The Nestedness Property of Convex Ordered Median Problem on a Tree

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Abstract

In a typical location problem there is a set of demand points (customers) embedded in some metric space and the objective is to locate a specified number of servers optimizing some criterion, which usually depends on the distances between the demand points and their respective servers. Traditionally, most papers focus on location problems where a server (facility) is modeled by a point in the metric space. However, during the last years there has been a growing interest in studying the location of connected structures, which cannot be represented by isolated points in the space. In some of the problems there is a constraint on the total length of the extensive facility, and the objective is then to minimize some monotone function of the distances between the demand points and the facility. These problems are called tactical problems, in contrast to strategic problems, where the total length of the facility is also a decision variable. For tactical location problems the existence of a nestedness property may be of special interest. An intuitive definition of nestedness is that a tactical solution with a shorter length constraint is a subset of a tactical solution with a longer length constraint. Recent studies of the nestedness property were motivated by concrete decision problems related to routing or network design. For instance, in order to improve the mobility of the population and reduce traffic congestion, many existing rapid transit networks are being updated by extending or adding lines. Other potential applications appear in hierarchical network design such as the case where a high power transmission or a cable communication network must be extended.

In this paper we study nestedness properties of tactical location problems on tree graphs where the new extensive facility is required to be connected. Topologically, the selected server has to be a subtree. Constraints are expressed in terms of total length of the extensive facility. The function unifies and generalizes the most common criteria used in location theory, e.g., median and center functions. We prove the existence of a nestedness property for tactical location problems defined on the trees.

There have been attempts to prove the nestedness property of a network location problem with some special cases of COM (convex ordered median) objectives. However, to our best knowledge there is no known result on whether the nestedness property holds for a general COM objective on a tree. Also, all the known studies consider only the nestedness results with respect to point solution, and not the general case of the nestedness property as it is described above. This work fills the gap by proving the "full" nestedness property for general COM objective on a tree network.

As a first step the nestedness property is proved for a simplified case of the real line, where the demand facilities are points on $\mathbb{R}^1$ and the extensive server facility to be located is an interval on $\mathbb{R}^1$. The main idea of the proof is the representation of the interval as a point in $\mathbb{R}^2$ and investigating its geometrical properties. The intuitive graphical approach to the proof is given. The next step is the proof of the nestedness property for a rooted tree problem, where the extended facility is a small subtree within some tree network rooted at a
specified node. In the final part of the thesis we prove the nestedness property for the general location model on a tree.
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1 Introduction

In a typical location problem there is a set of demand points embedded in some metric space and the objective is to locate a specified number of servers optimizing some criterion, which usually depends on the distances between the demand points and their respective nearest servers. Traditionally most papers focus on location problems where a server (facility) is representable by a point in the metric space. However, in the last years there has been a growing interest in studying the location of connected structures. Services which cannot be represented by isolated points in the space. These problems were motivated by concrete decision problems related to routing and network design. For instance, in order to improve the mobility of the population and reduce traffic congestion, many existing rapid transit networks are being updated by extending or adding lines. These lines can be viewed as new serving facilities. Sometimes connected structures to be located are called extensive facilities.

In some of the problems there is a constraint on the total length of the extensive facility, and the objective is then to minimize some monotone function of the distances between the demand points and the facility. These problems are called tactical problems, in contrast to strategic problems, where the total length of the facility is a decision variable [5].

For tactical location problems the nestedness property may be of special interest. An intuitive definition of nestedness is that a tactical solution with a shorter length constraint is a subset of a tactical solution with a longer length constraint.

Finally, turning to the objective function, the dependence of the distances of the demand points to the selected facility has commonly been expressed either as the minsum (median), or the minmax (center) criteria. (An exception is [5], where the authors consider the centdian objective which is a convex combination of the median and the center criteria.)

The goal of this paper is to study a nestedness property of tactical location problems on a tree, using the convex ordered median objective. This objective function unifies and generalizes the most common criteria mentioned above, e.g., median and center.

In the simplified case of a line, the demand facilities are points on $\mathbb{R}^1$ and the extensive server facility to be located is an interval on $\mathbb{R}^1$. 
2 Examples

2.0.1 A facility location problem with the nestedness property

The following is an example of a location problem which possesses the nestedness property.

Example 2.0.1 Given two customers located at points 0 and 1 on $\mathbb{R}^1$. Find the location $a$ of the closed interval $[a, a + t]$ of length $t$ which minimizes the maximum distance to the customers. The closed interval $[a, a + t]$ is called the extensive facility, and the parameter $t$ is called facility length.

This is the classic (unweighted) center location problem. The well known solution is to locate the extensive facility in the middle of the interval $[0, 1]$.

Solution:

$$a_{\text{opt}} = 0.5 - \frac{t}{2}$$

For $t_1 = 0$, the point location problem, we obtain the unique optimal solution $a_1 = 0.5$.

For $t_2 = 0.2$, we obtain the unique optimal solution $a_2 = 0.3$, and the interval is $[0.4, 0.6]$.

For $t_3 = 0.32$, we obtain the unique optimal solution $a_2 = 0.33$, and the interval is $[0.34, 0.66]$.

This example is illustrated in Fig. 1.

In the presented example the nestedness property is held, the $t = 0$-solution is a subset of the $t = 0.2$-solution. And this in turn is a subset of the $t = 0.32$-solution. The main reason for the nestedness property to hold, in the authors opinion, is the convexity of the objective function.

In the following example the nestedness property is not held, i.e., the longest optimal extended facility doesn’t contain the shortest one. The main reason for this is the non-convexity of the objective function.

2.0.2 A facility location problem without the nestedness property

The following example illustrates a facility location problem which doesn’t possess the nestedness property.

Example 2.0.2 Given are two customers located at points 0 and 1 on $\mathbb{R}^1$. Find the location of the extensive facility $[a, a + t]$ of length $t$ which minimizes the following function of distances to customers

$$f(d_o, d_1) = \left| \sin\left(\frac{5}{3}\pi d_o\right) \right| + \left| \sin\left(\frac{5}{2}\pi d_1\right) \right|.$$ (1)

where $d_o, d_1$ are the distances of $[a, a + t]$ to the customers at 0 and 1 respectively.

Solution:

$$d_o = \max(a, 0)$$

$$d_1 = \max(1 - a - t, 0).$$
Figure 1: Example 2.0.1 - the nestedness property is held

The plot of the objective function value vs extensive facility location is given in Fig. 2.

For $t_1 = 0$, the point location problem, we obtain optimal solution with most left value $a_1 = 0.5$.

For $t_2 = 0.2$, we obtain optimal solution $a_2 = 0.4$, and the interval is $[0.4, 0.6]$.

For $t_3 = 0.32$, we obtain optimal solution $a_3 = 0.68$, and the interval is

for a given $t$
find $a$ that minimizes:

$f = \max(d_0, d_1)$
This example is illustrated on Fig. 3.

Figure 2: Example 2.0.2 - objective function value vs extensive facility left boundary

In the above example the nestedness property does not hold, i.e., the longest optimal extended facility doesn’t contain the shortest one. The main reason for not having the nestedness property is the non-convexity of the objective function.
Figure 3: Example 2.0.2 - the nestedness property is not held.

for a given $t$
find $a$ that minimizes:

$$f(d_0, d_1) = |\sin\left(\frac{5}{3}\pi d_0\right)| + |\sin\left(\frac{5}{3}\pi d_1\right)|$$

solution:

- $t_1 = 0 \quad a_1 = 0.6$
- $t_2 = 0.2 \quad a_2 = 0$
- $t_3 = 0.32 \quad a_3 = 0.68$
3 Definitions

3.1 The tactical location problem on the line

Let $V = \{v_1, \ldots, v_n\}$ be a set of $n$ demand points (customers) on the real line. Suppose WLOG that $v_1 < v_2 < \ldots < v_n$, and $v_1 = 0$, $v_n = 1$.

Let $W = \{w_1, \ldots, w_n\}$ be a set of $n$ positive weights associated respectively with the $n$ demand points. Let $\bar{x} = [a, b], 0 \leq a \leq b \leq 1$, be an interval on the real line.

Define the weighted distance from an interval to a point $v_i$:

$$f_i(\bar{x}) = \begin{cases} w_i(a - v_i) & \text{if } v_i < a \\ w_i(v_i - b) & \text{if } v_i > b \\ 0 & \text{if } a \leq v_i \leq b \end{cases}$$

(2)

One can write:

$$f_i(\bar{x}) = \max(w_i(a - v_i), w_i(v_i - b), 0).$$

(3)

Hence $f_i(\bar{x})$ is piecewise linear and convex in $\mathbb{R}^2$.

**Definition 3.1.1** Let $\bar{y} = (y_1, \ldots, y_m)$ be a vector in $\mathbb{R}^m$. Define $\theta(\bar{y}) = (\theta_1(\bar{y}), \theta_2(\bar{y}), \ldots, \theta_m(\bar{y}))$ to be the vector in $\mathbb{R}^m$, obtained by sorting the $m$ components of $y$ in nonincreasing order, i.e., $\theta_1(\bar{y}) \geq \theta_2(\bar{y}) \geq \cdots \geq \theta_m(\bar{y})$. Given a nonnegative vector $\lambda = (\lambda_1, \ldots, \lambda_m) \in \mathbb{R}^m$, satisfying $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_m \geq 0$, the convex ordered median function (COM) $\mathcal{F} : \mathbb{R}^m \rightarrow \mathbb{R}$ is defined as

$$\mathcal{F}(y_1, \ldots, y_m) = \sum_{i=1}^{m} \lambda_i \theta_i(\bar{y}).$$

(4)

We use a subindex in brackets to denote the $i$-th component of a vector after sorting its components in descending order.

$$y_{(i)} = \theta_i(\bar{y}).$$

(5)

The COM function is then written as:

$$\mathcal{F}(y_1, \ldots, y_m) = \sum_{i=1}^{m} \lambda_i y_{(i)}.$$

(6)

**Definition 3.1.2** Let $V = \{v_1, \ldots, v_n\}$ be a set of $n$ demand points on the real line. Let $W = \{w_1, \ldots, w_n\}$ be a set of $n$ positive weights associated with the $n$ demand points. Let $\bar{x} = [a, b]$ be an interval on the real line. Let $f_{(1)}(\bar{x}), \ldots, f_{(n)}(\bar{x})$ be an array of sorted weighted distances from $\bar{x}$ in descending order ($f_{(i)}(\bar{x})$ is the $i$-th weighted distance from $\bar{x}$ to one of the nodes of $V$). Let $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$ be real non-negative numbers.

We define a COM objective, according to [5]
From definition 3.1.1 the COM objective can be written as

$$F(x) = \sum_{k=1}^{n} \lambda_k f_k(x). \quad (7)$$

We also use the notation $F(a,b)$ meaning $F(x)$ for $x = (a,b)$.

Using the presented objective function definition, we define a facility location problem on the line. One has to find the location of the extensive facility (closed interval) of length $t$ to minimize the COM objective function.

**Definition 3.1.3** The tactical COM problem on the line is the following minimization problem:

Find $x^*(t) = [a, b]$ such that:

$$F(x^*) = \min_{x \in C(t)} F(x) \quad (9)$$

**Example 3.1.1** Consider a problem with three demand points $v_1 = 0$, $v_2 = 0.5$, $v_3 = 1$. Let the weights be $w_1 = 3$, $w_2 = 20$, $w_3 = 7$, and the extensive facility length, $t = 0.1$. Consider extensive facility $x^1 = (a_1,b_1)$, $a_1 = 0.7$, $b_1 = 0.8$. $x^1$ is an interval on the real line. This configuration is illustrated in Fig. 4. The weighted distances of $x^1$ are:

$$f_1(x^1) = w_1(a_1 - v_1) = 2.1$$

$$f_2(x^1) = w_2(a_1 - v_2) = 4$$

$$f_3(x^1) = w_3(v_3 - b_1) = 1.4$$

Let us define $\lambda_1 = 2$, $\lambda_2 = 1$, $\lambda_3 = 0$. Since $f_2(x^1) > f_1(x^1) > f_3(x^1)$, the objective function value for $x^1$ is $F(x^1) = \lambda_1 f_2(x^1) + \lambda_2 f_1(x^1) + \lambda_3 f_3(x^1) = 10.1$.

**Example 3.1.2** Let the demand point set $V$, the weights $W$ and coefficients $\{\lambda_i\}$ be as in Example 3.1.1. Let the extensive facility length, $t = 0.5$. Let $x^2 = (a_2,b_2)$, $a_2 = 0.3$, $b_2 = 0.8$ be an interval on the real line. This configuration is illustrated in Fig. 5. The weighted distances of $x^2$ are:

$$f_1(x^2) = w_1(a_2 - v_1) = 0.9$$

$$f_2(x^2) = 0$$

$$f_3(x^2) = w_3(v_3 - b_2) = 1.4$$

Since $f_3(x^2) > f_1(x^2) > f_2(x^2)$, the objective function value for $x^2$ is $F(x^2) = \lambda_1 f_3(x^2) + \lambda_2 f_1(x^2) + \lambda_3 f_2(x^2) = 3.7$.

We refer to solutions $x^1$ and $x^2$ as intervals on $\mathbb{R}^1$ and note that $x^1 \subseteq x^2$. In this case we say $x^1$ is nested in $x^2$.

In some cases there can be more than one optimal solution for the problem (9). In such a case we will seek for the optimal solution closest to the zero point. We name it as the "left" solution and denote it by subindex $l$. 

7
Definition 3.1.4 \( \mathcal{I}_l(t) = [a_l(t), b_l(t)] \) (subscript \( l \) is for low) is a solution of (9) satisfying

\[
a_l(t) = \min \{ a : \mathcal{I}^*(t) = [a, b] \text{ is a solution of (9)} \}. \tag{10}
\]

Sometimes it’s useful to view an interval \([a, b]\) as a point in \( \mathbb{R}^2 \), where \( a \) is the abscissa and \( b \) is the ordinate. We denote by \( X \) the region in \( \mathbb{R}^2 \) which contains all possible intervals that lie between 0 and 1.

\[
X = \{(b, a) \in \mathbb{R}^2 : 0 \leq a \leq b \leq 1, \}. \tag{11}
\]

When working with \( \mathbb{R}^2 \) we use the notation \( d(\mathbf{y}, \mathbf{z}) \) for denoting the Euclidean distance between the two points \( \mathbf{y}, \mathbf{z} \in \mathbb{R}^2 \) and the notation \( d(\mathbf{y}, C) \) for denoting the Euclidean distance between the point \( \mathbf{y} \in \mathbb{R}^2 \) and the closed set \( C \subset \mathbb{R}^2 \).

\[
d(\mathbf{y}, C) = \min_{\mathbf{z} \in C} d(\mathbf{y}, \mathbf{z}) \tag{12}
\]
3.2 The nestedness property of the problem on the line

**Definition 3.2.1** A COM problem on the line possesses the **nestedness** property if for all pairs of extensive facility lengths $t_1, t_2$, $t_1 < t_2$, there exists a corresponding pair of optimal solutions of (9) $\pi(t_1) = (a(t_1), b(t_1))$ and $\pi(t_2) = (a(t_2), b(t_2))$, such that:

\[
\begin{align*}
    a(t_2) &\leq a(t_1) \\
    b(t_2) &\geq b(t_1).
\end{align*}
\]

Note that nestedness is a property of the minimization problem, and not of the optimal solution. If for every two extensive facility lengths there exist two nested optimal solutions, the problem is said to possess the nestedness property.

When dealing with nestedness, the existence of multiple optimal solutions leads to unnecessary complications. So, using definition 3.1.4 we slightly refine the problem (9), such that in the formulation the optimal solution is unique.

**Definition 3.2.2** The refined tactical COM problem on the line with
parameter $t$ is the following minimization problem:

Find $\pi^*_1(t) = [a_l(t), b_l(t)]$

such that $a_l(t) = \min \{a : x = [a, b] \text{ is an optimal solution of (9)}\}$  \hspace{1cm} (14)
4 Graphical Approach

4.1 Visualization on $\mathbb{R}^2$

4.1.1 Interval as a point on $\mathbb{R}^2$

An interval $[a, b]$ on the real line can be viewed as a point in $\mathbb{R}^2$.

