $D_l \leq D \leq \frac{L}{K}$

$$d^{*,D}_{M,N}(r) \leq d^{*,(FC)}_{M,N}(r). \quad \text{(B.45)}$$

In addition it can be easily shown that for $N = (K-1)M + 1 + l$, where $l = 0, \ldots, 2M-3$

$$\left\lfloor \frac{l}{2} \right\rfloor + 1 \leq \frac{N}{K+1} \leq \frac{(K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K} \quad \text{(B.46)}$$

by considering the cases where $l$ is even and odd, i.e. the cases where $l = 2b$ and $l = 2b + 1$. For the case $\frac{MN - (\left\lfloor \frac{l}{2} \right\rfloor + 1)}{M + N - 1 - 2\left\lfloor \frac{l}{2} \right\rfloor} \leq D \leq D_l$ assume $d^{*,K\cdot D}_{K,M,N}(K\cdot r)$ rotates around anchor point with multiplexing gain $m$. In this case there are two possibilities. The first possibility is $\left\lfloor \frac{l}{2} \right\rfloor + 2 \leq m \leq \frac{L}{K}$ where $m \in \mathbb{Z}$. When this is the case, we get from Corollary 3.1 that in the range $\frac{MN - (\left\lfloor \frac{l}{2} \right\rfloor + 1)}{M + N - 1 - 2\left\lfloor \frac{l}{2} \right\rfloor} \leq D \leq D_l$

$$d^{*,D}_{M,N}(\left\lfloor \frac{l}{2} \right\rfloor + 1) = d^{*,K\cdot D}_{K,M,N}(\left\lfloor \frac{l}{2} \right\rfloor + 1) \leq d^{*,K\cdot D}_{K,M,N}(\left\lfloor \frac{l}{2} \right\rfloor + 1). \quad \text{(B.47)}$$

For the second possibility $0 \leq m \leq \left\lfloor \frac{l}{2} \right\rfloor + 1$ we get from (B.46) and Theorem 3.3 that

$$d^{*,K\cdot D}_{K,M,N}(K\cdot m) = d^{*,(FC)}_{K,M,N}(K\cdot m) \geq d^{*,(FC)}_{M,N}(m) \geq d^{*,D}_{M,N}(m). \quad \text{(B.48)}$$

In addition $d^{*,D}_{M,N}(D) = d^{*,K\cdot D}_{K,M,N}(K\cdot D) = 0$. Since these are straight lines we get that in the range $\frac{MN - (\left\lfloor \frac{l}{2} \right\rfloor + 1)}{M + N - 1 - 2\left\lfloor \frac{l}{2} \right\rfloor} \leq D \leq D_l$

$$d^{*,D}_{M,N}(r) \leq d^{*,K\cdot D}_{K,M,N}(K\cdot r). \quad \text{(B.49)}$$

By induction, for $\frac{MN - (s-1)s}{M + N - 1 - 2(s-1)} \leq D \leq \frac{MN - (s+1)s}{M + N - 1 - 2s}$, $s = \left\lfloor \frac{l}{2} \right\rfloor, \ldots, 1$, assuming $d^{*,K\cdot D}_{K,M,N}(K\cdot r) \geq$
\( d_{M,N}(r) \), where \( D(s) = \frac{MN-s(s+1)}{M+N-1-2s} \) we get from similar arguments to (B.46)-(B.49) that

\[
d_{M,N}^{s,D}(r) \leq d_{K,M,N}^{s,K-D}(K \cdot r).
\]

Finally for \( 0 < D \leq \frac{MN}{N+1} \), from the same arguments as in (B.50) we also get

\[
d_{M,N}^{s,D}(r) \leq d_{K,M,N}^{s,K-D}(K \cdot r).
\]

Hence, from (B.49), (B.50) and (B.51) we get that in the range \( 0 < D \leq D_l \)

\[
\max_D \min \left\{ d_{M,N}^{s,D}(r), d_{K,M,N}^{s,K-D}(K \cdot r) \right\} = \max_D d_{M,N}^{s,D}(r).
\]

Since \( D_l \geq \frac{MN-\left(\frac{1}{2}\right)\left(\frac{1}{2}+1\right)}{M+N-1-2\left(\frac{1}{2}\right)} \) (B.44), and also from (B.45), (B.52) we get based on Corollary 3.1

\[
\max_D \min \left\{ d_{M,N}^{s,D}(r), d_{K,M,N}^{s,K-D}(K \cdot r) \right\} = d_{M,N}^{s,(FC)}(r) = d_{K,M,N}^{s,(IC)}(r) \quad 0 \leq r \leq \left\lfloor \frac{l}{2} \right\rfloor + 1.
\]

Now we wish to find \( d_{K,M,N}^{s,(IC)}(r) \) for \( \frac{(K-1)M+\left\lfloor \frac{l+1}{2} \right\rfloor}{K} \leq r \leq \frac{l}{K} \). Let us denote \( r_l = \frac{(K-1)M+\left\lfloor \frac{l+1}{2} \right\rfloor}{K} \).

Since

\[
d_{K,M,N}^{s,K-D}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor) = d_{K,M,N}^{s,(FC)}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor),
\]

we get (B.37)

\[
NM - (K \cdot r_l - 1) r_l \leq D_l \leq \frac{NM - (K \cdot r_l + 1)}{KM + N - 1 - 2(K \cdot r_l - 1)}.
\]

Based on Corollary 3.1 we get in the range \( 0 < D \leq D_l \)

\[
d_{K,M,N}^{s,K-D}(K \cdot r) \leq d_{K,M,N}^{s,(FC)}(K \cdot r).
\]

For \( D_l < D \leq \frac{NM}{K+\frac{1}{2}r_l+1} \) assume \( d_{M,N}^{s,D}(r) \) rotates around anchor point with multiplexing gain \( \frac{m}{K} \), where \( m \in \mathbb{Z} \). In case \( 0 \leq m < (K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor \), based on Corollary 3.1 and Lemma 3.5 we get

\[
d_{M,N}^{s,D}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor) \geq d_{M,N}^{s,D_l}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor) = d_{K,M,N}^{s,(FC)}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor) \geq d_{K,M,N}^{s,K-D}((K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor).
\]

In case \( (K-1)M + \left\lfloor \frac{l+1}{2} \right\rfloor \leq m \leq L \) we get from (B.46) and Theorem 3.3 that

\[
d_{M,N}^{s,D}(m) = d_{M,N}^{s,(FC)}(m) \geq d_{M,N}^{s,(FC)}(K \cdot m) \geq d_{K,M,N}^{s,K-D}(K \cdot m).
\]

We also get \( d_{M,N}^{s,D}(D) = d_{K,M,N}^{s,K-D}(K \cdot D) = 0 \). Since these are straight lines, we get for \( D_l < D \leq
\]
\[
\begin{align*}
N M - \frac{r l}{K} (r l + 1) & \leq d^* \left( K - M, N \right) (r l) \leq \frac{N M - \frac{r l}{K} (s - 1)}{K M + N - 1 - 2 (s - 1)} \quad s = (K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor + 1, \ldots, L - 1,
\end{align*}
\]

Similarly to (B.50) it can be shown by induction for
\[
\begin{align*}
MN - \frac{r l}{K} (s - 1) & \leq D \leq \frac{MN - \frac{r l}{K} (s + 1)}{K M + N - 1 - 2 s} \quad s = (K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor + 1, \ldots, L - 1
\end{align*}
\]
that
\[
d^* \left( M, N \right) (r l) \geq d^* \left( K - M, N \right) (K \cdot r).
\] (B.58)

Hence from (B.55), (B.58) and (B.59) we get
\[
\max_D \min \left\{ d^* \left( M, N \right) (r l), d^* \left( K - M, N \right) (K \cdot r) \right\} = d^*_{K - M, N} (r l) \leq r \leq \frac{L}{K}.
\] (B.60)

The remaining open point for \( N = (K - 1) M + 1 + l, l = 0, \ldots, 2M - 3 \) is the case
\[
\left\lfloor \frac{l}{2} \right\rfloor + 1 = \frac{(K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor}{K}.
\] (B.61)

First we would like to find when this equality takes place. For this we consider two cases. First let us consider \( l = 2b \). For this case (B.61) takes the following form
\[
K \cdot (b + 1) = (K - 1) M + b
\]
which leads to
\[
b = M - \frac{K}{K - 1}.
\]
Since \( b \geq 0, M \geq 1 \) and \( K \geq 2 \) are integers, we get that this equality can only hold when \( K = 2 \). In this case we get \( M = b + 2 \) and \( N = 3 (b + 1) \). Since both \( M \geq 1 \) and \( N \geq 1 \), we get that \( b \geq 2 \). Hence by assigning \( s = b + 1 \) we get (B.61) for \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \), where \( s \geq 1 \) is an integer. For the second case we consider \( l = 2b + 1 \). In this case by assigning in (B.61) we get \( b = M - 1 \). However we know that \( l = 2b + 1 \leq 2M - 3 \), and so \( b \leq M - 2 \). Hence for \( l = 2b + 1 \) (B.61) can not take place. From (B.46), (B.61) we get
\[
\left\lfloor \frac{l}{2} \right\rfloor + 1 = \frac{N}{K + 1} = \frac{(K - 1) M + \left\lfloor \frac{l + 1}{2} \right\rfloor}{K}.
\] (B.62)

In addition, (B.61) holds only for \( l = 2b \). For this case simply by assigning \( l = 2b \) we get
\[
D^*_{\left\lfloor \frac{L}{2} \right\rfloor} = D_l = D^*_{r_l}.
\] (B.63)

Hence, we are interested in finding \( d^*_{K - M, N} (r) \) for the case \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \), where \( s \geq 1 \) is an integer. For \( D > D_l \) we get \( d^*_{s+1,3,s} (r) \leq d^*_{s+1,3,s} (r) \). On the other hand for \( 0 < D < L \), we get
Hence the optimal solution must be attained for $d^{s,2}_{s+1,3,s} (r)$ rotates around anchor point with multiplexing gain $m \leq \frac{N}{K+1}$. Hence, by similar arguments to the ones used in (B.48) we get $d^{s,2}_{s+1,3,s} (m) \leq d^{s,2}_{2(s+1),3,s} (2 \cdot m)$, which leads to $d^{s,2}_{s+1,3,s} (r) \leq d^{s,2}_{2(s+1),3,s} (2 \cdot r)$ for $0 < D < D_1$. Hence in the range $0 \leq r \leq \frac{N}{K+1}$ the optimal solution is $d^{s,2}_{s+1,3,s} (r)$. Hence we get

$$d^{s,(IC)}_{K,M,N} (r) = d^{s,(IC)}_{2,s+1,3,s} (r) = \begin{cases} d^{s,(FC)}_{s+1,3,s} (r) & 0 \leq r \leq \frac{N}{K+1} = s \\ d^{s,(FC)}_{2(s+1),3,s} (2 \cdot r) & s \leq r \leq 3 \cdot s. \end{cases} \tag{B.64}$$

So far we have shown that

$$\max_D \min_{K,M,N} \left\{ d^{s,D}_{M,N} (r), d^{s,K,D}_{K,M,N} (K \cdot r) \right\} = d^{s,(IC)}_{K,M,N} (r). \tag{B.65}$$

Now we wish to show that this is also the solution to (3.8). We begin with the case for which $d^{s,(IC)}_{K,M,N} (r) = d^{s,(FC)}_{M,N} (r)$. This is the case for $N \geq (K+1) M - 1$, and also for $N = (K - 1) M - 1 + l$, $l = 0, \ldots, 2M-3$ when $0 \leq r \leq \lfloor \frac{l}{2} \rfloor + 1$. As a base line we consider the case $D_1 = \ldots, D_K = D^*_r$, where $D^*_r$ is the optimal NDCU per user in (B.65). Without loss of generality assume user $i$ has $D_i \neq D^*_r$. In this case based on Corollary 3.1 we get

$$\min_{A \subseteq \{1, \ldots, K\}, D_i \neq D^*_r} \left( d^{s,\sum_{a \in A} D_a}_{|A|,M,N} (|A| \cdot r) \right) \leq d^{s,(FC)}_{M,N} (r) = \max_D \min_{K,M,N} \left\{ d^{s,D}_{M,N} (r), d^{s,K,D}_{K,M,N} (K \cdot r) \right\}. \tag{B.66}$$

Hence the optimal solution must be $d^{s,(IC)}_{K,M,N} (r)$, attained for $D_1 = \ldots = D_K = D^*_r$. Now we consider the case where $d^{s,(IC)}_{K,M,N} (r) = d^{s,(FC)}_{K-M,N} (K \cdot r)$, for which $N = (K - 1) M + 1 + l$, where $l = 0, \ldots, 2M-3$ and $\frac{(K-1) M + 1 + l}{K} \leq r \leq \frac{l}{K}$. In this case the optimal solution in (B.65) for the $K$ users pulled together is attained for $K \cdot D^*_r$. Let us assume that $\sum_{i=1}^{K} D_i \neq K \cdot D^*_r$. In this case we get

$$\min_{A \subseteq \{1, \ldots, K\}, \sum_{i=1}^{K} D_i \neq K \cdot D^*_r} \left( d^{s,\sum_{a \in A} D_a}_{|A|,M,N} (|A| \cdot r) \right) \leq d^{s,(FC)}_{K-M,N} (K \cdot r) = \max_D \min_{K,M,N} \left\{ d^{s,D}_{M,N} (r), d^{s,K,D}_{K,M,N} (K \cdot r) \right\}. \tag{B.67}$$