For convenience, let $b$ be the ordinate, and $a$ be the abscissa. We represent the point on $\mathbb{R}^2$ as $(b, a)$. The locus of all feasible solutions of (9) with parameter $t_1$ is the $45^\circ$-slope line crossing the ordinate axis at $t_1$ (see Fig. 6).

The region of interest is all possible locations of all optimal solutions, this is the region $X$, defined in (11). It is a triangle with vertices $(0, 0), (1, 0), (1, 1)$.

4.1.2 The nestedness property

Consider $t_2 > t_1$ and optimal solutions $x(t_1)$ and $x(t_2)$. According to (13), $x(t_2)$ is nested in $x(t_1)$ if it lies within the rectangle whose upper left corner is $x(t_1)$ (see Fig. 6).

4.1.3 Distances to demand points

The unweighted distance from interval $[a, b]$ to demand point $v_i$, given at (2) (assuming $w_i = 1$), is the Euclidean distance between the point $(b, a)$ and the rectangle with upper left corner at $[v_i, v_i]$.

To obtain the weighted distance we simply multiply it by the weight.

**Example 4.1.1** Let the demand point set $V$, weights $W$ and coefficients $\lambda_i$ be as in Example 3.1.1. Let $\mathbf{x}^1, \mathbf{x}^2$ be as in Example 3.1.1 and $\mathbf{x}^3 = (a_3, b_3), a_3 = 0.1, b_3 = 0.3$.

Consider the distances of intervals the $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ to the demand point $v_2$.

The weighted distances from $\mathbf{x}^1, \mathbf{x}^2, \mathbf{x}^3$ to $v_2$ are:

$$f_2(\mathbf{x}^1) = w_2(a_1 - v_2)$$
$$f_2(\mathbf{x}^2) = 0$$
$$f_2(\mathbf{x}^3) = w_2(v_2 - b_3)$$

*Fig. 7 illustrates the relation between the unweighted distance $f_2(\mathbf{x})$ and the Euclidean distance between a point and a rectangle.*

4.1.4 Voronoi diagram

For each point in the plane we order the distances to all the demand points (rectangles) in descending order. Then, we introduce a partition of the plane such that each region is a locus of points which have the same order of distances to rectangles. Such a partition can be viewed as a generalization of weighted $n$-th order Voronoi diagram. For a given set $Q$ of points in $\mathbb{R}^2$ the standard $n$-th order Voronoi diagram is defined as a partition of $\mathbb{R}^2$ to cells, such that
Figure 6: Nestedness Property Illustration. The yellow region is called the nestedness rectangle of $x^*(t_1)$

each cell is defined as the set of points having a particular set of $n$ points from $Q$ as its $n$ nearest neighbors. The difference between the standard $n$-order Voronoi diagram and the suggested partition is as follows:

- we replace the set of points $Q$ by the rectangle defined in Section 4.1.3,
- we use weighted distances,
- a cell is defined by the order of distances, and not only the $n$ smallest distances.

For each point $\bar{x} \in X$ we calculate the weighted distances to all $n$ demand points - $f_1(\bar{x}), \ldots, f_n(\bar{x})$, recalling that the distance between the point $\bar{x}$ and demand point $v_i$ is an Euclidean distance between the point and the rectangle with upper left vertex at $v_i, v_i$, see Fig. 7. For each point the demand points are sorted according to their weighted distance to that point. Namely, for $\bar{x} \in X$
we obtain a vector (sequence): 

\[ s(x) = (v(1), v(2), ..., v(n)) \]

Where \( v(1) \) is the demand point with the largest weighted distance to \( x \), \( v(2) \) is the demand point with the second largest weighted distance to \( x \), finally, \( v(n) \) is the demand point with the smallest weighted distance to \( x \). The cell on the Voronoi diagram is the location of all the points \( x \) which have the same vector (sequence) \( s(x) \).

**Example 4.1.2** Fig. 8 illustrates such a partition for the set of demand points defined in Example 3.1.1.

For all points \( z \) in the region which contains \( x \), \( f_2(z) \geq f_1(z) \geq f_3(z) \).

### 4.1.5 Location of optimal solutions

For each point \( z \) in the interior of a cell on the Voronoi diagram the order of weighted distances \( f_i \) is the same, the COM function \( F(z) \) is a linear function.
above this region. Thus, the optimal solution can not be located in the interior of a Voronoi cell. The only possible locations of optimal solutions is the intersection of the line $a = b - t$ with the boundary of the Voronoi cell. This is illustrated in Example 4.2.1 and Fig. 9.

4.2 Intuitive proof of the nestedness property on the line

First, we deal with the case of multiple optimal solutions. In the case of more than one optimal solution we will take the most "left" solution. Namely from all intervals $(a, b)$ which are optimal solutions of (9) we will take the interval with the minimal $a$.

Consider the partition of the plane into polyhedral regions which was presented in the previous section. In each region the objective function $F$ is linear above the region, therefore, for each $t$, an optimal solution is the intersection of the line $a = b - t$ with the region boundaries.
Example 4.2.1 Let the demand point set $V$, weights $W$ and coefficients $\{\lambda_i\}_{i=1}^n$ be as in Example 3.1.1. Consider the problem (9) with four options for extensive serving facility length $t$.

$t \in \{0, t_1, t_2, t_3\}$

For each $t$, an optimal solution is the intersection of the line $a = b - t$ with the region boundaries.

Fig. 9 illustrates this principle. The only possible optimal solutions for different values of $t$ are marked in red.

Consider the function $\pi_t^v(t)$ which assigns to each length $t$ the "left" optimal solution of (9) with total length $t$. An important fact that will be proven later is that $\pi_t^v(t)$ is a continuous function of $t$ in $\mathbb{R}^1$ to $\mathbb{R}^2$, (it is also piecewise linear), therefore the plot of $\pi_t^v(t)$ on $\mathbb{R}^2$ is a Jordan curve. This line cannot be in the interior of the regions obtained by a partition of the plane, thus it lies within the boundaries of the regions.

We calculated the line of optimal solutions $\pi_t^v(t)$ for the previous example:

Example 4.2.2 Let the demand point set $V$, weights $W$ and coefficients $\{\lambda_i\}$ be as in Example 3.1.1. Optimal solution for each $t$ is plotted as the red line on Fig. 10.

We sign the lines containing the boundaries of the regions in the partition as $s_k$ lines, and their corresponding equations can be one of the following:

$\{(b, a) \in \mathbb{R}^2 : w_i(v_i - b) = w_j(v_j - b), \quad j \neq i\}$

$\{(b, a) \in \mathbb{R}^2 : w_i(a - v_i) = w_j(a - v_j), \quad j \neq i\}$

$\{(b, a) \in \mathbb{R}^2 : w_i(v_i - a) = w_j(b - v_j), \quad j \neq i\}$

An important property of the $s_k$ lines is that the slope of those lines can be 0, $-\infty$ or negative, but not positive (that will be shown later in Section 5.2.1). As it was mentioned above, all optimal solutions $\pi_t^v(t)$, $t \in [0, 1]$ form a Jordan curve in $\mathbb{R}^2$. Since every optimal solution belongs to one of $s_k$-lines, this Jordan curve is a union of intervals, while every interval is a part of some $s_k$-line. Thus the Jordan curve of the optimal solutions has a non-positive slope in each point while plotted on the $bo$ plane. The $a$-coordinate is non-increasing with $t$ and the $b$-coordinate is non-decreasing with $t$. That means that for every $t_1, t_2, t_2 > t_1$, $\pi^s(t_2)$ will be inside the nestedness rectangle of $\pi^s(t_1)$ - see Fig. 6.

This is illustrated in the next example.

Example 4.2.3 Let the demand point set $V$, weights $W$ and coefficients $\{\lambda_i\}$ be as in Example 3.1.1.

Suppose the optimal solution for all $t$, $\pi^s(t)$ is known and plotted on the $\mathbb{R}^2$ plane. For each $t_k$ the optimal solution $\pi^s(t_k)$ is the intersection of the $\pi^s(t)$-line with the line $a = b - t_k$; The optimal solutions $\pi^s(0), \pi^s(t_1), \pi^s(t_2), \pi^s(t_3)$ and their respective "nestedness rectangles" are plotted on Fig. 11.

Observing the plot on Fig. 11 we see the following: The red line on the right side of $\pi^s(t_2)$ never exits the rectangle with left corner at $\pi^s(t_2)$. There, as it was shown in 4.1, $\pi^s(t_2)$ is nested in every optimal solution for $t > t_2$. 

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Figure 9: Illustration of the Example 4.2.1
Figure 10: Optimal solutions for Example 3.1.1
Figure 11: Nestedness property illustrated by three optimal solutions for Example 3.1.1
5 Formal Proof

5.0.1 Proof guidelines

Based on previous work [4] we formulate optimization problems (9) and (14) as parametrized LP problems (paragraph 5.1.1). The continuity of the optimal solution \( \pi_l^*(t) \) with respect to the extensive facility length \( t \) follows from the properties of the parametrized LP problem (paragraph 5.1.2). The continuity of \( \pi_l^*(t) \) is proved in Theorem 5.1.1. As illustrated in Section 4.2 the optimal solution can lie only on at least one of the \( s_h \) lines (Lemma 5.2.2) which have a non-positive slope (Lemma 5.2.1). This is because of convexity and piecewise linearity of the objective function (see Section 5.2). Given two extensive facility lengths \( t_1, t_2 \), close enough to each other, the corresponding optimal solutions, \( \pi_l^*(t_1) \) and \( \pi_l^*(t_2) \) lie on the same \( s_h \) line. Then, the nestedness property follows from the negative slope of \( s_h \) line. We call this property nestedness in close neighborhood. Theorem 5.3.1 in Section 5.3 gives the formal proof of nestedness in closed neighborhood. The last step is to expand the nestedness property to all two values of extensive facility lengths. This is done in the proof of the main Theorem 5.3.2 using the Monotonicity Lemma 10.2.1. The schematic view of the proof is given in Fig. 12.

5.1 Continuity of the optimal solution

In this section continuity of the optimal solution is proved based on properties of parametrized LP problem - Section 5.1.3. LP formulation of the optimization problems (9) and (14) derived in Section 5.1.1, and the properties of the derived parametrized LP problems summarized in Section 5.1.2.

5.1.1 LP formulation of the problem

In this section linear programming formulations of (9) and (14) are presented.

The LP formulation of the convex COM problem is given in details in [4], here we summarize the principle results of the paper. To facilitate the discussion we first introduce some notation. For any real number \( c \) define \( c^+ = \max(c, 0) \). As a reminder, for \( \overline{y} = (y_1, \ldots, y_m) \), \( \overline{\theta}(\overline{y}) = (\theta_1(\overline{y}), \theta_2(\overline{y}), \ldots, \theta_m(\overline{y})) \) is defined to be the vector in \( \mathbb{R}^m \) obtained by sorting the \( m \) components of \( \overline{y} \) in non-increasing order, i.e., \( \theta_1(\overline{y}) \geq \theta_2(\overline{y}) \geq \ldots \geq \theta_m(\overline{y}) \). \( \theta_i(\overline{y}) \) will be referred to as the \( i \)-th largest component of \( \overline{y} \). Finally, for \( k = 1, \ldots, m \), define \( \Theta_k(\overline{y}) = \sum_{i=1}^{k} \theta_i(\overline{y}) \), the sum of the \( k \) largest components of \( \overline{y} \).

Let \( \overline{h} \) be a vector in \( \mathbb{R}^d \). Given a collection \( \{g_i(\overline{h})\}_{i=1}^{n} \) of \( n \) functions, defined on \( \mathbb{R}^d \), and a polyhedral set \( Q \subset \mathbb{R}^d \), consider the problem of minimizing the sum of the \( k \) largest functions from the collection of \( n \) functions over \( Q \),

Find \( \min_{\overline{h}} \Theta_k(g_1(\overline{h}), \ldots, g_n(\overline{h})) \)
subject to \( \overline{h} \in Q \subset \mathbb{R}^d \). \( (15) \)
According to [4] the problem (15) can be formulated as:

\[
\min (k\rho_k + \sum_{i=1}^{n} \gamma_i)
\]
subject to:
\[
\gamma_i + \rho_k \geq g_i(h), \quad \gamma_i \geq 0, \quad i = 1, \ldots, n
\]
\[
h = (h_1, \ldots, h_d) \in Q.
\]

(16)

Note that from [4] the minimum is attained at \( \rho_k^* = \theta_k \left( g_1(\overline{h}), \ldots, g_n(\overline{h}) \right) \).

Let \((\alpha_1, \ldots, \alpha_n)\) be a non-negative vector. Consider the problem of minimizing a linear combination of \( \Theta_k \) with non-negative coefficients:

Find \[
\min_{\overline{h}} \sum_{j=1}^{n} \alpha_j \Theta_j \left( g_1(\overline{h}), \ldots, g_n(\overline{h}) \right)
\]
subject to \( \overline{h} \in Q \subset \mathbb{R}^d \).

(17)
Substituting the expression for $\Theta_j \left( g_1(\overline{h}), \ldots, g_n(\overline{h}) \right)$ from (16), we obtain
the following optimization problem:

\[
\begin{array}{l}
\text{Find} \\
\min_{\overline{h}} \sum_{j=1}^{n} \alpha_j \min \rho_j, \{\gamma_{i,j}\}_{i=1}^{n} \left( j\rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right)
\end{array}
\]
subject to:
\[
\begin{array}{l}
\gamma_{i,j} \geq 0, \; \forall i, j = 1, \ldots, n, \\
\gamma_{i,j} + \rho_j \geq g_i(\overline{h}), \; \forall i, j = 1, \ldots, n, \\
\overline{h} \in Q \subset \mathbb{R}^d.
\end{array}
\]

We reformulate the minimum in (18) as:

\[
\begin{array}{l}
\min_{\rho_i, \{\gamma_{i,j}\}_{i,j=1}^{n}} \sum_{j=1}^{n} \alpha_j \left( j\rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right).
\end{array}
\]

Using (18) with (19) we can write down the formulation of (9) as an LP problem. We perform the following substitutions:

- $\overline{h}$ is the interval $[a, b]$ in $\mathbb{R}$, which is represented as the vector $\overline{x} = (b, a)$ in $\mathbb{R}^2$,
- the polyhedral set $Q_t$ is the set of all points in $\mathbb{R}^2$, which can represent the interval of length $t$:
  \[Q_t = \{(b, a) : a \in [0, 1], b \in [0, 1], b = a + t\},\]
- functions $g_1(\overline{h}), \ldots, g_n(\overline{h})$ are positively weighted distances from the interval $[a, b]$ to the demand points $v_1, \ldots, v_n$, given at (3),
- we write the COM objective function (8) in the form:
  \[F(\overline{x}) = \sum_{i=1}^{n} (\lambda_i - \lambda_{i-1}) \Theta_i (f_1(\overline{x}), \ldots, f_n(\overline{x})),\]
where $f_i(\overline{x})$, $i = 1, \ldots, n$, are distance function defined in (3), and assuming $\lambda_0 = 0$, so $\alpha_j = \lambda_j - \lambda_{j-1} \geq 0$ from (8).
Thus, (9) can be written as:

Find $\mathbf{x}_a = (a, b, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{n,1}, \gamma_{1,2}, \ldots, \gamma_{n,2}, \ldots, \gamma_{1,n}, \ldots, \gamma_{n,n})$

that minimizes $\sum_{j=1}^{n} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right)$

subject to:

$\gamma_{i,j} \geq 0,$
$\gamma_{i,j} + \rho_j \geq f_i((b, a)),$
$\forall i, j = 1, \ldots, n,$
$a \leq b,$
$0 \leq a \leq 1,$
$0 \leq b \leq 1,$
$b \leq a + t.$

(20)

The last step is to replace the non-linear function $f_i((b, a))$ by linear constraints on $\gamma_{i,j} + \rho_j$:

$\gamma_{i,j} + \rho_j \geq 0,$
$\gamma_{i,j} + \rho_j \geq w_i(a - v_i),$  
$\gamma_{i,j} + \rho_j \geq w_i(v_i - b)$.