Hence the optimal solution must be $d^{s,(IC)}_{K,M,N} (r)$. Now let us consider the case $N < (K - 1) M + 1$. In this case the optimal solution in (B.65) is attained for $D^*_r = \frac{N}{K}$. Without loss of generality assume $D_i < \frac{N}{K}$. In this case we get from Corollary 3.1 that

$$\min_{A \subseteq \{1, \ldots, K\}, D_i < \frac{N}{K}} \left( d^{s,\sum_{a \in A} D_a}_{|A|,M,N} (|A| \cdot r) \right) \leq MN - KM \cdot r = \max_D \min_{K,M,N} \left\{ d^{s,D}_{M,N} (r), d^{s,K,D}_{K,M,N} (K \cdot r) \right\}. \tag{B.68}$$

which shows again that $d^{s,(IC)}_{K,M,N} (r)$ is the solution. Finally we consider the case where $d^{s,(IC)}_{K,M,N} (r) = d^{s}(r)$, i.e. the case where $N = (K - 1) M + 1 + l$, $l = 0, \ldots, 2M-3$ and $\lfloor \frac{l}{2} \rfloor + 1 \leq r \leq \frac{(K-1) M + 1 + l}{K}$. Without
loss of generality assume \( D_i \neq D_t \). In this case we get from Corollary 3.1
\[
\min_{A \subseteq \{1, \ldots, K\}, D_i \neq D_t} \left( \min_{D \in \Delta} d^* \left( \sum_{a \in A} D_a (|A| \cdot r) \right) \right) \leq d^* (r) = \max_D \min \left\{ d^*_{M, N} (r), d^*_{K, M, N} (K \cdot r) \right\}
\] (B.69)
which shows that \( d^*_{K, M, N} (r) \) is the optimal solution. This concludes the proof.

### B.6 Proof of Lemma 3.6

For \( N \geq (K + 1) M - 1 \) it can be easily shown based on Lemma 3.2 and Corollary 3.1 that
\[
d^*_{K, M, N} (r) = d^*_{M, N} (r) = d^*_{K, M, N} (r).
\] (B.70)
For \( N < (K - 1) M + 1 \) we get \( \frac{L}{K} = \frac{N}{K} \). In this case from (B.13), (B.14) and (B.15) we get
\[
d^*_{M, N} (r) < d^*_{K, M, N} (K \cdot r) \leq d^*_{K, M, N} (K \cdot r) \quad 0 < D \leq \frac{N}{K}
\] (B.71)
for \( 0 < r < D \). In addition since \( \frac{N}{K} < \frac{MN}{N + M - 1} \) and \( 0 < D \leq \frac{N}{K} \) we get from Corollary 3.1 that
\[
d^*_{K, M, N} (r) = MN - KMr < d^*_{M, N} (r) \quad 0 < r \leq \frac{N}{K}.
\] (B.72)
Since \( d^*_{K, M, N} (r) \) consists of \( d^*_{M, N} (r) \) and \( d^*_{K, M, N} (K \cdot r) \) we get from (B.71), (B.72) that
\[
d^*_{K, M, N} (r) < d^*_{M, N} (r) \quad 0 < r < \frac{N}{K}.
\]

For \( N = (K - 1) M + 1 + l \) where \( l = 0, \ldots, 2M - 3 \), recall that we denoted
\[
D_t = \frac{MN - \left\lfloor \frac{L}{2} \right\rfloor \cdot \left( \left\lfloor \frac{L}{2} \right\rfloor + 1 \right) - 2 \cdot \left( \left\lfloor \frac{L}{2} \right\rfloor + 1 \right) \cdot \left( \frac{L}{2} - \left\lfloor \frac{L}{2} \right\rfloor \right)}{N + M - 1 - l}
\]
and also \( r_t = \left( \frac{K - 1}{K} \right)^{\frac{K + 1}{K}} \). In (B.24) it was shown that \( D_t < \frac{MN - (\frac{L}{2} + 1)(\frac{L}{2} + 2)}{M + N - 1 - 2(\frac{L}{2} + 1)} \). Hence, from Corollary 3.1 we get
\[
d^* (r) = d^*_{M, N} (r) < d^*_{M, N} (r) \quad \left\lfloor \frac{L}{2} \right\rfloor + 1 < r \leq \frac{L}{K}.
\] (B.73)
On the other hand from (B.37) we get \( D_t > \frac{MN - (K - r_t - 1)}{K + N + M - 1 - 2(\frac{L}{2} + 1)} \). Hence based on Corollary 3.1 we get
\[
d^* (r) = d^*_{K, M, N} (K \cdot r) < d^*_{K, M, N} (K \cdot r) \quad 0 \leq r < (K - 1) M + \left\lfloor \frac{L}{2} \right\rfloor.
\] (B.74)
Since \( d^*_{K, M, N} (r) \) consists of \( d^*_{M, N} (r) \) and \( d^*_{K, M, N} (K \cdot r) \), we get from (B.73), (B.74)
\[
d^* (r) < d^*_{K, M, N} (r) \quad \left\lfloor \frac{L}{2} \right\rfloor + 1 < r < \frac{(K - 1) M + \left\lfloor \frac{L}{2} \right\rfloor}{K}.
\] (B.75)
The remaining open point for $N = (K - 1) M + 1 + l$, where $l = 0, \ldots, 2M - 3$ is the case

$$\left\lfloor \frac{l}{2} \right\rfloor + 1 = \frac{(K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K}.$$  

In Theorem 3.4 it was shown (see equation (B.62) appendix B.5) that we get equality for $K = 2, M = s + 1$ and $N = 3 \cdot s$, where $s \geq 1$ is an integer. According to Theorem 3.3, for this case the optimal DMT of finite constellations equals

$$d_{2,s+1,3,s}^*(FC) = \begin{cases} 
  d_{s+1,3,s}^*(r) & 0 \leq r \leq \frac{N}{K+1} = s \\
  d_{2(3s+1),3,s}^*(2 \cdot r) & s \leq r \leq 3 \cdot s.
\end{cases}$$

Hence, from (B.64) we get $d_{2,s+1,3,s}^*(FC) = d_{2,s+1,3,s}^*(IC)$. By simply assigning we get that in this case $N < (K + 1) M - 1$. This concludes the proof.

### B.7 Proof of Theorem 3.5

We begin by finding for $N \geq (K + 1) M - 1$ an upper bound on the DMT of the unconstrained multiple-access channel, that equals to the optimal DMT of finite constellations $d_{M,N}^*(FC) (\max (r_1, \ldots, r_K))$. The proof relies on the upper bound on the optimal DMT in the symmetric case $d_{K,M,N}^*(IC) (r)$.

For $N \geq (K + 1) M - 1$ it was shown in Lemma 3.6 that

$$d_{K,M,N}^*(IC) (r) = d_{M,N}^*(FC) (r).$$  

From Theorem 3.2 we get that the optimal DMT is upper bounded by

$$\max_{(D_1, \ldots, D_K) \in D} \min_{A \subseteq \{1, \ldots, K\}} d_{|A|,M,N}^{*,DA} (R_A).$$  

We wish to solve (B.77). We solve it by finding upper and lower bounds on (B.77) that coincide. For the rate tuple $(r_1, \ldots, r_K)$ recall the definition $r_{\max} = \max (r_1, \ldots, r_K)$. We begin by lower bounding the optimization problem terms. Based on Lemma 3.2 and the fact that $d_{i,M,N}^*(i \cdot r), i = 1, \ldots, K$ are straight lines as a function of $r$ we get

$$d_{|A|,M,N}^* \sum_{a \in A} D_a \left( \sum_{a \in A} r_a \right) \geq d_{|A|,M,N}^* \left( |A| \cdot r_{\max} \right) \geq d_{M,N}^* \sum_{a \in A} D_a (r_{\max}) \forall A \subseteq \{1, \ldots, K\}.$$  

Hence, we get

$$\min_{A \subseteq \{1, \ldots, K\}} d_{|A|,M,N}^* \sum_{a \in A} D_a \left( \sum_{a \in A} r_a \right) \geq \min_{A \subseteq \{1, \ldots, K\}} d_{M,N}^* \sum_{a \in A} D_a (r_{\max}).$$  

From Corollary 3.1 we know that

$$\max_D d_{M,N}^* (r_{\max}) = d_{M,N}^*(FC) (r_{\max}).$$
obtained for $D_{\text{max}} = \frac{MN - |r_{\text{max}}| + 1}{N + M - 1 - 2|r_{\text{max}}|}$. Hence, from (B.79), (B.80) we get

$$
\max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} d_{|A|, M, N}^* (R_A) \geq \max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} \frac{\sum_{a \in A} D_a}{|A|} (r_{\text{max}}) = d_{M, N}^* (r_{\text{max}}) \quad (B.81)
$$

obtained for $D_1 = \cdots = D_K = D_{\text{max}}$. Next we upper bound the optimization problem and show it coincides with the lower bound. Without loss of generality assume $r_i = r_{\text{max}}$. In this case we get

$$
\min_{A \subseteq \{1, \ldots, K\}} d_{|A|, M, N}^* \left( \sum_{a \in A} r_a \right) \leq d_{M, N}^* \left( r_{\text{max}} \right). \quad (B.82)
$$

From (B.80), (B.82) we can write

$$
\max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} d_{|A|, M, N}^* (R_A) \leq \max_{D_i} d_{M, N}^* (r_{\text{max}}) = d_{M, N}^* (r_{\text{max}}) \quad (B.83)
$$

obtained for $D_i = D_{\text{max}}$. Hence, from (B.81), (B.83) we get

$$
\max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} d_{|A|, M, N}^* (R_A) = d_{M, N}^* (r_{\text{max}}) \quad (B.84)
$$

which is the optimal DMT of finite constellations.

Now we show for $N < (K + 1) M - 1$ that the optimal DMT of the unconstrained multiple-access channel is suboptimal compared to the optimal DMT of finite constellations. We do that by showing that there exists a set $B$ of multiplexing gain tuples $(r_1, \ldots, r_K)$ for which

$$
\max_{(D_1, \ldots, D_K) \in \mathcal{D}} \min_{A \subseteq \{1, \ldots, K\}} d_{|A|, M, N}^* (R_A) < d_{K, M, N}^* (r_1, \ldots, r_K) \quad \forall (r_1, \ldots, r_K) \in B
$$

where $d_{K, M, N}^* (r_1, \ldots, r_K)$ is the optimal DMT of finite constellations. We divide the sub-optimality proof of $N < (K + 1) M - 1$ to several cases. We begin with the case $N < (K - 1) M + 1$. For this case we show the sub-optimality by considering symmetric multiplexing gain tuples, i.e. $r_1 = \cdots = r_K = r$. In this case the optimization problem (B.77) solution equals $d_{K, M, N}^* (IC)$. From Lemma 3.6 we get that

$$
d_{K, M, N}^* (IC) (r) < d_{K, M, N}^* (FC) (r) = d_{K, M, N}^* (r, \ldots, r) \quad 0 < r < \frac{N}{K}.
$$

Hence, in this case we have proved the sub-optimality based on the optimal DMT in the symmetric case. Now we prove the sub-optimality for the case $N = (K - 1) M + 1 + l$, where $l = 0, \ldots, 2M - 3$. In Lemma 3.6 we have showed for $r_1 = \cdots = r_K = r$ that

$$
d_{K, M, N}^* (IC) (r) < d_{K, M, N}^* (FC) \left( \left\lfloor \frac{l}{2} \right\rfloor + 1 \right) < \frac{(K - 1) M + \left\lfloor \frac{l+1}{2} \right\rfloor}{K}. \quad (B.85)
$$

Hence, for the case where $\left\lfloor \frac{l}{2} \right\rfloor + 1 \neq \frac{(K-1)M+l+1}{K}$ this shows the sub-optimality of any IC’s DMT. There-
fore, in order to complete the sub-optimality proof we are left only with the case \( \left\lfloor \frac{K}{2} \right\rfloor + 1 = \frac{(K-1)M + 1}{K} \).

In Theorem 3.4 we have shown that \( \left\lfloor \frac{K}{2} \right\rfloor + 1 = \frac{(K-1)M + 1}{K} \) only when \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \), where \( s \geq 1 \) is an integer. Note that in this case the upper bound on the optimal DMT of IC’s in the symmetric case equals to the optimal DMT of finite constellations. Hence, in this case we can not obtain the sub-optimality from the symmetric case and we need to find a set of multiplexing gain tuples \( B \) for which

\[
\max_{(D_1, D_2)} \min \left( d^*_{s+1,3,s} (r_1), d^*_{2(s+1),3,s} (r_1 + r_2), d^*_{s+1,3,s} (r_2) \right) < d^*_{s+1,3,s} (r_1, r_2) \quad (r_1, r_2) \in B.
\]

(B.86)

We defer the proof of (B.86) to appendix B.8. In a nutshell we are interested in finding a set such that the optimal DMT of finite constellations equals to the two user optimal DMT, i.e. \( d^*_{s+1,3,s} (r_1 + r_2) = d^*_{s+1,3,s} (r_1, r_2) \) where the IC’s single user expressions \( d^*_{s+1,3,s} (r_1) \) or \( d^*_{s+1,3,s} (r_2) \) will be smaller than \( d^*_{2(s+1),3,s} (r_1, r_2) \) for any \( D_1, D_2 \) for which \( d^*_{s+1,3,s} (r_1 + r_2) = d^*_{s+1,3,s} (r_1, r_2) \). Figure 3.5 shows the optimal DMT of finite constellations for the case \( K = 2, M = 3 \) and \( N = 6 \), and Figure 3.6 illustrates the aforementioned description of the proof method for the same setting.

### B.8 Final Part of the Proof of Theorem 3.5

In order to find the set \( B \) we first present several properties of \( d^*_{s+1,3,s} (r) \), i.e. the optimal DMT of IC’s in the symmetric case, for this case. First note that from Theorem 3.4 we get

\[
d^*_{s+1,3,s} (r) = \begin{cases} 
  d^*_{s+1,3,s} (r) & 0 \leq r \leq \frac{N}{K+1} = s \\
  d^*_{2(s+1),3,s} (2 \cdot r) & s \leq r \leq \min \left( s + 1, \frac{3}{2}s \right) = d^*_{2(s+1),3,s} (r).
\end{cases}
\]

An example of \( d^*_{s+1,3,s} (r) \) for the case \( M = 3, N = 6 \) and \( K = 2 \), i.e. \( s = 2 \), is given in Figure 3.5.

From simple assignment of the values of \( M, N \) and \( K \) we get that \( l = 2 (s - 1) \). We know from Lemma 3.5, Theorem 3.3 and (B.62) that

\[
d^*_{s+1,3,s} \left( \frac{N}{K+1} \right) = d^*_{s+1,3,s} \left( \frac{N}{K+1} \right) = d^*_{2(s+1),3,s} \left( \frac{K \cdot N}{K+1} \right) = d^*_{2(s+1),3,s} \left( \frac{K \cdot N}{K+1} \right) = d^*_{2(s+1),3,s} (2 \cdot r).
\]

Hence, from (B.63) and (B.87) we get

\[
d^*_{s+1,3,s} (r) = d^*_{2(s+1),3,s} (2 \cdot r).
\]

(B.88)

Finally, from Corollary 3.1 we get

\[
d^*_{s+1,3,s} (r) = d^*_{2(s+1),3,s} (2 \cdot r) = d^*_{s+1,3,s} (r) \quad s - 1 \leq r \leq \frac{N}{K+1} = s
\]

(B.89)
where \( s < r \)

\[
d_{s+1,3,s}^{*(2, FC)}(2 \cdot r) = d_{s+1,3,s}^{*(FC)}(2 \cdot r) \quad s \leq r \leq s + \frac{1}{2}
\]  

(B.90)
i.e., The last line of \( d_{s+1,3,s}^{*(FC)}(r) \) before \( \frac{N}{K} = s \), and the first line of \( d_{s+1,3,s}^{*(FC)}(2r) \) after \( s \) are equal. To sum up, for \( \lfloor \frac{r}{2} \rfloor + 1 = \frac{(K-1)M + \lfloor \frac{r}{2} \rfloor}{K} \) the optimal DMT of IC’s in the symmetric case is upper bounded by a piecewise linear function as expected, and we have found the straight line coincide with it for \( s - 1 \leq r \leq s + \frac{1}{2} \). We are interested in finding a set of multiplexing gain tuples \( B \), for which (B.86) is fulfilled. In a nutshell we are interested in finding a set such that the optimal DMT of finite constellations equals to the two user optimal DMT, where IC’s single user expressions will be smaller than the optimal DMT of finite constellations for any \( D_1 \), \( D_2 \) for which the IC’s two users expression equals to the optimal DMT of finite constellations. Figure 3.6 illustrates the aforementioned description of the proof method.

From (B.89) and Corollary 3.1 we know that

\[
d_{s+1,3,s}^{*(FC)}(r) = d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r) \quad s \leq r \leq s + 1.
\]  

(B.91)
Hence, for certain \( s < r_0 < s + \frac{1}{2} \), we are interested in the set for which \( r_1 = r_0 + \epsilon, r_2 = r_0 - \epsilon \) such that \( s < r_0 + \epsilon < s + \frac{1}{2} \) and also

\[
d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r_0) = d_{s+1,3,s}^{*(FC)}(2r_0) < d_{s+1,3,s}^{*(FC)}(r_0 + \epsilon) = d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r_0 + \epsilon)
\]  

(B.92)
where the first equality results from (B.90). Note that (B.92) can be fulfilled because from Corollary 3.1 we get \( d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r_0) = d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r_0) \) for \( r > s \). In order to translate this condition to \( \epsilon \) we write the following inequality

\[
d_{s+1,3,s}^{*(\frac{r}{2}+1)}(r_0 + \epsilon) = MN - \left( \lfloor \frac{r}{2} \rfloor + 1 \right) \cdot \left( \lfloor \frac{r}{2} \rfloor + 2 \right) - \left( N + M - 1 - 2 \cdot \left( \lfloor \frac{r}{2} \rfloor + 1 \right) \right) (r_0 + \epsilon) >
MN - \lfloor \frac{r}{2} \rfloor \cdot \left( \lfloor \frac{r}{2} \rfloor + 1 \right) - \left( N + M - 1 - 2 \cdot \left( \lfloor \frac{r}{2} \rfloor \right) \right) r_0 = d_{s+1,3,s}^{*(\frac{r}{2})}(r_0)
\]  

(B.93)
When assigning \( K = 2, M = s + 1 \) and \( N = 3 \cdot s \) we get

\[
\epsilon < \frac{r_0}{s} - 1.
\]  

(B.94)
Hence, the set of multiplexing gain tuples we are considering is

\[
B_{r_0} = \left\{ r_1, r_2 | r_1 = r_0 + \epsilon, r_2 = r_0 - \epsilon, 0 < \epsilon < \min \left( \frac{r_0}{s} - 1, s + \frac{1}{2} \right) \right\}
\]  

(B.95)
where \( s < r_0 < s + \frac{1}{2} \) is a parameter determining the set. From [30, Lemma 7] we get that the optimal
DMT of finite constellations equals
\[ d_{2s+1,3s}^*(r_1, r_2) = \min \left( d_{s+1,3s}^*(r_1), d_{s+1,3s}^*(r_2), d_{2s+1,3s}^*(r_1 + r_2) \right). \tag{B.96} \]

Considering \((r_1, r_2) \in B_{r_0}\), based on (B.92), (B.95) and the fact that \(d_{s+1,3s}^*(r)\) is a straight line, we get
\[ d_{2s+1,3s}^*(r_1, r_2) = \min \left( d_{s+1,3s}^*(r_0 + \epsilon), d_{s+1,3s}^*(r_0 - \epsilon), d_{2s+1,3s}^*(2r_0) \right) \]
\[ = d_{2(s+1),3s}^*(2r_0) \quad 0 < \epsilon < \min \left( r_0 + \frac{r_0}{s} - 1, s + \frac{1}{2} \right) - r_0. \tag{B.97} \]

Hence, in order to prove (B.86) we need to show for certain \(0 < r_0 < s + \frac{1}{2}\) that
\[ \max_{(D_1, D_2)} \min \left( d_{s+1,3s}^*(r_0 + \epsilon), d_{s+1,3s}^*(r_0 - \epsilon), d_{2(s+1),3s}^*(2r_0) \right) < d_{2(s+1),3s}^*(2r_0) \tag{B.98} \]
where \(0 < \epsilon < \min \left( r_0 + \frac{r_0}{s} - 1, s + \frac{1}{2} \right) - r_0\). We begin the proof by taking the symmetric case, i.e. \(D_1 = D_2\), as a baseline. We assign \(D_1 = D_2 = D_{r_1} = D_1^*\). From (B.90) we get that \(d_{2(s+1),3s}^*(2r_0) = d_{s+1,3s}^*(r_0) = d_{2(s+1),3s}^*(2r_0)\). Hence for the symmetric case we get
\[ \min \left( d_{s+1,3s}^*(r_0 + \epsilon), d_{s+1,3s}^*(r_0 - \epsilon), d_{s+1,3s}^*(r_0) \right) = d_{s+1,3s}^*(r_0 + \epsilon) < d_{2(s+1),3s}^*(2r_0). \tag{B.99} \]

Since \(s < r_0 < s + \frac{1}{2}\) is not an anchor point, we get from Corollary 3.1 and (B.90) that \(d_{s+1,3s}^*(2r_0) = d_{2(s+1),3s}^*(2r_0)\) if and only if \(D_1 + D_2 = 2D_{r_1}^* = 2D_1^*\). Hence, in order for \(d_{s+1,3s}^*(2r_0) \) to attain the optimal DMT of finite constellations, we must choose
\[ D_1 + D_2 = 2D_1^*. \tag{B.100} \]

From (B.92), (B.99) we know that
\[ d_{s+1,3s}^*(r_0 + \epsilon) < d_{2(s+1),3s}^*(2r_0) < d_{s+1,3s}^*(r_0 + \epsilon). \tag{B.101} \]

Hence from Corollary 3.1 we can deduce that there must exist \(D' = D_{r_1}^* + \epsilon\), where \(0 < \epsilon < D_{r_1}^* + 1 - D_1^*\), such that
\[ d_{s+1,3s}^*(r_0 + \epsilon) = d_{2(s+1),3s}^*(2r_0). \tag{B.102} \]

We divide the assignment of \(D_1\) into several cases. For the case \(0 < D_1 < D'\) we get from Corollary 3.1 and the fact that \(s < r_0 + \epsilon < s + \frac{1}{2}\) is not an anchor point
\[ d_{s+1,3s}^*(r_0 + \epsilon) < d_{s+1,3s}^*(r_0 + \epsilon) = d_{2(s+1),3s}^*(2r_0). \tag{B.103} \]
Hence in this range the optimal DMT of finite constellations is not obtained. For the case \( D_1 = D' = D_{\lfloor \frac{1}{2} \rfloor} + \epsilon' \), we have shown (B.102) that \( d_{s+1,3,s}^{*,D'} (r_0 + \epsilon) \) equals to the optimal DMT of finite constellations. According to (B.100) we need to assign \( D_2 = D'' = D_{\lfloor \frac{1}{2} \rfloor} - \epsilon' \) in order to get

\[
d_{s+1,3,s}^{*,D'} (r_0 + \epsilon) = d_{2(s+1),3,s}^{*,2D_{\lfloor \frac{1}{2} \rfloor}} (2r_0) = d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

So far we have shown that the first two terms in the left side of (B.98) can attain the optimal DMT of finite constellations for \( D_1 = D' \). We are left with the third term that equals to the straight line \( d_{s+1,3,s}^{*,D''} (r) \). We consider two cases. For the first case we assume \( D'' < r_0 - \epsilon \). In this case we get

\[
d_{s+1,3,s}^{*,D''} (r_0 - \epsilon) < 0 < d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

For the second case we assume \( D'' > r_0 - \epsilon \). From symmetry considerations it can be easily shown that the straight line \( d' (r) \) fulfills \( d' (s) = d_{s+1,3,s}^{*,(FC)} (s) = d_{s+1,3,s}^{*,D'} (s) \) and \( d' (D'') = 0 \), also fulfills

\[
d' (r_0 - \epsilon) = d_{s+1,3,s}^{*,D'} (r_0 + \epsilon) = d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

Since \( D'' < D_{\lfloor \frac{1}{2} \rfloor} \), according to Corollary 3.1 we get

\[
d_{s+1,3,s}^{*,D''} (s) < d_{s+1,3,s}^{*,D'} (s) = d' (s).
\]

Since \( d_{s+1,3,s}^{*,D''} (D'') = d' (D'') = 0 \) and these are straight lines we get

\[
d_{s+1,3,s}^{*,D''} (r) < d' (r) \quad 0 < r < D''
\]

and so according to (B.104)

\[
d_{s+1,3,s}^{*,D''} (r_0 - \epsilon) < d' (r_0 - \epsilon) = d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

Thus, the third term in the left side of (B.98) \( d_{s+1,3,s}^{*,D_2} (r_0 - \epsilon) \) is smaller than the optimal DMT of finite constellations. Finally, we consider the case \( D_1 > D' \). For this case we get \( D_2 < D'' < D_{\lfloor \frac{1}{2} \rfloor} \), which according to Corollary 3.1 leads to

\[
d_{s+1,3,s}^{*,D_2} (r_0 - \epsilon) < d_{s+1,3,s}^{*,D''} (r_0 - \epsilon) < d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

From (B.103), (B.104), (B.108) and (B.109) we have proved that

\[
\max_{(D_1,D_2)} \min \left( d_{s+1,3,s}^{*,D_1} (r_0 + \epsilon), d_{s+1,3,s}^{*,D_1+D_2} (2r_0), d_{s+1,3,s}^{*,D_2} (r_0 - \epsilon) \right) < d_{2(s+1),3,s}^{*,(FC)} (2r_0).
\]

This concludes the proof.
B.9 Proof of Theorem 3.6

We base our proof on the techniques developed by Poltyrev [20] for the AWGN channel and extended in Chapter 2 to colored channels in the point-to-point case. We begin by partitioning the error event into several disjoint events of errors for subsets of the users. We relate each of these error events to the point-to-point channel of the relevant users pulled together. Then we use the bounds derived in Chapter 2 to upper bound each of the error events probabilities.