(21)

The complete LP formulation of (9) is therefore:

Find $\mathbf{x}_a = (a, b, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{n,1}, \gamma_{1,2}, \ldots, \gamma_{n,2}, \ldots, \gamma_{1,n}, \ldots, \gamma_{n,n})$

that minimizes $\sum_{j=1}^{n} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right)$

subject to:

$\gamma_{i,j} \geq 0,$
$\gamma_{i,j} + \rho_j \geq 0,$
$\gamma_{i,j} + \rho_j \geq w_i(a - v_i),$  
$\gamma_{i,j} + \rho_j \geq w_i(v_i - b),$  
$\forall i, j = 1, \ldots, n,$
$a \leq b,$
$0 \leq a \leq 1,$
$0 \leq b \leq 1,$
$b \leq a + t.$

(22)

Note that for the optimal solution (denoted by $\mathbf{x}_a^*$) of (22) at least one of inequalities (21) have to be an equality. This is because both $\rho_j$ and $\gamma_{i,j}$ appear with a positive sign in the objective function for all $i, j$.

In general, we may have $\lambda_j = \lambda_{j+1}$ for some $j$. This case can result in an unbounded optimal solution set, which is inconvenient, since we prefer to work with bounded sets. We define the set of indexes

$I_\lambda = \{k : \lambda_k > \lambda_{k-1}\}$

and reformulate (22):
Find \( x_a = (a, b, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{n,1}, \gamma_{1,2}, \ldots, \gamma_{n,2}, \ldots, \gamma_{1,n}, \ldots, \gamma_{n,n}) \)
that minimizes \( \sum_{j \in I} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right) \)
subject to:
\[
\gamma_{i,j} \geq 0,
\gamma_{i,j} + \rho_j \geq 0,
\gamma_{i,j} + \rho_j \geq w_i(a - v_i),
\forall i, j = 1, \ldots, n,
\]
\( a \leq b, \)
\( 0 \leq a \leq 1, \)
\( 0 \leq b \leq 1, \)
\( b \leq a + t. \)

(24)

**Lemma 5.1.1** Consider the LP problem (24) with some parameter \( t \) and its optimal solution, \( \bar{x}_a = (a^*, b^*, \rho_1^*, \ldots, \rho_n^*, \gamma_{1,1}^*, \ldots, \gamma_{n,1}^*, \gamma_{1,2}^*, \ldots, \gamma_{n,2}^*, \ldots, \gamma_{1,n}^*, \ldots, \gamma_{n,n}^*) \). The values of the variables \( \rho_1^*, \ldots, \rho_n^* \) are non-negative.

**Proof** Assume that for some \( k \), \( \rho_k^* < 0 \).

Define (a proposed) solution \( \bar{x}'_a = (a', b', \rho_1', \ldots, \rho_n', \gamma_{1,1}', \ldots, \gamma_{n,1}', \gamma_{1,2}', \ldots, \gamma_{n,2}', \ldots, \gamma_{1,n}', \ldots, \gamma_{n,n}') \) be equal to \( \bar{x}_a \), except the variables \( \rho_k \) and \( \gamma_{i,k} \) where:
\[
\rho_k' = 0,
\gamma_{i,k}' = \gamma_{i,k}^* + \rho_k^*.
\]

(25)

It can be verified that conditions (21) are satisfied and \( \bar{x}'_a \) is feasible.

\[
\rho_k' + \sum_{i=1}^{n} \gamma_{i,k}' = 0 + \sum_{i=1}^{n} (\gamma_{i,k}^* + \rho_k^*) = n\rho_k^* + \sum_{i=1}^{n} \gamma_{i,k}^* < \rho_k^* + \sum_{i=1}^{n} \gamma_{i,k}^*,
\]

(26)
contradicting the optimality of \( \bar{x}_a^* \).

Using Lemma 5.1.1 we insert the constraints \( \rho_j \geq 0, \forall j = 1, \ldots, n \), into the LP formulation (24).
Find \( \bar{x}_a = (a, b, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{1,n}, \gamma_{1,2}, \ldots, \gamma_{n,2}, \ldots, \gamma_{1,n}, \ldots, \gamma_{n,n}) \)
that minimizes \( \sum_{j \in I} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^n \gamma_{i,j} \right) \)
subject to:

\[
\rho_j \geq 0, \\
\gamma_{i,j} \geq 0, \\
\gamma_{i,j} + \rho_j \geq 0, \\
\gamma_{i,j} + \rho_j \geq w_i(a - v_i), \\
\gamma_{i,j} + \rho_j \geq w_i(v_i - b), \\
\forall i, j = 1, \ldots, n, \\
a \leq b, \\
0 \leq a \leq 1, \\
0 \leq b \leq 1, \\
b \leq a + t.
\]

The optimization problem (27) is the parametrized LP problem. The extensive facility length \( t \), \( 0 \leq t \leq 1 \), is the parameter, and for each value of \( t \) there is a corresponding optimal objective value of (27). Denote this value by \( F^*(t) \).

\( F^*(t) \) - optimal objective value of problems (9) and (27) as a function of the extensive facility length \( t \).

Now we proceed to the LP formulation of (14). Namely, among all optimal solutions \( (b^*, a^*) \), \( b^* \leq a^* + t \), which result in an objective value \( F^*(t) \), choose the one with the smallest \( a \) value :

Find \( \bar{x}_{al} = (a_l, b_l, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{1,n}, \gamma_{1,2}, \ldots, \gamma_{n,2}, \ldots, \gamma_{1,n}, \ldots, \gamma_{n,n}) \)
that minimizes \( a_l \)
subject to:

\[
\sum_{j \in I} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^n \gamma_{i,j} \right) = F^*(t) \\
\rho_j \geq 0, \\
\gamma_{i,j} \geq 0, \\
\gamma_{i,j} + \rho_j \geq 0, \\
\gamma_{i,j} + \rho_j \geq w_i(a_l - v_i), \\
\gamma_{i,j} + \rho_j \geq w_i(v_i - b_l), \\
\forall i, j = 1, \ldots, n, \\
a \leq b, \\
0 \leq a_l \leq b_l, \\
0 \leq b_l \leq 1, \\
b \leq a_l + t.
\]

Since \( F^*(t) \) is an optimal objective value for an extensive facility length \( t \), there is no interval of length \( t \) with objective value less than \( F^*(t) \). So the equality constraint in (28) can be replaced by inequality.
Find \( \bar{x}_{al} = (a_l, b_l, \rho_1, \ldots, \rho_n, \gamma_{1,1}, \ldots, \gamma_{1,n}, \gamma_{2,1}, \ldots, \gamma_{2,n}, \ldots, \gamma_{n,1}, \ldots, \gamma_{n,n}) \)
that minimizes \( a_l \)
subject to:

\[
\sum_{j \in I_\lambda} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^n \gamma_{i,j} \right) \leq F^*(t)
\]
\( \rho_j \geq 0, \)
\( \gamma_{i,j} \geq 0, \)
\( \gamma_{i,j} + \rho_j \geq 0, \)
\( \gamma_{i,j} + \rho_j \geq w_i(a_l - v_i), \)
\( \gamma_{i,j} + \rho_j \geq w_i(v_i - b_l), \)
\( \forall i, j = 1, \ldots, n, \)
\( a \leq b, \)
\( 0 \leq a_l \leq b_l, \)
\( 0 \leq b_l \leq 1, \)
\( b_l \leq a_l + t. \)

(29)

(29) is the Linear Programming formulation of (14).

5.1.2 Properties of the LP form of the convex ordered median location problem on the line

In this paragraph we summarize basic properties of the parametrized LP problems (27) and (29). We focus on boundedness and feasibility.

**Lemma 5.1.2** The LP problem (27) is feasible, has a finite optimal objective value and bounded optimal solutions set for all \( t \in [0,1] \).

**Proof** Consider \( \bar{x}'_a = (a', b', \rho'_1, \ldots, \rho'_n, \gamma'_{1,1}, \ldots, \gamma'_{1,n}, \gamma'_{2,1}, \ldots, \gamma'_{2,n}, \ldots, \gamma'_{n,1}, \ldots, \gamma'_{n,n}) \) defined as:

\[
a' = b' = 0; \\
\rho_j = 0, \forall j = 1, \ldots, n; \\
\gamma_{i,j} = w_i v_i, \forall i, j = 1, \ldots, n.
\]

(30)

It can easily be verified that conditions (21) are satisfied and \( \bar{x}'_a \) is a feasible solution. Thus (27) is feasible.

The objective value for \( \bar{x}'_a \), \( \sum_{j \in I_\lambda} (\lambda_j - \lambda_{j-1}) \sum_{i=1}^n \gamma'_{i,j} \), is finite and positive. Therefore the optimal solution for (27) is finite. All optimization variables of (27) are restricted to be non-negative, therefore the optimal objective value is non-negative and the LP problem (27) is bounded. Since all coefficients in \( \sum_{j \in I_\lambda} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^n \gamma_{i,j} \right) \) are strictly positive, none of the variables \( \rho_j, \gamma_{i,j} \) will be equal to \( \infty \). Therefore (27) has a bounded optimal solution set. 

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Lemma 5.1.2 states some properties of the LP problem (27). The LP formulation (29) is preferred in the nestedness proof, since it assures the uniqueness of the optimal values of $a_l$ and $b_l$. The next lemma states the properties of the LP problem (29).

**Lemma 5.1.3** Problem (29), the LP formulation of (14), is feasible, has a finite optimal objective value and bounded optimal solution for all $t \in [0, 1]$.

**Proof** For some $t$, consider an optimal solution of (27), $\pi^*_a$. $\pi^*_a$ is a feasible solution for (29).

Assume $\pi^*_a$ is an optimal solution of (29) for some $t = t'$. Since $\pi^*_a$ is an optimal solution of (27), $\pi^*_a$ is finite from Lemma 5.1.2. Thus, (29) has a bounded optimal solutions set for all $t \in [0, 1]$.

### 5.1.3 Proof of the continuity theorem

**Theorem 5.1.1** Let $\pi^*_L(t) = (b^*_L(t), a^*_L(t))$ be the unique optimal solution of the minimization problem (14). $\pi^*_L(t) : \Re \to \Re^2$ is a continuous function of $t$ for all $t \in [0, 1]$. Namely:

\[
\forall t, \forall \epsilon > 0 \ \exists \Delta \text{ such that } |\pi^*_L(t) - \pi^*_L(t')| < \epsilon, \ \forall |t' - t| < \Delta. \tag{31}
\]

**Proof** Consider (27), the LP formulation of (9), and its optimal objective value as a function of the parameter $t$, $F^*(t)$.

The parametrized LP problem (27) is feasible and bounded (Lemma 5.1.2), and the only parametrized constraint $b - a \leq t$ has linear function of $t$ on the right-hand side.

Then, from Lemma 10.1.1, $F^*(t)$ is convex, therefore $F^*(t)$ is continuous in the interior of the domain, i.e., in $(0, 1)$.

Now, consider the problem (29), which is the LP formulation of (14). It is feasible and bounded (Lemma 5.1.3), and has the following two parametrized constraints:

\[
\sum_{j \in I} (\lambda_j - \lambda_{j-1}) \left( \rho_j + \sum_{i=1}^{n} \gamma_{i,j} \right) \leq F^*(t) \tag{32}
\]

Each of the two constraints of (32) has a continuous function on the right-hand side. Therefore from Lemma 10.1.1, the optimal solution $a^*_L(t)$ is a continuous function of $t$. At optimality, $b^*_L(t) = a^*_L(t) + t$, and therefore is also a continuous function of $t$.

### 5.2 Properties of the objective function

In this section we study the objective function of the tactical COM problem on the line, (9), $F(x) : \Re^2 \to \Re$. From definition 3.1.2 and equations (2),(4), and
the objective function can be written in an explicit form:
\[
F(\mathbf{x}) = \lambda_1 \theta_1 ([f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]) \\
+ \lambda_2 \theta_2 ([f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]) \\
+ \cdots \\
+ \lambda_n \theta_n ([f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]).
\]
(33)

Remember that \( \theta_i(\mathbf{y}) \) is the \( i \)-th largest component of the vector \( \mathbf{y} \). The definition of the weighted distance functions \( f_i(\mathbf{x}) \), \( i = 1, \ldots, n \), given at (2).

Denote by \( X \subset \mathbb{R}^2 \) the set of all feasible solutions of (9) for all values of \( t \):
\[
X = \{(b, a) : 0 \leq a \leq 1; 0 \leq b \leq 1; b \geq a\}.
\]
Consider some open connected set \( C \subset X \). Sufficient conditions for the objective function \( F(\mathbf{x}) \) to be linear on \( C \) are as follows:

- All weighted distances \( \{f_i(\mathbf{x})\} \) are linear on \( C \).
- The ordering of functions \( \{f_i(\mathbf{x})\} \) remains the same for all \( \mathbf{x} \in C \). Namely the position of functions \( f_i(x) \) when sorted in descending order doesn’t change on \( C \). In this case each of the functions \( \theta_i([f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]) \) picks the same component of \([f_1(\mathbf{x}), f_2(\mathbf{x}), \ldots, f_n(\mathbf{x})]\).

The first condition may not be satisfied on \( C \) if one of the following holds:
\[
\exists \mathbf{x} = (b, a) \in C : a = v_i; \text{ for some } i,
\]
(34)
\[
\exists \mathbf{x} = (b, a) \in C : b = v_i; \text{ for some } i,
\]
(35)
\[
\exists \mathbf{x} = (b, a) \in C : a = 0,
\]
(36)
\[
\exists \mathbf{x} = (b, a) \in C : b = 0.
\]
(37)

If none of the above holds, then each \( f_i(\mathbf{x}) \) is either \( w_i(a - v_i) \) or \( w_i(v_i - b) \), i.e., linear over \( C \).

The second sufficient condition refers to the ordering of weighted distances. Since the weighted distance functions \( \{f_i(\mathbf{x})\} \) are continuous, the ordering may change in points where a pair of functions have the same value. Namely \( f_i(\mathbf{x}) = f_j(\mathbf{x}) \) for some \( i \neq j \).

The second condition may not be satisfied on \( C \) if one of the following holds:
\[
\exists \mathbf{x} = (b, a) \in C : w_i(a - v_i) = w_j(a - v_j); \text{ for some } i \neq j,
\]
(38)
\[
\exists \mathbf{x} = (b, a) \in C : w_i(v_i - b) = w_j(v_j - b); \text{ for some } i \neq j,
\]
(39)
\[
\exists \mathbf{x} = (b, a) \in C : w_i(v_i - b) = w_j(a - v_j); \text{ for some } i \neq j.
\]
(40)

Each one of the equations (34)-(38), and (39), (40) defines a straight line in \( \mathbb{R}^2 \). If the set \( C \) is not crossed by any line defined by (34)-(38), (39), and (40), then \( F(\mathbf{x}) \) is linear over \( C \).

We denote the lines defined by one of (34), (38), (39), and (40) as \( sk \)-lines.
5.2.1 \( s_h \)-lines

The geometric proof of the nestedness property was based on a partitioning of \( \mathbb{R}^2 \) which resembles the Voronoi diagram. The \( s_h \)-lines are the only lines that can be part of such a partition. (No other line will appear in the diagram). Every facet of the partition on Fig. 8 belongs to one of those lines. The set of all \( s_h \)-lines is denoted by \( I \).

\[
I = \{ s_h \} \text{ be a finite set of lines in } X \subset \mathbb{R}^2 \text{ satisfying one of the following:}
\]

\[
s_h = \{(b, a) \in X : a = v_i, \text{ for some } v_i \in V\}
\]

\[
s_h = \{(b, a) \in X : b = v_i, \text{ for some } v_i \in V\}
\]

\[
s_h = \{(b, a) \in X : a = 0\}
\]

\[
s_h = \{(b, a) \in X : b = 1\}
\]

\[
s_h = \{(b, a) \in X : w_i(v_i - b) = w_j(v_j - b)\}, \text{ for some } v_i, v_j \in V, j \neq i
\]

\[
s_h = \{(b, a) \in X : w_i(a - v_i) = w_j(a - v_j)\}, \text{ for some } v_i, v_j \in V, j \neq i
\]

\[
s_h = \{(b, a) \in X : w_i(v_i - a) = w_j(b - v_j)\} \text{, for some } v_i, v_j \in V, j \neq i
\]

(41)

Finally, denote by \( A(s_h) \) the continuum set of points induced by an \( s_h \)-line. Also denote \( A(I) = \bigcup_{h} A(s_h) \).

The \( s_h \)-lines have the two important features, which serve us in the proof:

- when drawn in \( \mathbb{R}^2 \) plane, \( A(s_h) \) has a non-positive slope,
- every optimal solution of (14) lies on one of the \( s_h \)-lines.

These features are summarized in the lemmas below.

As shown above, the optimal solutions of (14) will be located at least on one of the \( s_h \)-lines. This is summarized in the following lemmas.

Lemma 5.2.1 Consider the minimization problem (14) and let \( b = q_0 + q_1 a \) be one of the \( s_h \) lines, plotted in the \( (b, a) \) plane. Then the line \( b = q_0 + q_1 a \) has a non-positive slope

Proof The line \( b = q_0 + q_1 a \) satisfies one of the conditions (34)-(40).

Conditions (34)-(39) imply that either \( a = \text{const} \) or \( b = \text{const} \), meaning the slope is either 0 or \( -\infty \).

Condition (40) leads to:

\[
\begin{align*}
    &w_i v_i - w_j a = w_j b - w_j v_j, \\
    &b = \frac{w_i v_i - w_j v_j}{w_j}, \\
    &q_0 = \frac{w_i v_i - w_j v_j}{w_j}, \\
    &q_1 = \frac{w_i}{w_j}.
\end{align*}
\]

(42)

Therefore \( q_1 \leq 0 \), since \( w_k \geq 0 \) for all \( k = 1, \ldots, n \).

Lemma 5.2.2 Consider the minimization problem (14) and \( I \), the set of \( s_h \)-lines induced by the problem.

Consider some point \( \overline{y}_0 = (b_0, a_0) \in X \) and \( \overline{y}_0 \notin A(I) \), then \( \overline{y}_0 \) is not the optimal solution of problem (14) for \( t = b_0 - a_0 \).
Proof Denote by \( d \) the Euclidean distance from \( y_0 \) to the nearest of the \( s_h \)-lines to it:
\[
d = \min_{s_h \in I} d(y_0, s_h).
\]

Denote by \( B_{0,d/2} \) the open ball of radius \( d/2 \) centered at \( y_0 \):
\[
B_{0,d/2} = \{ \bar{y} = (b, a) \in X : d(y_0, \bar{y}) < d/2 \}.
\]
Since \( B_{0,d/2} \cap A(I) = \emptyset \), none of the conditions in (41) holds at \( \bar{y} = (b, a) \in B_{0,d/2} \) for all \( i, j \). Namely:

- All weighted distances \( \{ f_i(\bar{y}) \} \) defined in (2), are linear over \( B_{0,d/2} \).
- All non-zero weighted distances \( f_i(\bar{y}) \) have distinct values:
\[
f_i(\bar{y}) \neq f_j(\bar{y}) \forall j \neq i, f_i(\bar{y}) > 0, f_j(\bar{y}) > 0, \bar{y} \in B_{0,d/2},
\]
which means that the order of \( \{ f_i(\bar{y}) \} \) remains constant over \( B_{0,d/2} \).

Therefore, the COM function \( F(\bar{y}) = F(a, b) \) is linear on \( B_{0,d/2} \). There exist scalars \( q_c, q_a, q_b \) such that:
\[
F(a, b) = q_c + q_a a + q_b b
\]
over \( B_{0,d/2} \). For \( t = b_0 - a_0 \), all feasible solutions lie on the line \( b = a + t \), the objective function value on this line is:
\[
F(a, b) = q_c + q_a t + (q_a + q_b)a.
\]

If \( q_a + q_b = 0 \), then \( a' = a_0 - \epsilon, b' = b_0 - \epsilon, \epsilon \ll d/2 \), for \( \epsilon \) sufficiently small, it is clear that \( (b', a') \) results in smaller objective, better than \( (b_0, a_0) \).

If \( q_a + q_b > 0 \), then \( a' = a_0 - \epsilon, b' = b_0 - \epsilon, \epsilon \ll d/2 \) will result in \( F(a', b') < F(a, b) \), meaning \((b_0, a_0)\) is not an optimal solution of (9) and not a feasible solution of (14).

Similar arguments will prove the result for the case \( q_a + q_b < 0 \). \( \blacksquare \)

5.3 Main nestedness theorem on the line

Now we formulate and prove the main nestedness theorem on the line. First we prove nestedness in a close neighborhood of the optimal solution.

**Theorem 5.3.1** Consider the minimization problem (14) with specified extensive facility length \( t_0 \), \( 0 < t_0 < 1 \). Denote by \( \pi^*_t(t) = [a^*_t(t), b^*_t(t)] \) the optimal solution of (14). Then for each \( t_0 \in (0, 1) \) there exists \( \epsilon = \epsilon(t_0) > 0 \) such that
\[
a^*_t(t_0) \geq a^*_t(t) \quad \forall t \in [t_0, t_0 + \epsilon] \tag{45}
\]
and
\[
a^*_t(t_0) \leq a^*_t(t) \quad \forall t \in [t_0 - \epsilon, t_0] \tag{46}
\]

\[29\]
**Proof** Let $\pi_l(0)$ be an optimal solution of (14) with parameter $t = t_0$.

From Lemma 5.2.2 there exists at least one of the $s_h$-lines that passes through $\pi_l(0)$ in the $\mathbb{R}^2$ plane. Denote the collection of all the $s_h$-lines passing through $\pi_l(t_0)$ by $I_0$:

$$I_0 = \{ s_h \in I : \rho(\pi_l(t_0), s_h) = 0 \},$$

(47)

where $\rho(\pi_l(t_0), s_h)$ is the Euclidean distance from the point $\pi_l(t_0)$ to an $s_h$-line in $\mathbb{R}^2$.

Let $\rho_0 > 0$ be the distance to the nearest $s_h$-line which does not pass through $\pi_l(t_0)$.

$$\rho_0 = \min \{ \rho(\pi_l(t_0), s_h), \; s_h \in I \setminus I_0 \}$$

(48)

Consider a closed ball around $\pi_l(t_0)$ of radius $\rho_0/2$.

$$B(\pi_l(t_0), \rho_0/2) = \{ z \in \mathbb{R}^2 : \rho(\pi_l(t_0), z) \leq \rho_0/2 \}$$

(49)

Since $\pi_l(t)$ is a continuous function of $t$ (Theorem 5.1.1), there exists $\epsilon > 0$ such that all optimal solutions for $t \in [t_0 - \epsilon, t_0 + \epsilon]$ are within $B(\pi_l(t_0), \rho_0)$.

Consider some $t_1, t_0 < t_1 < t_0 + \epsilon$. $\pi_l(t_1)$, the optimal solution for (14) with parameter $t_1$ within the ball $B(\pi_l(t_0), \rho_0/2)$. From Lemma 5.2.2 $\pi_l(t_1)$ lies on one of the $s_h$-lines, WLOG denote it by $s_1$. The line $s_1$ belongs to the set $I_0$, because only lines from $I_0$ pass through the ball $B(\pi_l(t_0), \rho_0/2)$.

Therefore $\pi_l(t_0)$ and $\pi_l(t_1)$ belong to the same line $s_1$. The line equation of $s_1$ is:

$$s_1 = \{(b, a) : b = q_0 + q_1 a \}$$

(50)

From Lemma 5.2.1 the value of $q_1$ is non-positive, i.e., $q_1 = 0$, or $q_1 = -\infty$, or $q_1 < 0$.

Equations (52) deal with the simple case of $q_1 = 0$: $q_1 = -\infty$, while basic properties of $\pi_l(t_0)$ and $\pi_l(t_1)$ are summarized in (51). Algebraic calculations for the case $q_1 < 0$ are given in (53) and summarized in (54).

\[
\begin{align*}
& b_l(t_0) = a_l(t_0) + t_0; \quad b_l(t_0) = q_0 + q_1 a_l(t_0); \\
& b_l(t_1) = a_l(t_1) + t_1; \quad b_l(t_1) = q_0 + q_1 a_l(t_1); \\
& t_1 > t_0;
\end{align*}
\]

(51)

\[
\begin{align*}
q_1 = 0 & \quad \Rightarrow \quad b = \text{const on } s_1 \quad \Rightarrow \quad a_l(t_1) < a_l(t_0) \quad \Rightarrow \quad b_l(t_1) = b_l(t_0) \\
q_1 = \infty & \quad \Rightarrow \quad a = \text{const on } s_1 \quad \Rightarrow \quad a_l(t_1) = a_l(t_0) \quad \Rightarrow \quad b_l(t_1) > b_l(t_0)
\end{align*}
\]

(52)

\[
\begin{align*}
& q_1 < 0; \\
& a_l(t_1) = q_1 (a_l(t_1) - a_l(t_0)); \\
& b_l(t_0) = a_l(t_0) + t_0; \quad b_l(t_1) = a_l(t_1) + t_1; \\
& a_l(t_1) - a_l(t_0) = q_1 (a_l(t_1) - a_l(t_0)) + q_1 (t_1 - t_0); \\
& a_l(t_1) - a_l(t_0) = \frac{q_1}{1-q_1} (t_1 - t_0) < 0; \\
& a_l(t_1) + t_1 - a_l(t_0) - t_0 = \frac{q_1}{1-q_1} (t_1 - t_0) + t_1 - t_0 \\
& b_l(t_1) - b_l(t_0) = \frac{1}{1-q_1} (t_1 - t_0) > 0.
\end{align*}
\]

(53)
\begin{equation}
\begin{aligned}
a_l(t_1) - a_l(t_0) &= \frac{q_1}{1 - q_1}(t_1 - t_0) < 0 \implies a_l(t_1) < a_l(t_0) \\
b_l(t_1) - b_l(t_0) &= \frac{1}{1 - q_1}(t_1 - t_0) > 0 \implies b_l(t_1) > b_l(t_0)
\end{aligned}
\end{equation}

It can be seen from (52) and (54) that condition (45) is satisfied. Similar arguments will prove the result for the case of equation (46).

We prove the main nestedness Theorem 5.3.2 using the previous result of nestedness in close neighborhoods.

**Theorem 5.3.2** Denote by \( \pi^*_l(t) = [a^*_l(t), b^*_l(t)] \) the optimal solution of (14). Then for every \( t_1, t_2 \), \( t_1 < t_2 \) the nestedness property holds on \( \pi^*_l(t_1) \) and \( \pi^*_l(t_2) \), i.e., :

\begin{equation}
\begin{aligned}
a^*_l(t_1) &\geq a^*_l(t_2) \\
b^*_l(t_1) &\leq b^*_l(t_2)
\end{aligned}
\end{equation}

**Proof of Theorem 5.3.2** From Theorem 5.3.1 we conclude that \( a^*_l(t) \), as well as \( b^*_l(t) \), satisfy the condition of the monotonicity Lemma 10.2.1. Inequality (55) follows from the statement of Lemma 10.2.1.
6 Definitions - Trees

6.1 Tactical problem

The following definition of the tactical subtree problem is taken from [5].

Let $T = (V, E)$ be an undirected tree graph with node set $V = \{v_1, \ldots, v_n\}$ and edge set $E = \{e_2, \ldots, e_n\}$. Each edge $e_j$, $j = 2, 3, \ldots, n$, has a positive length $l_j$, and is assumed to be rectifiable. In particular, an edge $e_j$ is identified as an interval of length $l_j$, so that we can refer to its relative interior points. Sometimes the notation $e_{ij}$ of $(v_i, v_j)$ will be used, meaning the edge connecting the nodes $v_i$ and $v_j$. We assume that $T$ is embedded in the Euclidean plane.

We denote by $A(T)$ the mapping of $T$ into $\mathbb{R}^2$. In this mapping the edge $e_i$, connecting the nodes $v_j$ and $v_k$, is the interval of length $l_i$ between points $v_j$ and $v_k$ in $\mathbb{R}^2$. So $A(T)$ is a compact set of points in $\mathbb{R}^2$, which is the union of $n - 1$ intervals. Two intervals can intersect only at common nodes.

A closed subset $Y \subset A(T)$ with a finite number of connected components can be viewed as an embedding of a graph in $\mathbb{R}^2$. This graph is denoted by $\tilde{G}(Y)$ and called subgraph, the node and edge sets of the subgraph are denoted by $\tilde{V}$ and $\tilde{E}$ respectively. The node set $\tilde{V}$ consists of the nodes of $T$ which are in $Y$ plus the relative boundary points (sometimes called endpoints) of $Y$ which are not vertices of $T$. Correspondingly, $Y$ is the embedding of $\tilde{G}$, $Y = A(\tilde{G})$. If $Y \subset A(T)$ is a connected set, the resulting subgraph, $\tilde{G}(Y)$ is a tree. This tree is called a subtree of $T$ and denoted by $T(Y) = (\tilde{V}, \tilde{E})$ Correspondingly, $Y$ is the embedding of $\tilde{T}$, $Y = A(T)$.

We define the length or size of a subgraph $\tilde{G}$, $L(\tilde{G})$ as a sum of lengths of all the intervals $A(\tilde{G})$ consists of. The length of a subtree $L(\tilde{T})$ is the sum of the lengths of its edges. The length of the tree graph $T$ is assumed, without loss of generality (WLOG), to be 1.

We define the distance between two nodes $v_1, v_2 \in V$ to be the length of the unique path in $T$ between them: $\delta(v_1, v_2) = L(P(v_1, v_2))$. Since any two points $a, b \in A(T)$, can be viewed as nodes of the augmented tree graph with node set $V \cup \{a, b\}$, we define a distance between any two points in $A(T)$ in a similar way:

$$\delta(a, b) = L(P(a, b)), \forall a, b \in A(T).$$

The distance between a node $v_i$ and a subgraph $\tilde{G}$ is defined as the distance from $v_i$ to a closest point on $\tilde{G}$ and denoted as $\delta_i \triangleq \delta(v_i, \tilde{G})$.

In location theory the nodes $v_i$ are sometimes called demand facilities, subtree $\tilde{T}$ is called extensive facility or server facility and $L(\tilde{T})$ is the extensive facility length.

In our location model the nodes of the tree are viewed as demand points (customers), and each node $v_i$ is associated with a nonnegative weight $w_i$. The set of potential servers consists of subtrees. There is a transportation cost function of the customers (which are assumed to be at the nodes of the tree) to the serving facility, the cost for node $v_i$ is $w_i \delta(v_i, T)$. The overall transportation cost from the subtree $\tilde{T}$ to all demand points is defined as convex ordered median.
(COM) function on weighted distances: \( F(w_1 \delta(v_1, \tilde{T}), \ldots, w_n \delta(v_n, \tilde{T})) \) (see definition of \( F \) at 3.1.1).

**Definition 6.1.1** Given a tree graph \( T \), the non-negative weight vector \( w = (w_1, \ldots, w_n) \), the extensive facility length \( t \), and the COM function \( F(y_1, \ldots, y_n) \). The **tactical** subtree problem with a COM objective is the problem of finding a subtree (facility) \( \tilde{T} \) of length less than or equal to \( t \), minimizing the COM objective:

\[
\text{Find subtree } \tilde{T} \\
\text{which minimizes the objective:} \\
F(w_1 \delta(v_1, \tilde{T}), \ldots, w_n \delta(v_n, \tilde{T}}).
\]

Given a real \( t \), and a node \( v_1 \in V \) the rooted (at \( v_1 \)) **tactical** subtree problem with an ordered median objective is a problem of finding a subtree (facility) \( \tilde{T} \), containing the node \( v_1 \), of length smaller than or equal to \( t \), minimizing the COM objective.