When the ML decoder makes an error it means that the decoded word is different from the transmitted signal for at least one of the users. Hence, we can break the error probability into the following sum of disjoint events

\[ \Pr_e(H_{\text{eff}}^{(l)}, K), \rho) = \sum_{\alpha \subseteq \{1, \ldots, K\}} \Pr_e(H_{\text{eff}}^{(l), (s)}, \rho) \]  

(B.111)

where \( \Pr_e(H_{\text{eff}}^{(l), (s)}, \rho) \) is the probability of error to words that induce error on the users in \( s \). Note that the event of error to users in \( s \) depends only on \( H_{\text{eff}}^{(l), (s)} \) and not on \( H_{\text{eff}}^{(l), (1, \ldots, K)} \). We wish to upper bound \( \Pr_e(H_{\text{eff}}^{(l), (s)}, \rho) \) for any \( s \subseteq \{1, \ldots, K\} \).

Based on [20] we have the following upper bound on the error probability of the joint ML decoder when transmitting \( x' \in S_{K-D_1T_1} \)

\[ \Pr_e(x) \leq \Pr(\|\tilde{n}_\infty\| \geq R) + \sum_{L \in \text{Ball}(x', 2R) \cap S_{K-D_1T_1} \setminus \{x'\}} \Pr(\|L - x' - \tilde{n}_{ex}\| < \|\tilde{n}_{ex}\|) \]  

(B.112)

where \( S_{K-D_1T_1} \) is the \( K \cdot D_1 \cdot T_1 \)-complex dimensional effective IC of the \( K \) users, \( \text{Ball}(x', 2R) \) is a \( K \cdot D_1 \cdot T_1 \)-complex dimensional ball of radius \( 2R \) centered around \( x' \), and \( \tilde{n}_\infty \) is the effective noise in the \( K \cdot D_1 \cdot T_1 \)-complex dimensional hyperplane where the effective IC resides. Instead of calculating (B.112), we focus on upper bounding the probability of decoding words that lead to an error only for the users in \( s \subseteq \{1, \ldots, K\} \) (B.111). This will lead to an upper bound on the error probability. Hence, we begin by considering the error probability of \( x' \) to words that are different from \( x' \) only in the entries of the users in \( s \). Based on our ensemble, this is the error event of users in \( s \) almost surely (with probability 1). This error event is equivalent to the error event of a word \( x'' \), which is a vector of length \( |s| \cdot D_1 \cdot T_1 \) that resides within an \( |s| \cdot D_1 \cdot T_1 \)-complex dimensional IC \( S_{|s|} \cdot D_1 \cdot T_1 \), when \( x'' \) equals to \( x' \) in the entries of the users in \( s \), and the other words in \( S_{|s|} \cdot D_1 \cdot T_1 \) are equal, in the entries of the users in \( s \), to words in \( S_{K-D_1T_1} \), that lead to an error for the users in \( s \). Hence, we wish to upper bound the error probability of \( x'' \in S_{|s|} \cdot D_1 \cdot T_1 \). Based on the expressions in (B.112) we get that this upper bound can be written as

\[ \Pr(\|\tilde{n}'_\infty\| \geq R') + \sum_{L \in \text{Ball}(x'', 2R') \cap S_{|s|} \cdot D_1 \cdot T_1 \setminus \{x''\}} \Pr(\|L - x'' - \tilde{n}'_{ex}\| < \|\tilde{n}'_{ex}\|) \]  

(B.113)

where \( \text{Ball}(x'', 2R') \) is a \( |s| \cdot D_1 \cdot T_1 \)-complex dimensional ball of radius \( 2R' \) centered around \( x'' \), and \( \tilde{n}'_\infty \) is the effective noise in the \( |s| \cdot D_1 \cdot T_1 \)-complex dimensional hyperplane where \( S_{|s|} \cdot D_1 \cdot T_1 \) resides.

Next we upper bound the average decoding error probability of an ensemble of finite constellations,
which later we will extend to ensemble of IC’s. Note that the upper bounds on the error probability of IC’s in (B.111), (B.112) also apply to finite constellations. Assume user \( j \) code-book contains \([\gamma_{tr}^{(j)}]b^{2D_tT_i}\) words, where each word is drawn independently and uniformly within cube_{\(D_tT_i\)}(\( b \)), \( j = 1, \ldots, K \). Recall from 3.2 that \( \gamma_{tr}^{(j)} = \rho^T \). The \( K \) users constitute together an ensemble of \( \prod_{j=1}^{K} [\gamma_{tr}^{(j)}]b^{2D_tT_i}\) words, where a word in the ensemble is sampled from a uniform distribution in cube_{\(D_tT_i\)}(\( b \)) (not all words are drawn independently). In fact any subset of the users \( s \subseteq \{1, \ldots, K\} \) corresponds to an ensemble of \( \prod_{i \in s} [\gamma_{tr}^{(i)}]b^{2D_tT_i}\) words, where again a word in the ensemble is sampled from a uniform distribution, this time in cube_{\(|s|\cdot D_tT_i\)}(\( b \)). Hence, the number of codewords that are different in the entries of the users in \( s \) is upper bounded by \( \prod_{i \in s} [\gamma_{tr}^{(i)}]b^{2D_tT_i}\). These words are in fact drawn independently in the entries of the users in \( s \). Based on these arguments and since the ML decoder decides on the word with minimal Euclidean distance from the observation, we get for each word in the ensemble that the probability of error for users in \( s \subseteq \{1, \ldots, K\} \) is upper bounded by the average decoding error probability of an ensemble consisting of \( \prod_{i \in s} [\gamma_{tr}^{(i)}]b^{2D_tT_i}\) words drawn independently and uniformly within cube_{\(|s|\cdot D_tT_i\)}(\( b \)), with effective channel \( H_{eff}^{(j),(s)} \). In Theorem 2.3 an upper bound on the average decoding error probability of this ensemble was derived. By choosing for any \( s \subseteq \{1, \ldots, K\} \)

\[
R^2_{(s)} = R^2_{eff} = \frac{2|s| \cdot D_t \cdot T_i}{2 \pi e} \rho^{-\frac{\sum_{i \in s} r_i}{|s| D_t} - \sum_{i=1}^{|s| D_t} \eta_i^{(s)}} \frac{\eta_i^{(s)}}{|s| D_t T_i},
\]

we get for the ensemble the following upper bound on the probability of error for users in \( s \)

\[
\overline{P_{err}^{(s)}}(\rho, \eta_i^{(s)}) \leq D'(|s| \cdot D_t \cdot T_i) \rho^{-T_i(|s| \cdot D_t - \sum_{i \in s} r_i) + \sum_{i=1}^{|s| D_t} \eta_i^{(s)}} \forall s \subseteq \{1, \ldots, K\} \tag{B.114}
\]

where \( D'(|s| \cdot D_t \cdot T_i) \geq 1 \) and \( \eta_i^{(s)} \geq 0, i = 1, \ldots, |s| \cdot D_t \cdot T_i \).

So far we have upper bounded the probability of error of users in \( s \), in an ensemble of finite constellations, for any \( s \subseteq \{1, \ldots, K\} \). Now we extend this ensemble of finite constellations into an ensemble of IC’s with density \( \gamma_{tr}^{(j)} \) for user \( j \), where \( j = 1, \ldots, K \). We show that extending the ensemble of finite constellations to ensemble of IC’s does not change the upper bound on the error probability. Let us consider for user \( j \) a certain finite constellation from the ensemble \( C_0^j(\rho, b) \subseteq cube_{D_tT_i}(b) \). In accordance, for the ensemble of users relates to \( s \) let us denote a certain finite constellation from the effective ensemble by \( C_0^s(\rho, b) \subseteq cube_{|s|\cdot D_tT_i}(b) \). We extend each finite constellation into IC by extending each user finite constellation in the following manner

\[
IC^j(\rho, D_t \cdot T_i) = C_0^j(\rho, b) + (b + b') \cdot Z^{2D_tT_i} \tag{B.115}
\]

where without loss of generality \(^1\) we assumed that \( cube_{D_tT_i}(b) \in C^{D_tT_i} \). Therefore for the users in \( s \subseteq \{1, \ldots, K\} \) we get an effective IC

\[
IC^s(\rho, |s| \cdot D_t \cdot T_i) = C_0^s(\rho, b) + (b + b') \cdot Z^{2|s| \cdot D_tT_i}. \tag{B.116}
\]

\(^1\) In case \( cube_{D_tT_i}(b) \) is a rotated cube within \( C^{D_tT_i} \), then the replication is done according the corresponding \( M \cdot T_i \times D_t \cdot T_i \) matrix with orthonormal columns.
At the receiver we get

\[ IC(s)(\rho, |s| \cdot D_l \cdot T_i, H^{(l),s}_{\text{eff}}) = H^{(l),s}_{\text{eff}} \cdot C_0(\rho, b) + (b + b')H^{(l),s}_{\text{eff}} \cdot Z^{2|s| \cdot D_l \cdot T_i}. \tag{B.117} \]

By extending each finite constellation in the ensemble into an IC according to the method presented in (B.116), (B.117) we get a new ensemble of IC’s. We would like to set \( b \) and \( b' \) to be large enough such that the ensemble average decoding error probability has the same upper bound as in (B.114), and the users densities are equal to \( \gamma^{(j)}_{tr} \) up to a coefficient, where \( j = 1, \ldots, K \). First we would like to set a value for \( b' \).

For a word within the set \( \{H^{(l),s}_{\text{eff}} \cdot C_0(s)(\rho, b)\} \), increasing \( b' \) decreases the error probability inflicted by the codewords outside the set \( \{H^{(l),s}_{\text{eff}} \cdot C_0(s)(\rho, b)\} \), for any \( s \subseteq \{1, \ldots, K\} \). In Theorem 2.3 we have shown that for any \( \eta^{(s)}_i \geq 0 \), by choosing \( b' = \sqrt{\frac{|s| \cdot D_l \cdot T_i \cdot \eta^{(s)}_i}{\pi e \cdot D_l \cdot T_i \cdot \sum_{i \in s} r_i}} \), we get for \( \rho \geq 1 \)

\[ Pe(H^{(l),s}_{\text{eff}}), \rho) = E_{C_0}(P_e^{IC}(H^{(l),s}_{\text{eff}} \cdot C_0(s))) \leq D(|s| \cdot D_l \cdot T_i) \rho^{-T_i(|s| \cdot D_l - \sum_{i \in s} r_i)} \cdot |s| \cdot D_l \cdot T_i \cdot \eta^{(s)}_i \tag{B.118} \]

where \( E_{C_0}(P_e^{IC}(H^{(l),s}_{\text{eff}} \cdot C_0(s))) \) is the average decoding error probability of the ensemble defined in (B.117), and \( D(|s| \cdot D_l \cdot T_i) \geq D'(|s| \cdot D_l \cdot T_i) \). Hence, choosing \( b' \) to be the maximal value between \( \sqrt{\frac{|s| \cdot D_l \cdot T_i \cdot \eta^{(s)}_i}{\pi e \cdot D_l \cdot T_i \cdot \sum_{i \in s} r_i}} + \epsilon \), where \( s \subseteq \{1, \ldots, K\} \) will enable to satisfy (B.118) for any \( s \).

Next, we set the value of \( b \) to be large enough such that for each user, each IC density from the ensemble in (B.117), \( \gamma^{(j)}_r \), equals \( \gamma^{(j)}_{rc} \) up to a factor of 2, where \( j = 1, \ldots, K \). By choosing \( b = b' \cdot \rho^s \) we get

\[ \gamma^{(j)}_r = \gamma^{(j)}_r \cdot (\frac{b}{b + b'})^{2D_l \cdot T_i} = \gamma^{(j)}_r \cdot \frac{1}{1 + \rho^{-s}}. \]

Hence, for \( \rho \geq 1 \) we get

\[ \frac{1}{2} \gamma^{(j)}_r \leq \gamma^{(j)}_r \leq \gamma^{(j)}_r. \tag{B.119} \]

As a result we also get

\[ \mu^{(j)}_{tr} \leq \mu^{(j)}_{tr} = \frac{(\gamma^{(j)}_r - \mu^{(j)}_r)}{2\pi e \cdot 2} \leq 2\mu^{(j)}_r. \]

Hence, from (B.111) and (B.118) we get that

\[ Pe(H^{(l),s}_{\text{eff}}, \rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} D(|s| \cdot D_l \cdot T_i) \rho^{-T_i(|s| \cdot D_l - \sum_{i \in s} r_i)} \cdot |s| \cdot D_l \cdot T_i \cdot \eta^{(s)}_i. \tag{B.120} \]

and from (B.119) we get that user \( j \) has multiplexing gain \( r_j \) as required, where \( j = 1, \ldots, K \). This concludes the proof.
B.10 Proof of Lemma 3.8

$H_{\text{eff}}^{(l),[s]}$ is a block diagonal matrix. Hence the determinant of $|H_{\text{eff}}^{(l),[s]}|^\dagger_H \cdot H_{\text{eff}}^{(l),[s]}|$ can be expressed as

$$|H_{\text{eff}}^{(l),[s]}|^\dagger_H \cdot H_{\text{eff}}^{(l),[s]}| = \prod_{i=1}^{T_i} \hat{H}_i^\dagger \cdot \hat{H}_i. \quad (B.121)$$

Assume $\hat{H}_i = (\hat{h}_1, \ldots, \hat{h}_{m_i})$, i.e. $\hat{H}_i$ has $m$ columns. In this case we can state that the determinant

$$|\hat{H}_i^\dagger \cdot \hat{H}_i| = \|\hat{h}_1\|^2 \|\hat{h}_{2.1}\|^2 \cdots \|\hat{h}_{m.1m-1}\|^2.$$ 

Note that $\hat{H}_i$ has more rows than columns. The columns of $\hat{H}_i$ are subset of the columns of the channel matrix $H$. Hence, in order to quantify the contribution of a certain column of $H$, $h_j$, $j = 1, \ldots, K \cdot M$, to the determinant we need to consider the blocks where it occurs. We know that the contribution of $h_j$ to these determinants can be quantified by taking into account the columns to its left in each block, i.e. by taking into account $\{\hat{h}_1, \ldots, \hat{h}_{j-1}\}$.

Based on (3.23) and (3.24) we can quantify the contribution of $h_j$ to $|H_{\text{eff}}^{(l),[s]}|^\dagger_H \cdot H_{\text{eff}}^{(l),[s]}|$ by

$$\|h_j\|^2 \sum_{j=0}^{B.125} \prod_{k=0}^{j-1} \|\hat{h}_{j\pm j-1,\ldots,\pm j-k}\|^2 2h^{(l)}_j(k) \equiv \rho - \sum_{k=0}^{j-1} b^{(l)}_j(k) \min_r \{k+r, N\} \xi_{r,j} \quad (B.122)$$

where $b^{(l)}_j(k)$ is the number of occurrences of $h_j$ in the blocks of $H_{\text{eff}}^{(l),[s]}$, with only $\{\hat{h}_{j-1}, \ldots, \hat{h}_{j-k}\}$ to its left. $b^{(l)}_j(0)$ is the number of occurrences of $h_j$ with no columns to its left. Hence, the determinant is obtained by multiplying the contribution of each column in $H_{\text{eff}}^{(l),[s]}$

$$|H_{\text{eff}}^{(l),[s]}|^\dagger_H \cdot H_{\text{eff}}^{(l),[s]}| = \prod_{j=1}^{M} \|h_j\|^2 \sum_{j=0}^{B.125} \prod_{k=0}^{j-1} \|\hat{h}_{j\pm j-1,\ldots,\pm j-k}\|^2 2h^{(l)}_j(k) \equiv \rho - \sum_{k=0}^{j-1} b^{(l)}_j(k) \min_r \{k+r, N\} \xi_{r,j}. \quad (B.123)$$

Now we lower bound the determinant (B.123) by lower bounding the contribution of each column. Let us consider column $h_{a \cdot M + b}$, $a = 0, \ldots, |s| - 1$, $b = 1, \ldots, M$. From Lemma 3.7 we know that $h_{a \cdot M + b}$ occurs $N - M + 1$ times with $\{h_1, \ldots, h_{a \cdot M + b - 1}\}$ to its left, i.e. $b^{(l)}_{a \cdot M + b}(a \cdot M + b - 1) = N - M + 1$. In addition, $h_{a \cdot M + b}$ occurs in $\hat{H}_{N - M + 2a + 1}$, $v = 1, \ldots, \min(M - l - 1, b - 1)$, with

$$\{h_1, \ldots, h_{a \cdot M + b - 1}\} \setminus \left( \bigcup_{z=0}^{a} h_{z \cdot M + 1}, \ldots, h_{z \cdot M + v} \right) \quad (B.124)$$

to its left, i.e. when $v$ is increased by one the number of columns to its left reduces by $a + 1$. Finally, $h_{a \cdot M + b}$ occurs in $\hat{H}_{N - M + 2a}$, $v = 1, \ldots, \min(M - l - 1, M - b)$, with

$$\{h_1, \ldots, h_{a \cdot M + b - 1}\} \setminus \left( \bigcup_{z=1}^{a} h_{z \cdot M + v + 1}, \ldots, h_{z \cdot M} \right). \quad (B.125)$$
to its left (for \(a = 0\) it occurs with \(\{b_1, \ldots, b_{a-1}\}\) to its left), i.e. when \(v\) is increased by one the number of columns to its left reduces by \(a\). We wish to quantify the change in the determinant when reducing columns, and relate it to the PDF in (3.22). In order to analyze the performance we would like the set of columns in (B.124) to be a subset of the set of columns in (B.125), which is not the case. Hence, we assume a columns reduction that gives a lower bound on the determinant induced by the reduction in (B.124) and (B.125). We assume for \(\hat{H}_{N-M+2v}, v = 1, \ldots, \min(M - l - 1, M - b)\) that \(h_{aM+b}\) occurs with \(\{h_1, \ldots, h_{aM+b-1}\}\) to its left instead of (B.125). In this case, by adding columns to (B.125) we get a lower bound on the contribution of \(h_{aM+b}\) to the determinant in each of its occurrences, that equals to

\[
\rho^{-\min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b}},
\]

(B.126)

for any \(v = 1, \ldots, \min(M - l - 1, M - b)\). On the other hand for (B.124) we assume that only the left most column is reduced when increasing \(v\), instead of the \(a + 1\) columns. This leads to lower bound to the contribution of (B.124) to the determinant that equals to

\[
\rho^{-\min_{z \in \{aM+b-v, \ldots, N\}} \xi_{z,aM+b} \cdot v = 1, \ldots, \min(M - l - 1, b - 1)}.
\]

(B.127)

Hence, we get that the set of columns corresponding to (B.127) is a subset of the set of columns corresponding to (B.126). Thus, from (B.126),(B.127) we get the following lower bound on the determinant

\[
|H^{(l),|s|\dagger}_\text{eff} \cdot H^{(l),|s|\dagger}_\text{eff} | \geq \prod_{a=0}^{\lfloor r_{\text{max}} \rfloor} \prod_{b=1}^{M} \rho^{-(N-M+1+\min(M-l-1, M-b))} \min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b} \cdot \\
\prod_{b' = 2}^{M} \rho^{-\sum_{i=1}^{\min(M-l-1, b'-1)}} \min_{z \in \{aM+b'-i, \ldots, N\}} \xi_{z,aM+b'},
\]

(B.128)

\section*{B.11 Proof of Theorem 3.7}

In order to lower bound the DMT of the transmission scheme we use the upper bound on the average decoding error probability from Theorem 3.6 and the lower bound on the determinant of \(|H^{(l),|s|\dagger}_\text{eff} \cdot H^{(l),|s|\dagger}_\text{eff} |\) (B.128), to get a new upper bound on the error probability. We average the new upper bound on the realizations of \(H\) to obtain the transmission scheme DMT.

First let us denote \(l = \lfloor r_{\text{max}} \rfloor\). Recall from Theorem 3.6 that the upper bound on the error probability applies when \(n_i^{(s)} \geq 0\), for every \(i = 0, \ldots, |s| \cdot D_l \cdot T_l\) and for any \(s \subseteq (1, \ldots, K)\). In our analysis we assume that \(\xi_{i,j} \geq 0\) for \(i = 1, \ldots, N, j = 1, \ldots, K \cdot M\). We wish to show that it leads to \(n_i^{(s)} \geq 0\), i.e. we can use the upper bound on the error probability. We know that \(H^{(l),|s|\dagger}_\text{eff}\) is a block diagonal matrix, where the set of columns of each block is a subset of \(\{h_{a1}, \ldots, h_{aK \cdot M}\}\). Let us denote the set of indices of the columns
of $H$ that take place in $H_{\text{eff}}^{(l),(s)}$ by $a(s)$. In this case we get from trace considerations

$$
\sum_{i=1}^{N} \sum_{j \in a(s)} \rho^{-\xi_{i,j}} = \sum_{i=1}^{N} \rho^{-\eta_{i}^{(s)}} \quad \forall s \subseteq \{1, \ldots, K\}
$$

which leads to

$$
\sum_{i=1}^{N} \sum_{j \in a(s)} \rho^{-\xi_{i,j}} = \sum_{i=1}^{N} \rho^{-\eta_{i}^{(s)}} \quad \forall s \subseteq \{1, \ldots, K\}.
$$

(B.129)

From (B.129) we get that $\xi_{i,j} \geq 0$ for $i = 1, \ldots, N$, $j = 1, \ldots, K \cdot M$ if and only if $\eta_{i}^{(s)} \geq 0$ for any $s \subseteq \{1, \ldots, K\}$ and $i = 1, \ldots, |s| \cdot D_{i} \cdot T_{i}$. Therefore, we can use the upper bound from Theorem 3.6.

The upper bound on the error probability consists of the sum of $\overline{Pe}(\eta^{(s)}, \rho)$ for all $s \subseteq \{1, \ldots, K\}$. We wish to show that the DMT of each of the terms is lower bounded by $d_{M,N}^{\text{return}}(FC) (r_{\text{max}})$. First note that $\forall s \subseteq \{1, \ldots, K\}$ we can write

$$
\overline{Pe}(\eta^{(s)}, \rho) = \min \left( 1, D ([|s| \cdot D_{i} \cdot T_{i}) \rho^{-T_{i}([|s| \cdot D_{i} \cdot \sum_{s \in a, r_{i}} r_{i}]) \cdot |H_{\text{eff}}^{(l),(s)}|^{l} H_{\text{eff}}^{(l),(s)} - 1} \right)
$$

$$
\leq \min \left( 1, D ([|s| \cdot D_{i} \cdot T_{i}) \rho^{-|s| \cdot T_{i}(D_{i} - r_{\text{max}}) \cdot |H_{\text{eff}}^{(l),(s)}|^{l} H_{\text{eff}}^{(l),(s)} - 1} \right).
$$

(B.130)

where the inequality comes from the fact that assuming all users transmit at the maximal multiplexing gain increases the error probability. By assigning $D_{i} = \frac{M \cdot N - l - (l+1)}{N + M - 1 - 2 \cdot l}$ and $T_{i} = N + M - 1 - 2 \cdot l$ we get

$$
\overline{Pe}(\eta^{(s)}, \rho) \leq \min \left( 1, D ([|s| \cdot D_{i} \cdot T_{i}) \rho^{-|s| \cdot (MN - l - (l+1) - (N + M - 1 - 2l) \cdot r_{\text{max}}) \cdot |H_{\text{eff}}^{(l),(s)}|^{l} H_{\text{eff}}^{(l),(s)} - 1} \right).
$$

(B.131)

From (3.18) we know that $E_{H} (\overline{Pe}(\eta^{(s)}, \rho)) = E_{H} (\overline{Pe}(\eta^{(1,\ldots,|s|)}, \rho))$, i.e., the term corresponding to the first $|s|$ users. Hence, for all terms with the same $|s|$ we can consider

$$
\overline{Pe}(\eta^{(1,\ldots,|s|)}, \rho) \leq \min \left( 1, D ([|s| \cdot D_{i} \cdot T_{i}) \rho^{-|s| \cdot (MN - l - (l+1) - (N + M - 1 - 2l) \cdot r_{\text{max}}) \cdot |H_{\text{eff}}^{(l),(s)}|^{l} H_{\text{eff}}^{(l),(s)} - 1} \right).
$$

(B.132)

Based on (B.128) let us define

$$
A(a \cdot M + b, l) = (N - b + 1) \min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b}
$$

(B.133)

for $b = 1, a = 0, \ldots, |s| - 1$, and

$$
A(a \cdot M + b, l) = (N - b + 1) \min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b} + \sum_{i=1}^{\min(M-l-1,b-1)} \min_{z \in \{aM+b-i, \ldots, N\}} \xi_{z,aM+b}
$$

(B.134)

for $b = 2, \ldots, M$ and $a = 0, \ldots, |s| - 1$. From the bounds in (B.126), (B.127), (B.128) and also since $N - M + 1 + \min (M - l - 1, M - b) \leq N - b + 1$, we get that $\rho^{-A(a \cdot M + b, l)}$ gives a lower bound on the
contribution of $\frac{h}{n} \cdot M + b$ to the determinant. As a result we get the following upper bound

$$|H_{\text{eff}}^{(l)}| |s| |H_{\text{eff}}^{(l)}| |s| |s|^{-1} \leq \prod_{a=0}^{M} \prod_{b=1}^{M} \rho^{A(a \cdot M+b,l)}.$$  \hfill (B.135)

By assigning in the bound from (B.132) we get

$$\mathcal{P}(E_{\{\eta^{(1,\ldots,|s|)}}, \rho) \leq \rho^{-\left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot d \xi_{i,j}$$  \hfill (B.136)

where $(x)^{+}$ equals $x$ for $x \geq 0$ and 0 else.