Given two optimal solutions (subtrees) \( \tilde{T}_1, \tilde{T}_2 \) to tactical subtree problems with lengths \( t_1, t_2 \), respectively, \( 0 \leq t_1 < t_2 \leq 1 \), we say that they satisfy the **nestedness** property if \( A(\tilde{T}_1) \subset A(\tilde{T}_2) \).

**Definition 6.1.2** A tactical COM problem on the tree, defined in definition 6.1.1, possesses the **nestedness** property if for all pairs of extensive facility lengths \( t_1, t_2, t_1 < t_2 \), there exists a corresponding pair of optimal solutions \( \tilde{T}_1^* \) and \( \tilde{T}_2^* \) such that \( A(\tilde{T}_1^*) \subset A(\tilde{T}_2^*) \).

### 6.2 Representation of a subtree by distances to the leaf vector

A alternative representation of a subtree is by the vector of its distances to the leaves of the tree \( T \). Denote the set of leaf nodes of \( T \) by \( V_L \) and assume, WLOG, that the nodes \( V_L = \{v_2, v_3, \ldots, v_{N_T}\} \) (\( v_1 \) is saved to be the tree root for the rooted problem to be defined later).

For a subtree \( \tilde{T}^\alpha = (\tilde{V}^\alpha, \tilde{E}^\alpha) \), define a vector of distances from \( \tilde{T}^\alpha \) to the leaves of the tree \( T \) as \( \tilde{\alpha}^\alpha \):

\[
\tilde{\alpha}^\alpha \triangleq \tilde{a}^{(\tilde{T}^\alpha)} \triangleq \left( \delta(v_2, \tilde{T}^\alpha), \ldots, \delta(v_{N_T}, \tilde{T}^\alpha) \right).
\]

**Lemma 6.2.1** Let \( \tilde{T}^\alpha \) be a subtree in \( A(T) \). Denote by \( \tilde{\alpha}^\alpha \) the vector of distances from the leaves of \( T \) to \( \tilde{T}^\alpha \), \( \tilde{\alpha}^\alpha = \delta(T^\alpha) \), then:

1. The subtree \( \tilde{T}^\alpha \) is uniquely defined by its vector of distances to leaves, i.e., for any two subtrees \( \tilde{T}^1, \tilde{T}^2 \), \( \tilde{\alpha}^{(\tilde{T}^1)} = \tilde{\alpha}^{(\tilde{T}^2)} \iff \tilde{T}^1 = \tilde{T}^2 \).
2. For any two subtrees \( \tilde{T}^1, \tilde{T}^2 \), \( \tilde{\alpha}^{(\tilde{T}^1)} \leq \tilde{\alpha}^{(\tilde{T}^2)} \iff A(\tilde{T}^2) \subset A(\tilde{T}^1) \).
Proof

Proof of 1.
Consider, by contradiction, two different subtrees \( \tilde{T}_1 \neq \tilde{T}_2 \) with the same vector of distances to the leaves, \( \delta^1 = \delta^2 \). Let \( v_Q \in V_L \) be a leaf. Let \( \hat{v}_1 \in \tilde{T}_1 \) be a node on \( \tilde{T}_1 \) closest to \( v_Q \) and \( \hat{v}_2 \in \tilde{T}_2 \) be a node on \( \tilde{T}_2 \) closest to \( v_Q \). Both nodes have the same distances to \( v_Q \):

\[
\delta(v_Q, \hat{T}_1) = \delta(v_Q, \hat{v}_1) = \delta(v_Q, \hat{v}_2) = \delta(v_Q, \tilde{T}_2).
\]

We can assume WLOG that \( \hat{v}_1 \neq \hat{v}_2 \), otherwise we can select another \( v_Q \).
Consider \( v_Q \) as a root of \( T \). Since \( \hat{v}_1 \notin P(\hat{v}_2, v_Q) \) and \( \hat{v}_2 \notin P(\hat{v}_1, v_Q) \), there is a node \( v^* \) which is the common node of \( P(\hat{v}_2, v_Q) \) and \( P(\hat{v}_1, v_Q) \) farthest away from \( v_Q \). Now consider some leaf \( v_R \in V_L \), which is a descendant of \( \hat{v}_2 \).

\[
\begin{align*}
\delta(v_R, \tilde{T}_1) &= \delta(\hat{v}_1, v_R) \geq \delta(\hat{v}_1, v^*) + \delta(v^*, \hat{v}_2) + \delta(v_R, \tilde{T}_2) \\
\delta(v_R, \tilde{T}_1) &> \delta(v_R, \tilde{T}_2),
\end{align*}
\]

which is a contradiction.

Proof of 2.
Consider subtrees \( \tilde{T}_1, \tilde{T}_2 \) with the vectors of distances to leaves, \( \delta^1 \leq \delta^2 \). Let \( v_Q \in V_L \) be a leaf node. Let \( \hat{v}_1 \in \tilde{T}_1 \) be a node on \( \tilde{T}_1 \) closest to \( v_Q \) and \( \hat{v}_2 \in \tilde{T}_2 \) be a node on \( \tilde{T}_2 \) closest to \( v_Q \). The distances from those nodes to \( \hat{v}_Q \) is satisfy:

\[
\delta(v_Q, \hat{T}_1) = \delta(v_Q, \hat{v}_1) \leq \delta(v_Q, \hat{v}_2) = \delta(v_Q, \tilde{T}_2).
\]

If \( \hat{v}_1 \in P(\hat{v}_2, v_Q) \) then we change the selection of leaf \( v_Q \) and repeat. If there is no leaf \( v_Q \) such that \( \hat{v}_1 \notin P(\hat{v}_2, v_Q) \), we conclude that \( \tilde{T}_2 \subset \tilde{T}_1 \). This is because \( \tilde{T}_1 \) can be obtained by deleting the paths \( P(\hat{v}_1, v_Q) \) from \( T \) for all leaves \( v_Q \in V_L \).

Consider the case \( \hat{v}_1 \notin P(\hat{v}_2, v_Q) \). There is the node \( v^* \) which is a last common node of \( P(\hat{v}_2, v_Q) \) and \( P(\hat{v}_1, v_Q) \). Now consider some leaf \( v_R \), which is a descendant of \( \hat{v}_2 \) on \( \tilde{T} \), rooted at \( v_Q \).

\[
\begin{align*}
\delta(v_R, \tilde{T}_1) &= \delta(\hat{v}_1, v_R) \geq \delta(v^*, \hat{v}_1) + \delta(v^*, \hat{v}_2) + \delta(v_R, \tilde{T}_2) \\
\delta(v_R, \tilde{T}_1) &> \delta(v_R, \tilde{T}_2),
\end{align*}
\]

which is a contradiction. (Actually, Part 1 of the claim is a consequence of Part 2.)

6.3 Nestedness property

Let \( X^*(t) \) be an optimal solution set of the tactical subtree problem with extensive facility length \( t \), defined in Section 6.1. In general, \( X^*(t) \) is not a singleton, which makes it difficult to prove the nestedness property for optimal solutions. To overcome this obstruction we will restrict the whole optimal solution set \( X^*(t) \) to one point, which is the lexicographic minimum of the distances to leaves vector.
Definition 6.3.1 Given two vectors \( u, v \in \mathbb{R}^n \), we say that \( u \) is lexicographically less than \( v \) if the first non-zero component of \( v - u \) is positive. We denote it by \( u \leq_{\text{LEX}} v \).

It is obvious that for two non-equal vectors \( u, v \) either \( u \leq_{\text{LEX}} v \) or \( v \leq_{\text{LEX}} u \) holds.

Definition 6.3.2 We say that vector \( u \) is a lexicographic minimum of a set \( C \in \mathbb{R}^k \) if \( u \in C \) and \( u \leq_{\text{LEX}} v \), \( \forall v \neq u, v \in C \). The notation for lexicographic minimum is \( u = \text{lexmin} C \).

The following lemma summarizes the properties of the lexicographic minimum. It is given without the proof.

Lemma 6.3.1 For every bounded and closed set \( X \in \mathbb{R}^k \), its lexicographic minimum \( \text{lexmin} X \), exists, it is unique and located on the set’s boundary.

Moreover, if \( X \) is a subset of an affine set, \( \text{lexmin} X \notin \text{relint}(X) \), where \( \text{relint}(X) \) denotes the relative interior of \( X \).

Example 6.3.1 Consider a tree graph \( T = (V, E) \) as shown on Fig. 13, and assume that the length of each edge is 0.25. The weights assigned to the nodes are \( w_1 = 100, w_2 = 3, w_3 = w_4 = 1, w_5 = 0 \), the coefficients of the COM function are \( \lambda_1 = \lambda_2 = 3, \lambda_3 = 1, \) and \( \lambda_4 = \lambda_5 = 0 \). The goal is to find an optimal subtree of length \( t = 0.7 \) which minimizes the COM objective.

Solution:

The complete solution is presented at Example 10.4.1.

It can be seen from Fig. 13 that the subtree can be defined by a vector of three distances to the leaves \((\delta_2, \delta_3, \delta_4)\).
The optimal solution is not unique, each subtree of length 0.7 satisfying \( w_2\delta_2 \leq w_3\delta_3, w_2\delta_2 \leq w_4\delta_4 \) is an optimal subtree. The optimal objective value is 0.9.

Consider the set \( X \) in the \((\delta_2, \delta_3, \delta_4)\) space, which is the set of all optimal

\[
\begin{align*}
\delta^1 &= \delta(T^*1) = (0, 0.25, 0.05) \\
\delta^2 &= \delta(T^*2) = (0, 0.05, 0.25) \\
\delta^3 &= \delta(T^*3) = (0.04, 0.13, 0.13)
\end{align*}
\]

Note that \( \delta^2 \overset{LEX}{<} \delta^1 \overset{LEX}{<} \delta^3 \).

Figure 14: Example 10.4.1 - Three Optimal Solutions

Three different optimal solutions are presented at Fig. 14. The corresponding three vectors of the distances to leaves are:
solutions. \( X \) is defined by (from Example 10.4.1):
\[
\begin{align*}
\frac{3}{4} \delta_3 + \delta_4 & \geq 0.225 \\
\frac{3}{4} \delta_4 + \delta_3 & \geq 0.225 \\
0 & \leq \delta_3 \leq 0.25 \\
0 & \leq \delta_4 \leq 0.25 \\
\delta_3 + \delta_4 & \leq 0.3 \\
\delta_2 & = 0.3 - \delta_3 - \delta_4.
\end{align*}
\]

We can say that \( \delta_2 = \text{lexmin} \ X \).

The following theorem summarizes the main result of this paper.

**Theorem 6.3.1** Denote by \( H(t) \) the set of all optimal solutions to the tactical (unrooted) subtree problem \( (56) \) defined on some tree \( T \) with maximal subtree length \( t \) (see [3])

For each subtree \( \tilde{T}^Y \in H(t) \) we assign a vector \( \delta^Y \in \mathbb{R}^{N_T-1} \) which is a vector of distances to leaves for subtree \( \tilde{T}^Y \), according to \( (57) \). Let \( D^H(t) \subset \mathbb{R}^{N_T-1} \) be a collection of all distances to leaves vectors of \( H(t) \), \( D^H(t) = \{ \delta^Y : \tilde{T}^Y \in H(t) \} \).

Let \( \delta^L(t) = \text{lexmin} \{ D^H(t) \} \), and \( \tilde{T}^L(t) \) be its corresponding subtree. Then the following nestedness property holds:

\[
A \left( \tilde{T}^L(t_1) \right) \subset A \left( \tilde{T}^L(t_2) \right) \quad \forall \ 0 \leq t_1 < t_2 \leq 1.
\]

The proof of the Theorem 6.3.1 appears at Section 7.4.

### 6.4 Definition of the tactical rooted problem

Let \( T \) be a tree rooted at \( v_1 \). For each node \( v_j, j \neq 1 \) we denote its parent node by \( p(v_j) \) and the edge connecting \( v_j \) with \( p(v_j) \) by \((p(v_j), v_j)\), another notation to be used for that edge is \( e_j, e_j = (p(v_j), v_j) \). The length of \( e_j \) is denoted by \( l_j \).

For each edge of the rooted tree \( T \) assign a variable \( x_j, 0 \leq x_j \leq l_j \). The interpretation of \( x_j \) is as follows: Suppose that \( x_j > 0 \) and let \( \xi(x_j, e_j) \) be the point on edge \( e_j \), whose distance from \( p(v_j) \) is \( x_j \).

The only part of \( e_j \) included in the selected subtree rooted at \( v_1 \) is the subedge \( P(p(v_j), \xi(x_j, e_j)) \).

The variables \( x_j \) are called edge variables, and the vector of \( n - 1 \) edge variables is denoted by \( \bar{\tau} = (x_2, \ldots, x_n) \).

The union of all subedges, \( \bigcup_{j \neq 1} P(p(v_j), \xi(x_j, e_j)) \), represents a subgraph, which is denoted by \( \tilde{G}_\bar{\tau} \).

This subgraph is a subtree, rooted at \( v_1 \), if the following connectivity equation holds:

\[
x_j(l_i - x_i) = 0, \quad \forall i, j, \text{ s.t. } v_i = p(v_j), \quad j = 2, \ldots, n.
\]
When (63) holds, we will denote the resulted subtree as $\tilde{T}_x$, where $\tilde{T}_x = \bigcup_{j \neq 1} P(p(v_j), \xi(x_j, e_j))$.

The length of the tree $\tilde{T}_x$ is $\sum_{j=2}^n x_j$.

We call a vector $x = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ subgraph if $0 \leq x_j \leq l_j$, $j = 2, \ldots, n$. A subgraph is called a subtree if the connectivity condition (63) holds.

Consider the following linear transformation $D(x) : \mathbb{R}^{n-1} \to \mathbb{R}^{n-1}$:

$$D(x) = (d_2(x), \ldots, d_n(x))$$

where

$$d_j(x) \triangleq \sum_{v_k \in P(v_1, v_j)} (l_k - x_k).$$

(64)

If the vector of edge variables $x$ represents the subtree $\tilde{T}_x$, then $d_j(x)$ is a distance from the node $v_j$ to the tree $\tilde{T}_x$, $d_j(x) = \delta(v_j, \tilde{T}_x)$.

The tactical rooted COM optimal tree problem is:

Find $x = (x_2, x_3, \ldots, x_n)$ that minimizes $F(w_2d_2, w_3d_3, \ldots, w_nd_n)$ such that:

$$\sum_{j=2}^n x_j \leq t,$$

$$x_j \geq 0, j = 2, \ldots, n,$$

$$x_j \leq l_j, j = 2, \ldots, n,$$

and the connectivity condition (63) holds.

(65)

We call a vector $x = (x_2, \ldots, x_n) \in \mathbb{R}^{n-1}$ admissible if $\sum_{j=2}^n x_j \leq t$ and $0 \leq x_j \leq l_j$, $j = 2, \ldots, n$. Note that a subgraph $\tilde{G}_x$ of an admissible $x$ induces a closed subset of $A(T)$ of length less than or equal to $t$, which contains $v_1$.

J. Puerto and A. Tamir studied the solutions of (65) in [5]. Below are two useful results from that paper.

**Proposition 6.4.1** [5] Let $T = (V, E)$ be an undirected tree graph with node set $V = \{v_1, \ldots, v_n\}$. Let $x = (x_2, \ldots, x_n)$ be admissible. Then there exists a solution $x' = (x'_2, \ldots, x'_n)$ to (65), such that:

$$w_jd_j(x') \leq w_jd_j(x), \quad j = 2, \ldots, n.$$

**Corollary 6.4.2** [5] Consider the problem of selecting a subtree of total length less than or equal to $t$, containing $v_1$, and minimizing an isotone function $f(w_2d_2(x), \ldots, w_nd_n(x))$ of the weighted distances $(w_2d_2(x), \ldots, w_nd_n(x))$ of the nodes $v_2, \ldots, v_n \in V$ from the selected subtree. Then the following is a valid
formulation of the problem:

Find $\bar{x} = (x_2, x_3, \ldots, x_n)$
that minimizes $f(w_2d_2(\bar{x}), \ldots, w_nd_n(\bar{x}))$
such that:
\[ \sum_{j=2}^{n} x_j \leq t \]
\[ 0 \leq x_j \leq l_j, \ j = 2, \ldots, n. \]

According to Corollary 6.4.2 we formulate the tactical rooted COM optimal tree problem as follows:

Find $\bar{x} = (x_2, x_3, \ldots, x_n)$
that minimizes $F(w_2d_2(\bar{x}), w_3d_3(\bar{x}), \ldots, w_nd_n(\bar{x}))$
such that:
\[ \sum_{j=2}^{n} x_j \leq t \]
\[ 0 \leq x_j \leq l_j, \ j = 2, \ldots, n. \]

For the rooted subtree problem the subtree is defined as a set of edge variables. In this case the vector of distances to leaves is obtained by a linear mapping on edge variables $\bar{x}$.