Based on (B.136) the average over the channel realizations can be upper bounded by

$$E_{H} \left( \mathcal{P}(E_{\{\eta^{(s)}}, \rho) \right) = E_{H} \left( \mathcal{P}(E_{\{\eta^{(1,\ldots,|s|)}}, \rho) \right)$$

$$\leq \int_{\xi_{i,j} \geq 0} \rho^{-\left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot d \xi_{i,j}$$

$$= \int_{\xi_{i,j} \geq 0} \rho^{-\left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot d \xi_{i,j}$$

where $\xi_{i,j} \geq 0$ means $\xi_{i,j} \geq 0$ for $i = 1, \ldots, N$ and $j = 1, \ldots, K \cdot M$. We divide the integration range to two sets

$$\int_{\xi_{i,j} \in A} \rho^{-\left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot d \xi_{i,j} + \int_{\xi_{i,j} \in \overline{A}} \rho^{-\left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot d \xi_{i,j}$$

where $A = \left\{ \bigcap_{i=1}^{N} \bigcap_{j=1}^{K \cdot M} 0 \leq \xi_{i,j} \leq K \cdot M \cdot N \right\}$, $\overline{A} = \left\{ \bigcup_{i=1}^{N} \bigcup_{j=1}^{K \cdot M} \xi_{i,j} > K \cdot M \cdot N \right\}$, and for the second term in (B.138) we upper bounded the error probability per channel realization by 1.

We begin by lower bounding the DMT of the first term in (B.138). In a similar manner to [35] and Chapter 2, for very large $\rho$ and finite integration range, we can approximate the integral by finding the most dominant exponential term. Hence, for large $\rho$ the first term in (B.138) equals

$$\rho^{-\min_{\xi_{i,j} \in A} \left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot \xi_{i,j}$$

$$\min_{\xi_{i,j} \in A} \left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot \xi_{i,j}$$

Hence, by showing that

$$\min_{\xi_{i,j} \in \mathcal{A}} \left(|s|\cdot(N+M-1-2l)_{\max}\right)} - \sum_{i=1}^{N} A(i,l)^{\frac{K \cdot M}{|s|}} \cdot \xi_{i,j} \geq MN - l (l + 1) - (N + M - 1 - 2l)_{\max}$$

we get that the first term attains DMT which is lower bounded by $d_{M,N}^{\ast}(FC)$ (r_{\max}). In order to show (B.140)
we use the following lemma.

**Lemma B.1.** The solution for the minimization problem

\[
\min_{\xi_{i,j} \in A} \left( |s| \cdot (MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}}) - \sum_{i=1}^{\left|s\right|} A(i,l) \right) + \sum_{i=1}^{N} \sum_{j=1}^{K} \xi_{i,j}
\]

equals to the solution for the following minimization problem

\[
\min_{\alpha \in A'} \sum_{i=1}^{\left|s\right| - M} (N - i + 1) \alpha_i
\]

where \(\alpha = (\alpha_1, \ldots, \alpha_{|s| \cdot M})^T\), and the set \(A'\) fulfills the following two conditions: \(0 \leq \alpha_i \leq K \cdot M \cdot N\) for \(i = 1, \ldots, |s| \cdot M\) and also

\[
\sum_{a=0}^{\left|s\right|-1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}}.
\]

**Proof.** The proof is in appendix B.12. \(\square\)

Based on Lemma B.1 we can see that by proving

\[
\min_{\alpha \in A'} \sum_{i=1}^{\left|s\right| - M} (N - i + 1) \alpha_i \geq MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}} \tag{B.141}
\]

we also prove (B.140). Therefore, we wish to show that any vector \(\alpha \in A'\) fulfills this inequality. Consider a certain vector \(\alpha \in A'\). We define \(\beta_{a \cdot M + b} = \frac{(N + 1 - b) \cdot \alpha_{a \cdot M + b}}{|s|}\) for \(a = 0, \ldots, |s| - 1, b = 1, \ldots, M\). From this definition we get

\[
\sum_{a=0}^{\left|s\right|-1} \sum_{b=1}^{M} \beta_{a \cdot M + b} = \sum_{a=0}^{\left|s\right|-1} \sum_{b=1}^{M} \frac{(N - b + 1) \alpha_{a \cdot M + b}}{|s|} = MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}}. \tag{B.142}
\]

By assigning in (B.141) we get

\[
\sum_{a=0}^{\left|s\right|-1} \sum_{b=1}^{M} (N - a \cdot M - b + 1) \alpha_{a \cdot M + b} = \sum_{a=0}^{\left|s\right|-1} \sum_{b=1}^{M} \frac{|s| \cdot (N - a \cdot M - b + 1) \beta_{a \cdot M + b}}{N - b + 1}. \tag{B.143}
\]

We use the following lemma to prove (B.141).

**Lemma B.2.** Consider \(N \geq (|s| + 1) M - 1\), we get for any \(a = 0, \ldots, |s| - 1\) and \(b = 1, \ldots, M\)

\[
\frac{|s| \cdot (N - (a \cdot M + b) + 1)}{N - b + 1} \geq 1.
\]
Proof. The proof is in appendix B.13.

Since $K \geq |s|$ and $N \geq (K + 1) M - 1$ we can assign the inequality of Lemma B.2 in (B.143) to get

$$
\sum_{a=0}^{s-1} \sum_{b=1}^{M} (N - a \cdot M - b + 1) \alpha_{a:M+b} \geq \sum_{a=0}^{s-1} \sum_{b=1}^{M} \beta_{a:M+b} = MN - l (l + 1) - (N + M - 1 - 2l) r_{\max}
$$

(B.144)

where the equality results from (B.142). This proves (B.141) and so proves that the DMT of the first term in (B.138) is lower bounded by $d_{M,N}^{*,(FC)}(r_{\max})$.

Now we show that the second term in (B.138) is also lower bounded by $d_{M,N}^{*,(FC)}(r_{\max})$.

$$
\int_{\xi_{1,1} \in A} 1 \cdot \rho^{-\sum_{i=1}^{N} \sum_{j=1}^{K} \xi_{i,j} d_{\xi_{1,2}} \leq \int_{\xi_{1,1} > K \cdot M \cdot N} \rho^{-\xi_{1,1}} \geq \rho^{-K \cdot M \cdot N}.
$$

Since $d_{M,N}^{*,(FC)}(r_{\max}) \leq K \cdot M \cdot N$ the DMT of the second term in (B.138) is also lower bounded by $d_{M,N}^{*,(FC)}(r_{\max})$.

We have shown that for $l = \lfloor r_{\max} \rfloor$ the DMT of $E_{H}(\overline{Pe}^{(s)}(\mu), \rho)$ is lower bounded by $d_{M,N}^{*,(FC)}(r_{\max}) = MN - l (l + 1) - (M + N - 1 - 2l) r_{\max}$ for any $s \subseteq \{1, \ldots, K\}$. Since

$$
\overline{Pe}(H_{\text{eff}}^{(l),K}, \rho) \leq \sum_{s \subseteq \{1, \ldots, K\}} \overline{Pe}(\mu^{(s)}(\mu), \rho)
$$

we get that the DMT of the $K$ sequences of IC’s is also lower bounded by $d_{M,N}^{*,(FC)}(r_{\max})$. This concludes the proof.

B.12 Proof of Lemma B.1

Recall that the optimization problem

$$
\min_{\xi_{1,2} \in A} \left( |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{\max}) - \sum_{i=1}^{s} A(i, l) \right) + \sum_{i=1}^{N} \sum_{j=1}^{K \cdot M} \xi_{i,j}
$$

(B.145)

where

$$
A(a \cdot M + b, l) = (N - b + 1) \min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b}
$$

for $b = 1$ and $a = 0, \ldots, |s| - 1$, and

$$
A(a \cdot M + b, l) = (N - b + 1) \min_{z \in \{aM+b, \ldots, N\}} \xi_{z,aM+b} + \sum_{i=1}^{\min(M-l-1,b-1)} \min_{z \in \{aM+b-i, \ldots, N\}} \xi_{z,aM+b}
$$

(B.147)

for $b = 2, \ldots, M$ and $a = 0, \ldots, |s| - 1$. When $|s| \cdot M + 1 \leq j \leq K \cdot M$ and $1 \leq i \leq N$, we get that $\xi_{i,j}$ occurs only in the term $\sum_{i=1}^{N} \sum_{j=1}^{K \cdot M} \xi_{i,j}$ in (B.145), where $\xi_{i,j} \geq 0$. Thus, the minimization is obtained...
Therefore, we can rewrite the optimization problem

$$
\min_{\xi_{i,j} \in \mathcal{A}} \left( |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{\text{max}}) - \sum_{i=1}^{\lfloor s/2 \rfloor} A(i,l) \right)^+ + \sum_{i=1}^{N} \sum_{j=1}^{|s|-M} \xi_{i,j}. \quad (B.149)
$$

Now we wish to show that $\xi_{i,j} = 0$ for $j = 1, \ldots, |s| \cdot M$ and $i = 1, \ldots, j - 1$. Essentially, we show for $i < j$ that reducing $\xi_{i,j}$ affects (B.149) more than $- \min_{z \in \{1, \ldots, N\}} \xi_{z,j}$ does. First let us observe $\xi_{i,a \cdot M + b}$ for $i = 1, \ldots, a \cdot M + b - \min(M-l-1,b-1)$, where $a = 0, \ldots, |s| - 1$, $b = 1, \ldots, M$. Note that this values do not have any representation in $A(a \cdot M + b, l)$. Therefore, they do not affect $(\cdot)^+$ and only affect $\sum_{i=1}^{N} \sum_{j=1}^{|s|-M} \xi_{i,j}$. Thus, in order to obtain the minimum we must choose

$$
\xi_{i,a \cdot M + b} = 0 \quad i = 1, \ldots, a \cdot M + b - \min(M-l-1,b-1) - 1
$$

for any $a = 0, \ldots, |s| - 1$ and $b = 1, \ldots, M$. Note that the function in (B.149) is continues. In case $(\cdot)^+ = 0$ the function in (B.149) can be written as

$$
\sum_{a=0}^{\lfloor s/2 \rfloor} \sum_{b=1}^{M} \sum_{i=a \cdot M + b - \min(M-l-1,b-1)}^{M} \xi_{i,a \cdot M + b} \quad \quad (B.150)
$$

In this case as long as $(\cdot)^+ = 0$ reducing $\xi_{i,a \cdot M + b}$ for $a \cdot M + b - \min(M-l-1,b-1) \leq i \leq a \cdot M + b - 1$ and $a = 0, \ldots, |s| - 1$, $b = 2, \ldots, M$ also reduces (B.150). When $(\cdot)^+ > 0$ (B.149) can be written as

$$
\begin{align*}
&|s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{\text{max}}) \\
&+ \sum_{a=0}^{\lfloor s/2 \rfloor} \sum_{b=2}^{M} \sum_{i=1}^{\min(M-l-1,b-1)} \left( \xi_{a \cdot M + b - i,a \cdot M + b} - \min_{z \in \{a \cdot M + b - i, \ldots, N\}} \xi_{z,a \cdot M + b} \right) \\
&+ \sum_{a=0}^{\lfloor s/2 \rfloor} \sum_{b=1}^{M} \left( \sum_{z=a \cdot M + b}^{N} \xi_{z,a \cdot M + b} - (N - b + 1) \min_{z \in \{a \cdot M + b, \ldots, N\}} \xi_{z,a \cdot M + b} \right). \quad (B.151)
\end{align*}
$$

Since $\xi_{a \cdot M + b - i,a \cdot M + b} \geq \min_{z \in \{a \cdot M + b - i, \ldots, N\}} \xi_{z,a \cdot M + b}$, reducing $\xi_{a \cdot M + b - i,a \cdot M + b}$ also reduces (B.151). Since the function is continues, considering these two cases is sufficient in order to state that the minimum is obtained when

$$
\xi_{i,j} = 0 \quad j = 1, \ldots, |s| \cdot M, \quad i = 1 \ldots, j - 1. \quad (B.152)
$$

This is due to the fact that for any value of $\xi_{z,a \cdot M + b} \geq 0$, $a = 0, \ldots, |s| - 1$, $b = 1, \ldots, M$ and $z = a \cdot M + b, \ldots, N$ the terms in (B.150),(B.151) are reduced when decreasing $\{\xi_{a \cdot M + b - i,a \cdot M + b}\}_{i=1}^{\min(M-l-1,b-1)}$, and also since the function is continues. Note that from (B.151) we can see that decreasing $\sum_{z=a \cdot M + b}^{N} \xi_{z,a \cdot M + b}$ does not necessarily decrease the function. This is due to the fact that $N - b + 1 \geq N - (a \cdot M + b) + 1$. 