This mapping is denoted by $D^T(\bar{x}) : \mathbb{R}^{n-1} \rightarrow \mathbb{R}^{N_T-1}$ and it is a restricted version of the transformation $D^{\bar{x}}$ defined in (64).

$$\bar{\delta} = A_T\bar{x} + B_T$$

$A_T \in \mathbb{R}^{(N_T-1)-(n-1)}$, $B_T \in \mathbb{R}^{N_T-1}$

$$A_T(i,j) = \begin{cases} -1 & \text{if } v_{j+1} \in P(v_1, v_{i+1}) \\ 0 & \text{otherwise} \end{cases}$$

$$B_T(i) = \sum_{v_k \in P(v_1, v_{i+1})} l_k$$

$1 \leq i \leq N_T - 1,$
$1 \leq j \leq n - 1.$

Note that, in general, the mapping (68) is the affine transformation applied on edge variables $\bar{x}$, since the distances to leaf not defined for disconnected subgraph. In case of $\bar{x}$ admissible and satisfying (63), the transformation (68) gives the distances to leaves vector.

Theorem 6.4.3 Denote by $X(t) \in \mathbb{R}^{n-1}$ the set of all optimal solutions of the tactical rooted COM optimal tree problem (67), and let $D^X(t) = A_TX(t) + B_T$ be the image of $X(t)$ under $D^T(\bar{x})$ defined in (68). Let $\delta^X(t) = \text{lexmin}\{D^X(t)\}$, then:

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1. For each $t$, $0 \leq t \leq 1$, there exists a unique edge variables vector $x^L(t)$ such that

$$\delta^L(t) = A_T \cdot x^L(t) + B^T.$$ 

2. The following nestedness property holds:

$$x^L(t_1) \leq x^L(t_2), \quad \forall \ 0 \leq t_1 \leq t_2 \leq 1.$$  (69)
7 Nestedness of the Rooted Problem

7.1 Proof guide lines

We map the tree optimization problem to the optimization problem in the $\mathbb{R}^{n-1}$ space, where $n$ is the number of vertices in $T$. This is done using the problem definition in [3]. Moreover the rooted tree optimization problem is reformulated as a parametrized LP problem (using LP formulation of COM function in [4]) - Section 5.1.1. The continuity of the optimal solution $\pi^*_L(t)$ in $\mathbb{R}^{n-1}$ with respect to extensive facility length $t$ follows from the LP formulations - Section 7.2.2.

Investigating the properties of COM objective function, we conclude that the optimal solutions $\pi^*_L(t)$, when mapped to $\mathbb{R}^{n-1}$, can lie only on at least one of the $s_k$-hyperplanes. This is due to the convexity and the piecewise linearity of the COM function. Actually, given two extensive facility lengths $t_1, t_2$ close enough to each other, the corresponding optimal solutions $\pi^*_L(t_1), \pi^*_L(t_2)$ lie on the same line, which is the intersection of $n-2$ $s_k$-hyperplanes - Section 7.3.

Then, the nestedness property for those two solutions follows from geometric properties of the $s_k$-hyperplanes. We call this property nestedness in a closed neighborhood - Section 7.4. The last step is to expand the nestedness property to every two values of the extensive facility length. This is done using the Monotonicity Lemma 10.2.1. The schematic view of the proof is given in Fig. 15.

7.2 Continuity of the optimal solution

7.2.1 LP formulation of the rooted problem

It is possible to formulate (66) as a pure LP problem ([5]), using the LP formulation for the $k$-centrum and the COM objectives [4]. This provides an LP formulation for finding a subtree rooted at a distinguished point, whose length is at most $t$, minimizing a COM objective. For convenience we define $\lambda_n = 0$.

The LP formulation requires an augmented state space $\mathbb{R}^{(n-1)+(n-1)+(n-1)^2} = \mathbb{R}^{(n+1)(n-1)}$. We will denote a vector in this augmented space with subindex $a$.

Find $\pi_a = (x_2, \ldots, x_n, u_2, \ldots, u_n, y_{2,2}, \ldots, y_{n,n})$ that minimizes:

$$ z = \sum_{k=1}^{n-1} \left( \lambda_k - \lambda_{k+1} \right) \left( k u_k + \sum_{i=2}^{n} y_{i,k} \right) $$

subject to:

$$ y_{i,k} \geq 0, \quad \forall \ i = 2, \ldots, n, \ k = 1, \ldots, n-1, $$

$$ y_{i,k} \geq w_i d_i(x_2, \ldots, x_n) - u_k, \quad \forall \ i = 2, \ldots, n, \ k = 2, \ldots, n, $$

$$ 0 \leq x_i \leq l_i, \quad \forall \ i = 2, \ldots, n, $$

$$ \sum_{j=2}^{n} x_j \leq t. $$

7.2.2 Proof of the continuity result

Lemma 7.2.1 Let $X_a(t) \subset \mathbb{R}^n$ be the set of all optimal solutions of (70) with parameter $t$, and assume that $X_a(t)$ is bounded and non empty for all $t \in [0,1]$. 

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Let \( l \) be a non-negative integer, define a linear transformation \( D : \mathbb{R}^n \rightarrow \mathbb{R}^l \) by
\[
d_i = \sum_{j=1}^{n} d_{ij} x_j, \quad i = 1, \ldots, l
\]
Denote by \( D(t) \) the image of \( X(t) \) under transformation \( D \), and \( \overline{d}_L(t) = \text{lexmin} \, D(t) \). Then \( \overline{d}_L(t) \) is a piecewise linear, continuous function \( \mathbb{R}^1 \rightarrow \mathbb{R} \).

**Proof** Define \( z^*(t) \) as the objective function value of (70). From Lemma 10.1.1 in Appendix 10.1 \( z^*(t) \) is piecewise linear, continuous and convex function of \( t \).

Consider the following induction hypothesis:

Let \( d_{L,k}(t) \) be the \( k \)-th component of \( \overline{d}_L(t) \). \( d_{L,k}(t) \) is piecewise linear, continuous function of \( t \).
\( k = 1 \). Since \( z^*(t) \) is continuous and piecewise linear, we can divide the interval \([0, 1]\) to a finite number of closed intervals \([t_s, t_{s+1}]\), \( U_{s=1,\ldots,p} [t_s, t_{s+1}] = [0, 1] \), assuming \( z^*(t) \) is linear in each interval,

\[
z^*(t) = z_1(t_s)t + z_0(t_s) \quad \forall t \in [t_s, t_{s+1}].
\]

Consider the following parametrized LP problem, defined on the interval \([t_s, t_{s+1}]\):

Maximize

\[
\sum_{j=2}^{n} -d_{1j}x_j
\]

such that

\[
\sum_{j=2}^{n} c_jx_j \leq z_1(t_s)t + z_0(t_s),
\]

\[
\sum_{j=2}^{n} a_{ij}x_j \leq b_1^it + b_0^i, \quad i = 1, \ldots, m,
\]

\( t \in [t_s, t_{s+1}] \).

The optimal solution value of (71) is the value of the first component of \( d_{L,1}(t) : d_{L,1}(t) \).

Using Lemma 10.1.1, we conclude that \( d_{L,1}(t) \) is a piecewise linear, continuous function of \( t \) on the interval \([t_s, t_{s+1}]\). The continuity of \( d_{L,1}(t) \) follows from the continuity of \( z^*(t) \).

Assuming the induction hypothesis holds for \( k - 1 \). Then the functions \( z^*(t) \) and \( d_{L,i}(t), \quad i = 1, \ldots, k - 1 \), are piecewise linear and continuous. Thus the interval \([0, 1]\) can be divided into a finite number of closed intervals \([t_s, t_{s+1}]\), \( U_{s=1,\ldots,p} [t_s, t_{s+1}] = [0, 1] \), and

\[
z^*(t) = z_1(t_s)t + z_0(t_s),
\]

\[
d_{L,i}(t) = d_{L,i}^1(t_s)t + d_{L,i}^0(t_s), \quad i = 1, \ldots, k - 1,
\]

(72)

\( \forall t \in [t_s, t_{s+1}] \).

Consider the following parametrized LP problem, defined on the interval \([t_s, t_{s+1}]\):

Maximize

\[
d_k(t) = \sum_{j=1}^{n} -d_{kj}x_j
\]

such that

\[
\sum_{j=2}^{n} c_jx_j \leq z_1(t_s)t + z_0(t_s)
\]

\[
\sum_{j=2}^{n} d_{ij}x_j \leq d_{L,i}^1(t_s)t + d_{L,i}^0(t_s), \quad i = 1, \ldots, k - 1
\]

\[
\sum_{j=2}^{n} a_{ij}x_j \leq b_1^it + b_0^i, \quad i = 1, \ldots, m,
\]

\( t \in [t_s, t_{s+1}] \).
The optimal solution value of (73) is the value of the $k$-th component of $d_{L,k}(t)$, $d_{L,k}(t)$. Applying Lemma 10.1.1 to the above LP problem, we conclude that $d_{L,k}(t)$ is a piecewise linear, continuous function of $t$ on the interval $[t_s, t_{s+1}]$. The continuity of $d_{L,k}(t)$ follows from the continuity of $z^*(t)$ and $d_{L,i}(t)$ for $i = 1, \ldots, k - 1$.

We now proceed to prove the continuity of the optimal subtree $\pi_L(t)$ in $t$. For this purpose we define a metric space of all subtrees rooted at $v_1$. Let $A(T) \subset 2^A(T)$ be the set of all subtrees in $A(T)$ rooted at $v_1$. Define a metric on $A(T)$:

$$\rho(Y_1, Y_2) = L(Y_1 \oplus Y_2), \quad Y_1, Y_2 \in A(T).$$  \hspace{1cm} (74)

Define a mapping $\tilde{Y} : \mathbb{R}^{N_T-1} \to A(T)$ which assigns to each vector of distances to leaves $\tilde{d} \in \mathbb{R}^{N_T-1}$ it’s corresponding (rooted) subtree in $A(T)$. The domain of the mapping is all vectors that can represent distances to leaves of subtrees rooted at $v_1$.

**Lemma 7.2.2** $\tilde{Y} : \mathbb{R}^{N_T-1} \to A(T)$ is a continuous mapping.

**Proof**

We will show that

$$\forall d_0 \in \mathbb{R}^{N_T-1}, \quad \forall \epsilon > 0, \quad \exists \delta = \delta(\epsilon, d_0) > 0,$$

$$\text{such that : } ||\tilde{d} - d_0|| \leq \delta \Rightarrow \rho(\tilde{Y}(\tilde{d}), \tilde{Y}(d)) \leq \epsilon,$$

where $||\cdot||$ is the Euclidean norm.

Consider a subtree $\tilde{Y}(\tilde{d})$ and let $\pi^0 = (x_0^1, \ldots, x_0^n)$ be its vector of edge variables. Denote by $\tilde{V}(\tilde{d})$ the node set of $\tilde{Y}(\tilde{d})$. Denote by $\Delta_0$ the distance from the subtree $\tilde{Y}(\tilde{d})$ to the nearest node in $T$ which is not in $\tilde{V}(\tilde{d})$.

Define $\Delta_0$ to be the shortest distance between the leaves of $\tilde{Y}(\tilde{d})$ to the nodes of $T$, which are not leaves of $\tilde{Y}(\tilde{d})$.

$$\Delta_0 = \min_{v_j \in \tilde{V}(\tilde{d})} d(v_j, \tilde{Y}(\tilde{d}))$$  \hspace{1cm} (75)

Consider some positive $\delta < \Delta_0$ and construct subtrees $Y^-, Y^+$, represented by edge variables $\pi^-, \pi^+$ respectively. $Y^-$ is obtained by subtracting an interval of length $\delta$ from leaves of $\tilde{Y}(\tilde{d})$. $Y^+$ obtained by elongating $\tilde{Y}(\tilde{d})$ by an intervals of length $\delta$ in every "possible direction" from $\tilde{Y}(\tilde{d})$. The formal operations are as follows:

- $J^- = \{ j : x_j^0 \neq 0 \text{ and } x_j^0 = l_j \Rightarrow x_j^0 = 0 \text{ for all } v_k \text{ such that } v_j \in P(v_1, v_k) \}$
- $x_j^0 = x_j^0 - \delta, \forall j \in J^-$
- $J^+ = \{ j : x_j^0 < l_j \text{ and } x_j^0 = 0 \Rightarrow x_j^0 = l_k \text{ for all } v_k \in P(v_1, v_j) \}$
- $x_j^0 = x_j^0 + \delta, \forall j \in J^+$.

(76)
We denote the distances to $Y^-$ from the leaves of $T$ by $\bar{d}^\sim = (d^\sim_2, \ldots, d^\sim_{NT})$, and the distances to $Y^+$ from the leaves of $T$ by $\bar{d}^+ = (d^+_2, \ldots, d^+_N)$. From the construction of $Y^-, Y^+$:

\[
d^+_i \leq d^0_0 - \delta, \forall i = 2, \ldots, N_T,
\]
\[
d^+_i \geq d^0_0 + \delta, \forall i = 2, \ldots, N.
\]

(77)

Consider a tree $Y^*$ with distances to leaves $\bar{d}^*$, and $Y^- \not\subseteq Y^*$. From Lemma 6.2.1, $\bar{d}^* \not\subseteq \bar{d}^\sim$, i.e., $\exists k : d^*_{ik} > d^\sim_k \geq d^0_k - \delta$. Therefore, $||\bar{d}^* - d^0|| > \delta$. So $||\bar{d}^* - d^0|| \leq \delta$ implies that $Y^- \not\subseteq Y^*$. In a similar way it can be shown that $||\bar{d}^* - d^0|| \leq \delta \Rightarrow Y^* \subseteq Y^-$.

Consider a subtree $Y_1$, satisfying $Y^- \subseteq Y_1 \subseteq Y^+$. The subgraph $Y_1 \oplus Y^-$ consists of at most $n - 1$ intervals each of length less than or equal to $2\delta$. Then $\rho(Y_1, Y^-) < 2\delta n$. Thus, taking $\delta < \min(\frac{\epsilon}{2n}, \Delta_0)$ we obtain:

\[
||\bar{d}^* - d^0|| < \delta \Rightarrow Y^- \subseteq \bar{Y}(\bar{d}^*) \subseteq Y^+ \Rightarrow \rho(\bar{Y}(\bar{d}^*), Y^-) \leq 2n\delta < \epsilon/2
\]
\[
\rho(\bar{Y}(\bar{d}^*), \bar{Y}(\bar{d}^*)) \leq \rho(Y^-, \bar{Y}(\bar{d}^*)) + \rho(\bar{Y}(\bar{d}^*), Y^-) \leq \epsilon
\]

(78)

Now consider a mapping $\bar{X} : A(T) \rightarrow \mathbb{R}^{n-1}$, $\bar{X}(Y) = \pi = (x_2, \ldots, x_n)$, where $\{x_i\}$ are the edge variables of the subtree $Y$, rooted at $v_1$.

**Lemma 7.2.3** $\bar{X} : A(T) \rightarrow \mathbb{R}^{n-1}$ is a continuous mapping.

**Proof** We will show that:

Let $Y^0 \in A(T)$, $\forall \epsilon > 0$, $\exists \delta = \delta(\epsilon, Y^0) > 0$, such that:

\[
\rho(Y^0, Y^*) \leq \delta \Rightarrow ||\bar{X}(Y^0) - \bar{X}(Y^*)|| \leq \epsilon.
\]

Consider $\Delta_0$ defined in (75) and define $Y^-, Y^+$ according to (76), for some $\delta < \Delta_0$. Since $\rho(Y^0, Y^*) \leq \delta$, $Y^- \subseteq Y^* \subseteq Y^+$. Consider $||\bar{X}(Y^0) - \bar{X}(Y^-)||$:

\[
x^0_j - \delta \leq x^0_j \leq x^0_j + \delta \quad \text{if} \quad 0 < x^0_j < l_j,
\]
\[
x^0_j \leq \delta \quad \text{if} \quad x^0_j = 0,
\]
\[
x^0_j \geq l_j - \delta \quad \text{if} \quad x^0_j = l_j.
\]

(79)

Thus, $||\bar{X}(Y^0) - \bar{X}(Y^*)|| \leq 2n\delta$. By taking $\delta < \min(\frac{\epsilon}{2n}, \Delta_0)$ we obtain the continuity property.

Till now, for each $t \in (0, 1]$ we obtained a unique subtree, represented by the edge variable vector $\pi_L(t)$ and has the following properties:

1. $\pi_L(t)$ is an optimal solution of (66),
2. $\pi_L(t)$ satisfies the connectivity condition (63),
3. If \( \overline{d}_L(t) \) is a vector of distances to leaves of \( L(t) \), then \( \overline{d}_L(t) \) is lexicographically less than any other set of distances to leaves of the solutions of (66).

Using Lemmas 7.2.2 and 7.2.3 we can prove that \( \pi_L(t) \) is a continuous function w.r.t. \( t \).

**Lemma 7.2.4** Let \( X(t) \in \mathbb{R}^{n-1} \) be the set of all optimal solutions of problem (66), and let \( D(t) = A_T \cdot X(t) + B_T \) be an image of \( X(t) \) under the transformation (68). Let \( \overline{d}_L(t) = \text{lexmin} D(t) \), and \( \pi_L(t) \in X(t) \) be the vector of edge variables of subtree \( \hat{Y}(\overline{d}_L(t)) \). The following continuity property holds: for each \( \epsilon > 0 \) there is a \( \delta = \delta(\epsilon) \), \( \delta > 0 \), such that

\[
\pi_L(t') \in B(\pi_L(t), \epsilon); \quad \forall t' \in [t - \delta, t + \delta]
\]  

**Proof** Consider the LP formulation of (66) - (70). Let \( X_a(t) \in \mathbb{R}^{n^2-1} \) be the set of all optimal solutions of (70) with parameter \( t \). \( X_a(t) \) is nonempty for all \( t \in [0, 1] \), but possibly unbounded. There is a linear transformation from the space \( \mathbb{R}^{n^2-1} \) to the space of the vectors of distances to leaves \( \mathbb{R}^{Nt-1} \), which is an augmented variation of (68):

\[
D^a_T(\pi_a) = A^a_T \pi_a + B^a_T
\]

\[ A^a_T(i,j) = \begin{cases} -1 & \text{if } j \leq n - 1 \text{ and } v_{j+1} \in P(v_1, v_{i+1}) \\ 0 & \text{otherwise} \end{cases} \]

\[
B^a_T(i) = \sum_{v_k \in P(v_1, v_{i+1})} l_k
\]

Applying Lemma 7.2.1, we see that \( \overline{d}_L(t) \) is a piecewise linear, continuous function \( [0, 1] \rightarrow \mathbb{R}^{Nt-1} \).

\( \pi_L(t) \) can be written as a function composition \( \pi_L(t) = (X \circ \hat{Y} \circ \overline{d}_L)(t) \).
We use notation \( \circ \) for function composition, i.e., \( (u \circ v)(x) = u(v(x)) \). Since \( X, Y, \overline{d}_L \) are continuous, \( \pi_L(t) \) is continuous as well. \( \blacksquare \)

### 7.3 \( s_h \)-hyperplanes

Consider a COM objective function \( F(w_2 f_2(\pi), w_3 f_3(\pi), \ldots, w_n f_n(\pi)) \) in (66), this function has the following properties:

1. Let \( F(u_1, \ldots, u_k) : \mathbb{R}^k \rightarrow \mathbb{R} \) be a COM function. Consider convex closed connected set \( C \in \mathbb{R}^k \) and assume WLOG that for some \( u \in C \), \( u_1 \leq u_2 \leq \ldots \leq u_k \). If the order of elements \( u_1, \ldots, u_k \) remains unchanged for all members in \( C \), i.e., for example \( u_1 \leq u_2 \leq \ldots \leq u_k \), \( \forall u \in C \), then \( F(u_1, \ldots, u_k) \) is a linear function on \( C \).
2. Let \( F(w_2 d_2(x), w_3 d_3(x), \ldots, w_n d_n(x)) : \mathbb{R}^{n-1} \to \mathbb{R} \) be a transformation defined on \( \mathbb{R}^{n-1} \). Where \( d_i(x) : \mathbb{R}^{n-1} \to \mathbb{R} \) is a linear transformation defined in (64), and \( F \) is a COM function. Consider a closed connected set \( C \subseteq \mathbb{R}^{n-1} \) and assume WLOG that \( w_2 d_2(x) \leq w_3 d_3(x) \leq \ldots \leq w_n d_n(x) \), \( \forall x \in C \). Then \( F(w_2 d_2(x), w_3 d_3(x), \ldots, w_n d_n(x)) \) is a linear function on \( C \).

Let \( \tilde{I} = \{ \tilde{s}_h \} \) be a finite set of hyperplanes in \( \mathbb{R}^{n-1} \) satisfying one of the following:

\[
\tilde{s}_h = \{ \tilde{z} \in \mathbb{R}^{n-1} : z_i = 0 \text{ for some } i \},
\]

\[
\tilde{s}_h = \{ \tilde{z} \in \mathbb{R}^{n-1} : z_i = l_i \text{ for some } i \},
\]

\[
\tilde{s}_h = \{ \tilde{z} \in \mathbb{R}^{n-1} : w_j \sum_{v_k \in P(v_i, v_j)} (l_k - z_k) = w_i \sum_{v_k \in P(v_i, v_j)} (l_k - z_k) \text{ for some } i \neq j \}.
\]

(82)

For the above first two types of hyperplane \( \tilde{s}_h \) defines the bound of feasible solutions of (66). For the third type, \( \tilde{s}_h \) defines the hyperplane where \( w_i d_i(x) \) and \( w_j d_j(x) \) change their order in the COM function \( F(w_2 d_2(x), \ldots, w_n d_n(x)) : \mathbb{R}^{n-1} \to \mathbb{R} \). Another representation of the hyperplane \( \tilde{s}_h \) is a pair \( (\tilde{\nu}_h, b_h) \), \( \tilde{s}_h = \{ \tilde{z} \in \mathbb{R}^{n-1} : \tilde{\nu}_h \cdot \tilde{z} = b_h \} \). Two or more hyperplanes are called independent if their \( \tilde{\nu}_h \) vectors are linearly independent.

**Lemma 7.3.1** Let \( X(t) \subseteq \mathbb{R}^{n-1} \) be the set of all optimal solutions of (66), and let \( D^X(t) = A \cdot X(t) + B \) be an image of \( X(t) \) under the linear transformation (68). Let \( \tilde{d}_L(t) = \text{lexmin} D^X(t) \) and \( \tilde{\pi}^L(t) \) be its corresponding subtree representation in \( X(t) \). Then there are \( n - 2 \) independent hyperplanes, members of \( \tilde{I} \), which contain the point \( \tilde{\pi}^L(t) \).

**Proof** Denote by \( I' \) the set of all hyperplanes from \( \tilde{I} \) that pass through \( \tilde{\pi}^L(t) \),

\[
I' = \{ \tilde{s}_h \in \tilde{I} : \tilde{\pi}^L(t) \in \tilde{s}_h \}.
\]

(83)

Suppose \( |I'| = K \) and \( I' = \{ \tilde{s}_1, \tilde{s}_2, \ldots, \tilde{s}_K \} \). Define a matrix \( \mathcal{A} \in \mathbb{R}^{(K+1) \times (n-1)} \) and a vector \( \mathcal{B} \in \mathbb{R}^{K+1} \)

\[
\mathcal{A} = \begin{bmatrix}
\tilde{\nu}_1 \\
\tilde{\nu}_2 \\
\cdot \\
\cdot \\
\tilde{\nu}_K \\
1 & 1 & \cdots & 1
\end{bmatrix}, \quad \mathcal{B} = \begin{bmatrix}
1 \\
1 \\
\cdot \\
\cdot \\
1 \\
t
\end{bmatrix},
\]

(84)

where \( \tilde{\nu}_i, \ i = 1, \ldots, k, \) is the normal vector to hyperplane \( \tilde{s}_i \). We will prove that \( \text{rank}(\mathcal{A}) = n - 1 \).

Suppose, by contradiction, that \( \text{rank}(\mathcal{A}) < n - 1 \). The linear system \( \mathcal{A} \tilde{\pi} = \mathcal{B} \) has at least one solution: \( \tilde{\pi}_L(t) \). Since its not full rank, \( \mathcal{A} \tilde{\pi} = \mathcal{B} \) has an infinite number of solutions. Denote by \( Z \) the set of solutions of \( \mathcal{A} \tilde{\pi} = \mathcal{B} \). Since \( \tilde{\pi}_L(t) \in Z \), \( Z \) can be written as

\[
Z = \{ \tilde{\pi} \in \mathbb{R}^{n-1} : \tilde{\pi} = \tilde{\pi}_L(t) + k \mathcal{A} \}
\]
where \( \mathcal{K}_d \) is a basis matrix of \( \text{Ker}(d) \) and \( \text{rank}(\mathcal{K}_d) = n - 1 - \text{rank}(d) > 0 \), \( \bar{z} \in \mathbb{R}^{\text{rank}(\mathcal{K})} \). For \( \bar{z} \in Z \) denote by \( \rho(\bar{z}, \hat{s}_h) \) the Euclidean distance from point \( \bar{z} \) to the hyperplane \( \hat{s}_h \) in \( \mathbb{R}^{n-1} \). Let \( \epsilon_0, 0 < \epsilon_0 < \min_{\hat{s}_h \in F'} \rho(\bar{z}_L(t), \hat{s}_h) \), where \( \epsilon_0 \) is the distance to hyperplane nearest to \( \bar{z}_L(t) \) which does not contain \( \bar{z}_L(t) \). Let \( \epsilon_1 = \frac{\epsilon_0}{\|\mathcal{K}_d\|} \), where \( \|\mathcal{K}_d\| = \max_{\bar{z} \neq \emptyset} \frac{\|\mathcal{K}_d\bar{z}\|}{\|\bar{z}\|} > 0 \), since \( \mathcal{K}_d \) is full rank. Consider a set \( Z_{e_1} = \{ \bar{z} \in \mathbb{R}^{n-1} : \bar{z} = \bar{z}_L(t) + \mathcal{K}_d \bar{e}, \|\bar{z}\| \leq \epsilon_1 \} \).

**Proposition 1:** For all \( \bar{z} \in Z_{e_1} \), the connectivity condition (63) holds.

**Proof** For all \( \bar{z} \in Z_{e_1} \), \( \|\bar{z} - \bar{z}_L(t)\| = \|\mathcal{K}_d \bar{e}\| \leq \epsilon_0 \), which means \( \bar{z} \) is in a ball of radius \( \epsilon_0 \) around \( \bar{z}_L(t) \). Therefore, \( \bar{z} \) does not belong to hyperplanes which do not pass through \( \bar{z}_L(t) \). However, since \( Z_{e_1} \subset Z \), \( \bar{z} \in Z_{e_1} \) is a solution of \( d \bar{z} = \mathcal{B} \), and all hyperplanes from \( I \) that pass through \( \bar{z}_L(t) \), pass through \( \bar{z} \). Taking a close look at the components of \( \bar{z} \) and \( \bar{z}_L(t) \), we can see that

\[
\begin{align*}
x_L^i(t) &= l_i \quad \leftrightarrow \quad x_i = l_i \\
x_L^i(t) &= 0 \quad \leftrightarrow \quad x_i = 0.
\end{align*}
\]

which means that \( \bar{z}_L(t) \) is connected \( \leftrightarrow \bar{z} \) is connected, q.e.d.

**Proposition 2:** For all \( \bar{z} \in Z_{e_1} \), \( F(d_2(\bar{z}), \ldots, d_n(\bar{z})) = F(d_2(\bar{z}_L(t)), \ldots, d_n(\bar{z}_L(t))) \).

**Proof** \( Z_{e_1} \) is a closed connected set in \( \mathbb{R}^{n-1} \). None of the hyperplanes \( \hat{s}_h \in I \), pass through this set. Each hyperplane either contains the set \( Z_{e_1} \) or has an empty intersection with the set \( Z_{e_1} \), i.e., \( Z_{e_1} \subset \hat{s}_h \) if \( \bar{z}_L(t) \in \hat{s}_h \), or \( Z_{e_1} \cap \hat{s}_h = \emptyset \) otherwise. This means that the function \( F(d_2(\bar{z}), \ldots, d_n(\bar{z})) \) is linear on \( Z_{e_1} \). The minimal value of the function, \( \min_{\bar{z} \in Z_{e_1}} F(d_2(\bar{z}), \ldots, d_n(\bar{z})) \), is attained at the point \( \bar{z}_L(t) \) which is in the relative interior of \( Z_{e_1} \). So \( F(d_2(\bar{z}), \ldots, d_n(\bar{z})) \) is constant on \( Z_{e_1} \), q.e.d.

Now, let \( D^2_{e_1} \) be an image of \( Z_{e_1} \) under the transformation (68),

\[
\begin{align*}
D^{Z_{e_1}} = \hat{A}(\bar{z}_L(t) + \mathcal{K}_d \bar{e}) + \hat{\mathcal{B}}, & \quad \bar{z} \in \mathbb{R}^{\text{rank}(K_d)}, \|\bar{z}\| \leq \epsilon_1 \\
D^{Z_{e_1}} = \bar{d}_L^i(t) + \hat{A} \mathcal{K}_d \bar{e}, & \quad \bar{z} \in \mathbb{R}^{\text{rank}(K_d)}, \|\bar{z}\| \leq \epsilon_1.
\end{align*}
\]

(85)

Note, that \( \bar{d}_L^i(t) = \text{lexmin}\{D^X(t)\} \), and since \( D^{Z_{e_1}} \subset D^X(t) \), \( \bar{d}_L^i(t) = \text{lexmin}\{D^{Z_{e_1}}\} \).

If \( \hat{A} \mathcal{K}_d \) has a non-zero component, there is a contradiction to \( \bar{d}_L^i(t) = \text{lexmin}\{D^{Z_{e_1}}\} \).

If \( \hat{A} \mathcal{K}_d \) is identically zero, then all \( \bar{z} \in Z_{e_1} \) satisfy the connectivity conditions and this contradicts the uniqueness of \( \bar{z}_L(t) \) (Lemma 63.1).