139
and so the contribution of \((N - b + 1) \min_{z \in \{a \cdot M + b, \ldots, N\}} \xi_{z,a \cdot M + b}\) may be more significant than

\[
\sum_{z = a \cdot M + b}^{N} \xi_{z,a \cdot M + b}.
\]

Based on (B.152) we can rewrite the function in the following manner

\[
\left( |s| \cdot (MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}}) - \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \min_{z \in \{a \cdot M + b, \ldots, N\}} \xi_{z,a \cdot M + b} \right)^+ \\
+ \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} \sum_{z = a \cdot M + b}^{N} \xi_{z,a \cdot M + b}.
\]

(B.153)

From (B.153) we can see that the minimum is obtained when

\[
\xi_{z,a \cdot M + b} = \alpha_{a \cdot M + b} \quad a \cdot M + b \leq z \leq N
\]

for \(a = 0, \ldots, \lfloor s \rfloor - 1, b = 1, \ldots, M\). This is due to the fact that when the values are not equal, reducing the values to the minimal value will reduce \(\sum_{z = a \cdot M + b}^{N} \xi_{z,a \cdot M + b}\) while not changing \(\min_{z \in \{a \cdot M + b, \ldots, N\}} \xi_{z,a \cdot M + b}\). Therefore, we can write (B.153) as follows

\[
\left( |s| \cdot (MN - l(l + 1) - (N + M - 1 - 2l)r_{\text{max}}) - \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} \right)^+ \\
+ \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - (a \cdot M + b) + 1) \alpha_{a \cdot M + b}
\]

(B.155)

where \(0 \leq \alpha_i \leq K \cdot M \cdot N, i = 1, \ldots, \lfloor s \rfloor \cdot M\).

We wish to show that the minimum is obtained when

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot (MN - l(l + 1) - (N + M - 1 - 2l) r_{\text{max}}).
\]

Again, note that the function is continues. For the case where \((\cdot)^+ = 0\) we get

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - (a \cdot M + b) + 1) \alpha_{a \cdot M + b}.
\]

(B.156)

This is attained when

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} \geq |s| \cdot (MN - l(l + 1) - (N + M - 1 - 2l) r_{\text{max}}).
\]
Evidently for this case the minimal values occur at

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max}).
\]

On the other hand for the case \((\cdot)^+ > 0\) we get

\[
|s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max}) - \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (a \cdot M) \alpha_{a \cdot M + b}. \quad (B.157)
\]

Hence increasing \(\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (a \cdot M) \alpha_{a \cdot M + b}\) decreases the function as long as \((\cdot)^+ > 0\) which means

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} < |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max}).
\]

Hence, based on the fact that the function is continues we get again that for this case the minimal values occur at

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max}).
\]

The event

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max})
\]

where \(\alpha_i \geq 0, i = 1, \ldots, |s| \cdot M\), is within the range \(0 \leq \alpha_i \leq K \cdot M \cdot N, i = 1, \ldots, |s| \cdot M\). This is because in order to fulfil the equality we get

\[
\max(\alpha_1, \ldots, \alpha_{|s| \cdot M}) \leq \frac{|s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max})}{N - b + 1} \leq K \cdot M \cdot N.
\]

Therefore, the minimization problem solution is obtained for

\[
\min_{\mathcal{A}^*} \sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - (a \cdot M + b) + 1) \alpha_{a \cdot M + b}
\]

where the set \(\mathcal{A}^*\) is defined by the following two conditions: \(0 \leq \alpha_i \leq K \cdot M \cdot N, i = 1, \ldots, |s| \cdot M\), and

\[
\sum_{a=0}^{\lfloor s \rfloor - 1} \sum_{b=1}^{M} (N - b + 1) \alpha_{a \cdot M + b} = |s| \cdot (MN - l (l + 1) - (N + M - 1 - 2l) r_{max}).
\]
B.13 Proof of Lemma B.2

We begin by analyzing the case \( a = |s| - 1 \) and \( b = M \). For this case let us consider \( N = (|s| + 1) M - 1 \). In this case we get

\[
\frac{|s| (N - |s|) \cdot M + 1}{N - M + 1} = \frac{|s| (M)}{|s|M} = 1. \tag{B.158}
\]

Note that for \( c \geq d \geq 0 \) and \( x_2 > x_1 \geq c \) we get

\[
\frac{x_2 - c}{x_2 - d} \geq \frac{x_1 - c}{x_1 - d}. \tag{B.159}
\]

Hence, based on (B.159), (B.158), we get for \( N > (|s| + 1) M - 1 \)

\[
\frac{|s| (N - (|s|) \cdot M - 1)}{N - (M - 1)} \geq \frac{|s| (M)}{|s|M} = 1. \tag{B.160}
\]

So far we have proved the lemma for \( a = |s| - 1 \), \( b = M \) and \( N \geq (|s| + 1) M - 1 \). For the general case we consider \( \frac{|s|(a \cdot M + b - 1)}{N - (b - 1)} \). In this case we get

\[
\frac{|s| (N - (a \cdot M + b - 1))}{N - (b - 1)} = |s| \frac{(N + |s|M - a \cdot M - b) - (|s|M - 1)}{(N + |s|M - a \cdot M - b) - (M - 1)} \geq |s| \frac{(N + |s|M - a \cdot M - b) - (|s|M - 1)}{(N + |s|M - a \cdot M - b) - (M - 1)} \tag{B.161}
\]

where the inequality results from the fact that \( M - b \leq |s|M - a \cdot M - b \). From (B.159) and (B.160) we get that

\[
|s| \frac{(N + |s|M - a \cdot M - b) - (|s|M - 1)}{(N + |s|M - a \cdot M - b) - (M - 1)} \geq |s| \frac{N - (|s|M - 1)}{N - (M - 1)} \geq 1. \tag{B.162}
\]

From (B.161), (B.162) we get the proof of the lemma also for any \( a = 0, \ldots, |s| - 1 \) and \( b = 1, \ldots, M \). This concludes the proof.

B.14 Proof of Theorem 3.8

We prove that there exists \( K \) sequences of \( 2 \cdot D_1 \cdot T_1 \)-real dimensional lattices (as a function of \( \rho \)) that attains the optimal DMT for \( N \geq (K + 1) M - 1 \). We rely on the extension of the Minkowski-Hlawaka Theorem to the multiple-access channel presented in [18, Theorem 2]. We upper bound the error probability of the ensemble of lattices for each channel realization, and average the upper bound over all channel realizations to obtain the optimal DMT.

We consider \( K \) ensembles of \( 2 \cdot D_1 \cdot T_1 \)-real dimensional lattices, one for each user, transmitted using \( G_t^{(1,...,K)} \) defined in 3.4.2. For user \( i \), the first \( D_1 \cdot T_1 \) dimensions of the lattice are spread on the real part of the non-zero entries of \( G_t^{(i)} \), and the other \( D_1 \cdot T_1 \) dimensions of the lattice on the imaginary part of the non-zero entries of \( G_t^{(i)} \). The volume of the Voronoi region of the lattice of user \( i \) equals \( V_f^{(i)} = (\gamma_{tr}^{(i)})^{-1} = \rho^{-r_i T_1} \), i.e. multiplexing gain \( r_i \). Since the users are distributed, the effective lattice at the transmitter can be written
as \( \Lambda_{tr} = \Lambda_1 \times \Lambda_2 \times \cdots \times \Lambda_K \), where \( \Lambda_i \) is the lattice transmitted by user \( i \). At the receiver the channel induces a new lattice \( H_{eff}^{(l)K} \cdot \mathbf{x}' \), where \( \mathbf{x}' \in \Lambda_{tr} \). For lattices with regular lattice decoding, the error probability is equal among all codewords. Hence, it is sufficient to analyze the lattice’s zero codeword error probability. Without loss of generality let us assume that the receiver rotates \( \mathbf{y}_{ex} \) such that the channel can be rewritten as

\[
y_{ex} = B \cdot \mathbf{x} + \tilde{\mathbf{n}}_{ex}
\]

where \( B^\dagger B = H_{eff}^{(l)K} K^\dagger H_{eff}^{(l)K} \), and \( \tilde{\mathbf{n}}_{ex} \sim \mathcal{CN}(0, \rho^{-1} \cdot \frac{1}{2 \pi e} \cdot I_{K \cdot D_l \cdot T_l}) \).

We define the indication function of a 2 \( \cdot \) \( K \cdot D_l \cdot T_l \) dimensional ball with radius 2\( R \) centered around zero by

\[
I_{Ball(2R)}(\mathbf{x}) = \begin{cases} 
1, & \| \mathbf{x} \| \leq 2R \\
0, & \text{else}
\end{cases}
\]

In addition let us define the continues function of bounded support \( f_{rc}(\mathbf{x}) = I_{Ball(2R \cdot eff)}(\mathbf{x}) \cdot Pr(\| \tilde{\mathbf{n}}_{ex} \| > \| \mathbf{x} - \tilde{\mathbf{n}}_{ex} \| ) \). Based on (B.112) we can state that for each lattice induced at the receiver, \( \Lambda_{rc} \), the lattice zero codeword error probability is upper bounded by

\[
\sum_{\mathbf{x} \in \Lambda_{rc}, \mathbf{x} \neq 0} f_{rc}(\mathbf{x}) + Pr(\| \tilde{\mathbf{n}}_{ex} \| \geq R_{eff}).
\]  

(B.164)

where \( \frac{R_{eff}^2}{2K_l T_l \sigma_x^2} = \mu_{rc} = \rho \frac{\pi K_l T_l}{K_{eff}^\dagger H_{eff}^{(l)K} K^\dagger H_{eff}^{(l)K}} \). For regular lattice decoding we can equivalently consider

\[
y_{ex}' = B^{-1} \cdot y_{ex} = \mathbf{x} + \tilde{\mathbf{n}}_{ex}.
\]  

(B.165)

where \( \tilde{\mathbf{n}}_{ex} \sim \mathcal{CN}(0, (H_{eff}^{(l)K} K^\dagger H_{eff}^{(l)K})^{-1}) \), i.e. the lattice at the receiver remains \( \Lambda_{tr} \) and the effect of the channel realization is passed on to the additive noise. In addition let us denote an indication function over an ellipse centered around zero by

\[
I_{ellipse(B,2R)}(\mathbf{x}) = \begin{cases} 
1, & \| B \cdot \mathbf{x} \| \leq 2R \\
0, & \text{else}
\end{cases}
\]

By defining the continues function \( g_{rc}(\mathbf{x}) = I_{ellipse(B,2R \cdot eff)}(\mathbf{x}) \cdot Pr(\| B \tilde{\mathbf{n}}_{ex} \| > \| B(\mathbf{x} - \tilde{\mathbf{n}}_{ex}) \| ) \) we get the following upper bound on the error probability

\[
\sum_{\mathbf{x} \in \Lambda_{tr}, \mathbf{x} \neq 0} g_{rc}(\mathbf{x}) + Pr(\| B \cdot \tilde{\mathbf{n}}_{ex} \| \geq R_{eff})
\]  

(B.166)

that equals to the upper bound in (B.164). In addition, since \( f_{rc}(B \cdot \mathbf{x}) = g_{rc}(\mathbf{x}) \), and based on the fact that \( H_{eff}^{(l)K} \) is a block diagonal matrix we get

\[
\left| H_{eff}^{(l)(S)} H_{eff}^{(l)(S)} \right|^{-1} \cdot \int_{\mathbf{x} \in \mathbb{R}^{2 |S| \cdot D_l \cdot T_l}} f_{rc}(\mathbf{x}^{(S)}) \, d\mathbf{x}^{(S)} = \int_{\mathbf{x} \in \mathbb{R}^{2 |S| \cdot D_l \cdot T_l}} g_{rc}(\mathbf{x}^{(S)}) \, d\mathbf{x}^{(S)} \quad \forall S \subseteq \{1, \ldots, K\}
\]  

(B.167)
where $\varphi^{(S)}$ equals zero in the entries corresponding to $\{1, \ldots, K\} \setminus S$ and the other entries are in $\mathbb{R}^{|S| \cdot D_l \cdot T_l}$.

In [18, Theorem 2] Nam and El Gamal extended the Minkowski-Hlawka theorem to the multiple-access channel by using Loeliger ensembles of lattices [16] for each user. From this theorem we get that for a certain Riemann integrable function of bounded support $f(\varphi)$

$$E_{\Lambda_{tr}} \left( \sum_{\varphi \in \Lambda_{tr}, \varphi \neq 0} f(\varphi) \right) = \sum_{S \subseteq \{1, \ldots, K\}} \prod_{s \in S} \frac{1}{V_f} \int_{\varphi^{(S)} \in \mathbb{R}^{|S| \cdot D_l \cdot T_l}} f(\varphi^{(S)}) d\varphi^{(S)}. \quad (B.168)$$

For each channel realization $B$, the function $g_{rc}(\varphi)$ is bounded, and so by averaging over the Loeliger ensembles for the multiple-access channel, we get based on (B.166), (B.168) that the upper bound on the error probability using regular lattice decoding is

$$\sum_{S \subseteq \{1, \ldots, K\}} \prod_{s \in S} \frac{1}{V_f} \int_{\varphi^{(S)} \in \mathbb{R}^{|S| \cdot D_l \cdot T_l}} g_{rc}(\varphi^{(S)}) d\varphi^{(S)} + Pr(\|B \cdot \tilde{n}_{ex}\| \geq R_{eff}). \quad (B.169)$$

By assigning the relation of (B.167) in (B.169) we get

$$\sum_{S \subseteq \{1, \ldots, K\}} \rho^{T_l} \sum_{s \in S} r_s |H_{eff}^{(l), (S)}| H_{eff}^{(l), (S)^\dagger} |^{-1} \int_{\varphi^{(S)} \in \mathbb{R}^{|S| \cdot D_l \cdot T_l}} f_{rc}(\varphi^{(S)}) d\varphi^{(S)} + Pr(\|\tilde{n}_{ex}\| \geq R_{eff}). \quad (B.170)$$

Based on the bounds derived in Theorem 2.3, we can upper bound the integral of the first term in (B.170) by

$$\sum_{S \subseteq \{1, \ldots, K\}} \frac{4|S| \cdot D_l \cdot T_l}{2e^{V_f|S| \cdot D_l \cdot T_l}} \rho^{T_l} (|S| \cdot D_l - \sum_{s \in S} r_s) |H_{eff}^{(l), (S)}| H_{eff}^{(l), (S)^\dagger} |^{-1}.$$ 

Since we consider radius of $R_{eff}$, for large values of $\rho$ the second term in (B.170) is negligible compared to the first term (see the proof of Theorem 2.3). Hence, the remaining step is calculating the average over all channel realizations. We divide the average into two ranges $A$ and $\overline{A}$ as depicted in Theorem 3.7. For each channel realizations in $A$ we upper bound the error probability by one. As shown in Theorem 3.7, the probability of receiving channel realizations in this range has exponent that is lower bounded by the optimal DMT. For channel realizations in $\overline{A}$ we get that $g_{rc}(\varphi)$ has bounded support, and so we can use the Minkowski-Hlawka theorem to get the upper bound in (B.170). This bound coincides with the upper bound in Theorem 3.7 which was shown to obtain the optimal DMT. This concludes the proof.

### B.15 Proof of Corollary 3.2

We first consider the symmetric case $r_1 = \cdots = r_K = r_{\text{max}}$. Similarly to Corollary 2.3 we can state that if a sequence of $K$ lattices attains diversity order $d$ for symmetric multiplexing gain $r_{\text{max}} = 0$, it also attains diversity order

$$d \left( 1 - \frac{r_{\text{max}}}{D_{[r_{\text{max}}]T_{[r_{\text{max}}]}}} \right). \quad (B.171)$$
for any symmetric multiplexing gain $0 < r_{\text{max}} \leq D_{\lfloor r_{\text{max}} \rfloor} T_{\lfloor r_{\text{max}} \rfloor}$. This is due to the fact that changing $r_{\text{max}}$ merely has the effect of scaling the effective lattice at the receiver. From Theorem 3.8 we get that there exists a sequence of $K$ lattices (one for each user) that attains for symmetric multiplexing gain $r_{\text{max}} = l$ the optimal DMT $d_{M,N}^* (F_C) (l)$, where $l = 0, \ldots, M - 1$. In this case we also get from (B.171) and Theorem 3.8 that this sequence also attains the optimal DMT $d_{M,N}^* (F_C) (r_{\text{max}})$, when the symmetric multiplexing gain is in the range $l \leq r_{\text{max}} \leq l + 1$.

Now consider for the same sequence of lattices a multiplexing gains tuple $(r_1, \ldots, r_K)$ with $r_{\text{max}}$ as its maximal multiplexing gain. The performance can only improve compared to the symmetric case since some of the multiplexing gains of the users are smaller than $r_{\text{max}}$. Since the DMT can not be any larger than $d_{M,N}^* (F_C) (r_{\text{max}})$, which is already obtained in the symmetric case, we get that $d_{M,N}^* (F_C) (r_{\text{max}})$ is obtained by any multiplexing gains tuple with $r_{\text{max}}$ as its maximal value.
Bibliography


יחס הגומלין בין קצב דעיכת הסתברות השגיאה
ומספר דרגות החופש של קונסטלציות אינן
סופיות בערוצי דעיכה מרובי מיבור בויסות ההתאインターフאר

ייאיר יונה

הגיש לסנאט של אוניברסיטת תל-אביב
יחס הגומלין בינ’ קצב דעיכת הסתברות השגיאה
ומספר דרגות החופש של קונסטלציות או
סופיות בגרועי דעיכה מובי גניסות ויצאות

议案

ייאיר יונה

הנשטט של אוניברסיטת תל-אביב

ענוה וنعמה באוונגרסיטט ת"א בפקולטה להנדסה

בן-חנני פרופ’ מחבר פדר

אדר ב’ החשמי’
עבודת זו נועשת בהנחייה

פרופ' מאיר פדר
תקציר

יחס הגומלין בין קצב דעיכת הסתברות השגיאה ומספר דרגות החופש (DMT) האופטימאליentin כולל האופטימליים הם פונקציותوشארות תכונות של דנה ושאף של DMT האופטימלי הם מסגרת אחידה של משאר הגומלין בין קצב דעיכת הסתברות השגיאה ומספר דרגות החופש עבור ערכי יחס אות-לרעש גבוהים. ה-DMT האופטימאלי הינו פונקציה אשר נותנת למספר נתון של דרגות חופש את הקצב הדעיכתי המקסימלי של הסתברות השגיאה אותו ניתן להשיג. בנוסף, ה-DMT האופטימאלי מספק מסגרת אחידה באמצעותה ניתן להשוות שידורי ערוצי דעיכה מרובי כניסות ויציאות.

конסטלציה היא (ICs) נתון числе הים קבוצות ברות-מניה במרחב האוקלידי, אותן מאופיינות באמצעות הצפיפות שלהן שווה (עקרונית) למספר הנקודות הממוצע שנכנסות ביחידת נפח. פולטירב הגדיר מסגרת עבודה שבה המשדר משדר קבוצת נקודות של IC, אשר מקיימת מגבלת הספק, וmah - IC מפענח ביחס ל-IC מסגרת אחידה באמצעותה ניתן להשוות שידורי ערוצי דעיכה מרובי כניסות ויציאות ذو מעבר מרובי זה.ModuleName: DMT

בחלק השני של עבודה זו אנו מנתחים את ה-DMT של IC (בערוצי MAC). אנו מניחים כי ישנם K משתמשים, כאשר לכל משתמש יש M_ant אנטנות שידור ובמקלט יש N_ant אנטנות קליטה. בניגוד לערוצי נקודה לנקודה, אנו מראים כי קונסטלציות אין סופיות משיגות את ה-DMT האופטימאלי בערוצי MAC רק כאשר מספר המשתמשים הינו בתחום

\[ 1 \leq K \leq \max \left(1, \frac{N-M+1}{M} \right) \]

אם_nr

בחלק השלישי של עבודה זו אנו משתמשים בקודי שריג בצפיפות נמוכה (LDLC's) בערוצי דעיכה מרובי כניסות ויציאות. LDLC's הם קודי שריג אשר מאופיינים בדלילות של המטריצה ההפכית של המטריצה היוצרת של השריג. בתחילה אנו מפתחים אלגוריתם מקסום-מכפלות ל_LDLC's בערוצי נקודה לנקודה. עבור ערוץ AWGN אנו מראים קשר מעניין בין ההודעות המועברות באלגוריתם מקסום-מכפלות ואלגוריתם סכום-מכפלות. בנוסף, תוצאות נומריות ניתן לראות כי אלגוריתם מקסום-מכפלות משיג הסתברות שגיאה יותר טובה לבלוק מאלגוריתם סכום-מכפלות, עבור מימדים קטנים. לאחר מכן אנו מרחיבים את אלגוריתם מקסום-מכפלות לערוץ מרבי-כניסה大理石י באמצעות ההכללה של הנחת העץ. כאשר מניחים כי לסימבולים הסתברות א-פריורית גאוסית, מתקבל קשר מעניין בין ההודעות המועברות ושערוך שגיאה ריבועית ממוצעת מינימאלית. לבסוף, אנו מתכננים LDLC דליל מאוד ומציעים סכמת שידור לערוצי עם שתי אנטנות שידור ושתי אנטנות קליטה. השילוב בין ה-LDLC הדליל וסה Çalışת השידור מאפשר להשיג ביצועים ברי השוואה לקודים המובילים עבור ערוץ זה, תוך השגת פענוח בסיבוכיות נמוכה.
תוכן העניינים

1. מבוא ..............................................................................................................

2. ה-DMT האופטימאלי של קונסולטיציוואן-סופי (IC’s) (ברועתי קודה לינוקס)...,

3. על ה-DMT של IC’s ברועתי מורבי משטחשלם ........................................

4. ממבוא .........................................................................................................

5. הגדרות .......................................................................................................}

6. חסם עליון על קצב דעיכת ה-DMT ב- IC’s .................................................

7. השגת קצב דעיכה האופטימאלי ...............................................................}

8. הסכמות השידורים ..................................................................................

9. ההעונות האפקטיביות .............................................................................

10. חסם עליון על הסתברות השגיאה ...........................................................

11. השגת קצב הדעיכה האופטימאלי .........................................................

12. חסם עליון על קצב הדעיכה השגיאה ....................................................

13. תגובת ק keyof התכנית ..............................................................................

14. חסם עליון על ה-DMT האופטימאלי .......................................................}

15. דיווון .......................................................................................................}

16.شرיגים מול קונסולטיציוואן סופי והבוסאות על סריזים ......................

17. פרשנות גיאומטרית ל-DMT האופטימאלי ...........................................

18. דוגמה עבר המקהרה של שירות אנטווט שיזוור הקטנה ....................

19. הקצר בול מססר דרגת הحاول של IC’s סופיコンסולטיציוואן סופי........

20. על הע-DMT של IC’s ברועתי מורבי משטחשלם ....................................

21. ממבוא .......................................................................................................}

22. הגדרות .....................................................................................................

23. מודל ה-DMT ............................................................................................

24. סימונים נוספים ......................................................................................

25. חסם עליון על ה-DMT האופטימאלי .....................................................

26. אופטימאל בין DMT האופטימאלי במקהלה הסמוכה ...................

27. דיווון .......................................................................................................}

28. אופטימאל בין DMT האופטימאלי במקהלה הסמוכה .......................

29. דיווון .......................................................................................................}

30. אופטימאל בין DMT האופטימאל במקהלה הסמוכה .......................
השוואה לקונסטלציות סופיות

ディון: קמירות מול חוסר קמירות של ה-DMT האופטימאלי

השגת ה-DMT האופטימאלי עבור \( M = 1 \) \( K = N \)

תת-אופטימאליות עבור המקרה \( M = 1 \) \( K = N \)

השערת אורתוגונאלית היא תת-אופטימלית

 сум התשדורת על התשובה השגיאה

השידור האופטימלי

השידור האופטימלי

תקע

N \geq (K+1)M - 1

N < (K+1)M - 1

N \geq (K+1)M - 1

台州

ןולטימואיט

אלגוריתם הפרמטרי יעיל

portunCLDLC העבר עדכון

 BUILD CLDLC

Conclusions

תת-אופטימאליות

אלגוריתם הפרמטרי

يمة

שם

anielCLDLC העבר עדכון

Conclusions
ערוך גאוסי לבן חיבורי.................................................................4.6.1
ערוך דעיכה עם שני אנטנות(IntPtrレイヤ).........................................4.6.2
. סיכום ומסקנות.................................................................5.1
. הצות למחק עזרי.................................................................5.2
IC's בעור צף מנקודה לנקודה עם ריבוי כניסות ויציאות.........................5.2.1
IC's בעור צף מרובי משתמשים עם ריבוי כניסות ויציאות.........................5.2.2
IC's בעור צף דעיכה עם שתי אנטנות שידור וקליטה.........................5.2.3
 המסקנות.................................................................5.3
A. הוכחות והרחבות לפרק 2......................................................111
A.1 הוכחה של لما.................................................................111
A.2 הוכחה של لما.................................................................112
A.3 הוכחה של لما.................................................................112
A.4 הוכחה של لما.................................................................112
A.5 הוכחה של لما.................................................................122
A.6 הוכחה של لما.................................................................122
A.7 הוכחה של لما.................................................................127
A.8 הוכחה של لما.................................................................127
B. הוכחות והרחבות לפרק 2......................................................111
B.1 הוכחה של لما..................................................................111
B.2 הוכחה של لما..................................................................112
B.3 הוכחה של لما..................................................................112
B.4 הוכחה של لما..................................................................112
B.5 הוכחה של لما..................................................................111
B.6 הוכחה של لما..................................................................122
B.7 הוכחה של لما..................................................................122
B.8 הוכחה של لما..................................................................127
B.9 הוכחה של لما..................................................................127

<table>
<thead>
<tr>
<th>מספר הבית</th>
<th>טקסט</th>
</tr>
</thead>
<tbody>
<tr>
<td>126</td>
<td>8.B</td>
</tr>
<tr>
<td>130</td>
<td>9.B</td>
</tr>
<tr>
<td>133</td>
<td>10.B</td>
</tr>
<tr>
<td>134</td>
<td>11.B</td>
</tr>
<tr>
<td>138</td>
<td>12.B</td>
</tr>
<tr>
<td>142</td>
<td>13.B</td>
</tr>
<tr>
<td>142</td>
<td>14.B</td>
</tr>
<tr>
<td>144</td>
<td>15.B</td>
</tr>
<tr>
<td>147</td>
<td>ביבליוגרף</td>
</tr>
</tbody>
</table>