Consider one of the hyperplanes, say \( \hat{s}_h \), which contains \( \bar{z}_L(t) \), and suppose this hyperplane is of type 3, i.e., \( \hat{s}_h = \{ \bar{z} \in \mathbb{R}^{n-1} : w_i \sum_{v_k \in P(v_1,v_i)} (l_k - z_k) = w_j \sum_{v_k \in P(v_1,v_j)} (l_k - z_k) \text{ for some } i \neq j \} \). Since \( \bar{z}_L(t) \) is an edge variables vector, the distance from the corresponding subtree \( \hat{T}_{\bar{z}_L(t)} \) to node \( v_i \) is given by (64). The connectivity equation (63), implies that there is a node \( v_r \), on the path \( P(v_1,v_i) \), which has a non zero variable \( x_{L,r}(t) \), while all \( k \) ancestor nodes have
Let $L, k$ be nodes with $x_{L,k}(t) = l_k$, and all descendant nodes have $x_{L,k}^r(t) = 0$. Thus, the distance to node $v_i$ is:

$$d_i(x_{L}(t)) = \sum_{v_k \in P(v_1,v_r) \setminus v_r} (l_k - l_k) + l_r - x_{L,r}(t) + \sum_{v_k \in P(v_r,v_i) \setminus v_r} l_k$$

$$= l_r - x_{L,r}(t) + \sum_{v_k \in P(v_r,v_i) \setminus v_r} l_k$$

(86)

**Corollary 7.3.1** Let $\hat{I} = \{\hat{s}_h\}$ be a finite set of hyperplanes in $\mathbb{R}^{n-1}$ satisfying one of the following:

$$\hat{s}_h = \{z \in \mathbb{R}^{n-1} : z_i = 0 \text{ for some } i\},$$

$$\hat{s}_h = \{z \in \mathbb{R}^{n-1} : z_i = l_i \text{ for some } i\},$$

$$\hat{s}_h = \{z \in \mathbb{R}^{n-1} : A_{ri} - w_i z_i = A_{li} - w_j z_j \text{ for some } i,r,l,j\},$$

where for each pair $a,b$, $A_{ab} = \sum_{v_k \in P(v_b,v_a)} l_k$ for some $v_b \in P(v_1,v_a)$. (87)

Then there are $n - 2$ independent hyperplanes from $\hat{I}$, such that $x_L(t)$ is in their intersection.

$$\exists I' \subset \hat{I} \text{ such that: }$$

$$x_L(t) \in \bigcap_{\hat{s}_h \in I'} \hat{s}_h;$$

$$|I'| = n - 2$$

$I'$ is linearly independent (88)

**Example 7.3.1** Consider the rooted tactical COM problem on the tree defined in Example 6.3.1. We want to write down the set $\hat{I}$, i.e., all $s_h$ hyperplanes for this problem.

**Solution**

The number of the demand facilities in this problem is $n = 5$, so the set $\hat{I}$ consists of:

- $n - 1$ hyperplanes of type I,
- $n - 1$ hyperplanes of type II,
- $\binom{n-1}{2}$ hyperplanes of type III.

The set $\hat{I}$ has a cardinality of $\frac{n^2 + n - 2}{2}$, which in our case is equal to 14.

We write down the entire list of hyperplanes in $\hat{I}$ for the problem to be used.
in the examples in the next sessions.

\[ \tilde{I} = \{ s_h \}_{h=1}^{14} \]

**type I:**

\[
\begin{align*}
\mathcal{I} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_2 = 0 \} \\
\mathcal{I}_2 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_3 = 0 \} \\
\mathcal{I}_3 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_4 = 0 \} \\
\mathcal{I}_4 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_5 = 0 \} \\
\end{align*}
\]

**type II:**

\[
\begin{align*}
\mathcal{I}_5 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_2 = l_2 \} \\
\mathcal{I}_6 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_3 = l_3 \} \\
\mathcal{I}_7 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_4 = l_4 \} \\
\mathcal{I}_8 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : x_5 = l_5 \} \\
\end{align*}
\]

**type III:**

\[
\begin{align*}
\mathcal{I}_9 & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_2(l_2 - x_2 + l_5 - x_5) = w_3(l_3 - x_3) \} \\
\mathcal{I}_{10} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_2(l_2 - x_2 + l_5 - x_5) = w_4(l_4 - x_4) \} \\
\mathcal{I}_{11} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_2(l_2 - x_2 + l_5 - x_5) = w_5(l_5 - x_5) \} \\
\mathcal{I}_{12} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_3(l_3 - x_3) = w_4(l_4 - x_4) \} \\
\mathcal{I}_{13} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_3(l_3 - x_3) = w_5(l_5 - x_5) \} \\
\mathcal{I}_{14} & = \{ (x_2, x_3, x_4, x_5) \in \mathbb{R}^4 : w_4(l_4 - x_4) = w_5(l_5 - x_5) \} \\
\end{align*}
\]

7.4 Proof of the nestedness property for the rooted problem

**Proof** of Theorem 6.4.3

Let \( \bar{\pi}^L(t_2) \) be the optimal solution of (67) with parameter \( t = t_2 \). For simplicity we denote it by \( \bar{\pi}^2 \). Denote by \( \tilde{I}_2 \) the set of all \( s_h \)-hyperplanes that pass through \( \bar{\pi}^2 \). Let \( \epsilon > 0 \) be an Euclidean distance from \( \bar{\pi}^2 \) to the nearest \( s_h \)-hyperplane that does not pass through \( \bar{\pi}^2 \),

\[
\epsilon_0 = \min_{\bar{s}_h \in \tilde{I}_2} \rho(\bar{\pi}^2, \bar{s}_h). \tag{90}
\]

Select an open ball of radius \( \frac{\epsilon_0}{2} \) centered at \( \bar{\pi}^2 \), \( B(\bar{\pi}^2, \frac{\epsilon_0}{2}) \).

From the continuity of the optimal solution (Lemma (7.2.4)), there exists \( \delta > 0 \), such that all optimal solutions of the (67) are within the ball:

\[
\exists \delta > 0 \text{ such that:} \\
\pi^L(t) \in B(\bar{\pi}^2, \frac{\epsilon_0}{2}); \forall t \in [t_2 - \delta, t_2 + \delta] \tag{91}
\]

Consider some \( t_1 \in [t_2 - \delta, t_2 + \delta] \) and the corresponding optimal solution \( \pi^L(t_1) \). For simplicity we denote it by \( \pi^1 \). Denote by \( \tilde{I}_1 \) the set of all \( s_h \)-hyperplanes that pass through \( \pi^1 \). From the choice of \( \epsilon_0 \), (90) and the fact that \( \pi^1 \in B(\bar{\pi}^2, \frac{\epsilon_0}{2}) \), we conclude that \( \tilde{I}_1 \subset \tilde{I}_2 \), i.e., all hyperplanes that pass through \( \pi^1 \), also pass through \( \bar{\pi}^2 \).
According to Corollary 7.3.1, there is a set $I' \subset I^*_1$ of $n - 2$ independent $s_h$-hyperplanes. Also $I' \subset I^*_2$. In $\mathbb{R}^{n-1}$ a set of $n - 2$ independent hyperplanes defines a line. Therefore the line between $\mathbf{x}^1$ and $\mathbf{x}^2$ is defined as the intersection of $n - 2$ hyperplanes in $I'$. Both vectors $\mathbf{x}^1, \mathbf{x}^2$ are solutions of the following system of linear equations:

\[
\begin{align*}
\mathbf{a}^1 \cdot \mathbf{z} &= c_1 \\
\mathbf{a}^2 \cdot \mathbf{z} &= c_2 \\
\vdots & \quad \vdots \\
\mathbf{a}^{n-2} \cdot \mathbf{z} &= c_{n-2}
\end{align*}
\]

$\forall \mathbf{z} \in \mathbb{R}^{n-1}, \mathbf{a}^i \in \mathbb{R}^{n-1}, c_i \in \mathbb{R}, i = 1, \ldots, n - 2.$

(92)

The vectors $\mathbf{a}^i = (a_{i1}, \ldots, a_{in-1})$ and scalars $c_i$ are defined as follows:

- If the $h$-th hyperplane in $I'$ is of type 1, say $\hat{s}_h = \{ \mathbf{z} \in \mathbb{R}^{n-1} : z_m = 0 \text{ for some } m \}$, then

\[
\begin{align*}
c_h &= 0 \\
a_{h,k} &= 0 \quad \forall k \neq m \\
a_{h,m} &= 1.
\end{align*}
\]

(93)

- If the $h$-th hyperplane in $I'$ is of type 2, say $\hat{s}_h = \{ \mathbf{z} \in \mathbb{R}^{n-1} : z_m = l_m \text{ for some } m \}$, then

\[
\begin{align*}
c_h &= l_m \\
a_{h,k} &= 0 \quad \forall k \neq m \\
a_{h,m} &= 1.
\end{align*}
\]

(94)

- If $h$-th hyperplane in $I'$ is of type 3, say $\hat{s}_h = \{ \mathbf{z} \in \mathbb{R}^{n-1} : A_{ri} - w_i z_i = A_{lj} - w_j z_j \text{ for some } i, r, l, j \}$, (see (87) ) , then

\[
\begin{align*}
c_h &= A_{li} - A_{ri} \\
a_{h,k} &= 0 \quad k \neq i, k \neq j, \\
a_{h,i} &= -w_i, \quad a_{h,j} = w_j.
\end{align*}
\]

(95)

Denote by $\mathbf{\mu}, \mathbf{\mu} \in \mathbb{R}^{n-1}$, the homogeneous solution of system (92), assuming $\sum_{i=1}^{n-1} \mu_i = 1$.

The relation between any two solutions $\mathbf{x}^1, \mathbf{x}^2$, can be written in vector form:

\[
\mathbf{x}^2 = \mathbf{x}^1 + \mathbf{\mu}(t_2 - t_1),
\]

(96)

or per component

\[
x^2_i = x^1_i + \mu_i(t_2 - t_1), \quad i = 2, \ldots, n.
\]

(97)

Since , from Lemma 10.3.1, $\mu_i \geq 0$, $\mathbf{x}^2 \geq \mathbf{x}^1$ if (WLOG) $t_2 > t_1$. 

51
At this point, we proved the nestedness property for every two optimal solutions which are close enough. To complete the proof for every two optimal solutions $\mathbf{x}_L(t_1), \mathbf{x}_L(t_2), 0 \leq t_1 < t_2 \leq 1$, we apply the Monotonicity Lemma 10.2.1 on each component of $\mathbf{x}_L(t_1), \mathbf{x}_L(t_2)$. 

**Example 7.4.1** Consider the rooted tactical COM problem on the tree defined in Example 6.3.1. Consider three values of the extended facility length $t$: $t_1 = 0.7$, $t_2 = 0.75$, $t_3 = 0.8$. Below are the corresponding optimal solutions written in the form of the edge vector $\mathbf{x}_L(t) = (x_2, x_3, x_4, x_5)$. The graphical representation of the optimal subtrees is given in Fig. 16.

$$
\begin{align*}
\mathbf{x}_L(t_1) &= (0.25, 0.2, 0, 0.25) \\
\mathbf{x}_L(t_2) &= (0.25, 0.25, 0, 0.25) \\
\mathbf{x}_L(t_3) &= (0.25, 0.25, 0.05, 0.25).
\end{align*}
$$

Using the results of Example 7.3.1 we can assign to each optimal solution the hyperplanes from $\tilde{I}$ that pass through it:

$$
\begin{align*}
\mathbf{x}_L(t_1) \in s_3 \cap s_5 \cap s_8 &\Rightarrow I_1^* = \{s_3, s_5, s_8\} \\
\mathbf{x}_L(t_2) \in s_3 \cap s_5 \cap s_6 \cap s_8 \cap s_{13} &\Rightarrow I_2^* = \{s_3, s_5, s_6, s_8, s_{13}\} \\
\mathbf{x}_L(t_3) \in s_5 \cap s_6 \cap s_8 \cap s_{13} &\Rightarrow I_3^* = \{s_5, s_6, s_8, s_{13}\}.
\end{align*}
$$

Since $\mathbf{x}_L(t_1), \mathbf{x}_L(t_2)$ both lie within the hyperplanes $s_3, s_5, s_8$, we conclude that the unique line connecting those two points is defined as the intersection $s_3 \cap s_5 \cap s_8$. As a function of $t$ these lines are represented as:

$$
\begin{align*}
x_2(t) &= l_2 \\
x_3(t) &= t - l_2 - l_5 \\
x_4(t) &= 0 \\
x_5(t) &= l_5 \\
t &\in [0.7, 0.75].
\end{align*}
$$

It can be seen that all edge variables are non-decreasing with $t$, which gives the nestedness property: $\tilde{T}_{x_L(t_1)} \subset \tilde{T}_{x_L(t_2)}$.

Now, consider $\mathbf{x}_L(t_2), \mathbf{x}_L(t_3)$, both lie within the hyperplanes $s_5, s_6, s_8, s_{13}$. Note that since $w_5 = 0$, the hyperplanes $s_6$ and $s_{13}$ are identical, see Example 52.
7.3.1. The unique line connecting those two points is defined as the intersection $s_5 \cap s_6 \cap s_8$. We write down this line as a function of $t$:

\[
\begin{align*}
x_2(t) &= l_2 \\
x_3(t) &= l_3 \\
x_4(t) &= t - l_2 - l_3 - l_5 \\
x_5(t) &= l_5 \\
\end{align*}
\tag{101}
\]

All edge variables are non-decreasing with $t$, which gives the nestedness property:

\[\tilde{T}_{x^*_L(t_2)} \subset \tilde{T}_{x^*_L(t_3)}\]

**Example 7.4.2** Consider the rooted tactical COM problem on the tree defined in Example 6.3.1.

Fig. 17 presents the optimal objective value and edge vectors of optimal solutions for $t \in [0,1]$. Those are obtained by solving the LP problem (70) and finding the lexicographic minimum using the procedure described in Lemma 7.2.1. It can be seen that all edge variables are non-decreasing with $t$ and the nestedness property holds.

Fig. 18 shows for each $t$ the hyperplanes that pass through the optimal solution $x^*_L(t)$.

For $t = [0,0.5]$ the optimal solutions $x^*_L(t)$ are lying on the line defined by $s_2 \cap s_3 \cap s_{12}$.

For $t = [0.5,0.75]$ the optimal solutions $x^*_L(t)$ are lying on the line defined by $s_3 \cap s_5 \cap s_8$. 

Figure 17: Example 6.3.1 - Optimal Solution for $t \in [0,1]$
For $t = [0.75, 1]$ the optimal solutions $x^*_L (t)$ are lying on the line defined by $s_5 \cap s_6 \cap s_8, (s_{13} \text{ is identical to } s_6)$.

Figure 18: Example 6.3.1 - Assigning $s_k$-hyperplanes to optimal solutions
8 Nestedness of the General (Unrooted) Tree Problem

This section is focused on the proof of Theorem 6.3.1. The proof is based on the nestedness property of rooted trees. In order to proceed we need the following theorem:

**Theorem 8.0.1** Let $T = (V, E)$ be an undirected tree with positive edge lengths $\{l_j\}$, and $\sum_{j: e_j \in E} l_j = 1$. Denote by $T_z$ a subtree of $T$ (closed connected subset of $A(T)$) with total edge length $z$. Consider a set $\{T_z\}$, $0 < z \leq 1$, where for each $z$ there is a unique subtree $T_z$, and every two subtrees of $\{T_z\}$ satisfy the following:

$$T_a \cap T_b \neq \emptyset \rightarrow T_a \subset T_b \quad \forall a, b \in (0, 1], \ a < b.$$  

(102)

Then for every two members of $\{T_z\}$ we have,

$$\forall a, b \in [0, 1], \ a < b \quad T_a \subset T_b \quad \forall T_a, T_b \in \{T_z\}.$$  

(103)

**Proof** We first prove the following proposition

**Proposition 8.0.2** Given an integer $k \geq 2$, for all $p = 1, 2, 3, 4, \ldots, 2^k - 3$, the following property holds:

$$T_a \subseteq T_{2^k - p}, \quad \forall a, \frac{1}{2^k} \leq a \leq \frac{2^k - p}{2^k}. \quad (104)$$

**Proof:** By induction on $p$. For $p = 1$, since $\frac{1}{2^k} + \frac{2^k - 1}{2^k} = 1$, then $T_a$ and $T_{2^k - p}$ have a non-empty intersection (sum of lengths of edges for those two subtrees is greater than or equal 1). Therefore, by (102), it follows that (104) holds. Assuming (104) for $p$, we will show it holds for $p + 1$. Consider $a' = \frac{2^k - (p + 1)}{2^{k + 1}}$. Since $\frac{1}{2^k} \leq a' \leq \frac{2^k - p}{2^{k + 1}}$:

$$T_{a'} \subseteq T_{2^k - p}. \quad (105)$$

Now, choose $\frac{1}{2^k} \leq a \leq \frac{2^k - p}{2^{k + 1}}$, which by induction gives us

$$T_a \subseteq T_{2^k - p}. \quad (106)$$

The sum of the lengths of the edges of $T_a$ and $T_{a'}$ is not less than $\frac{2^k - p}{2^{k + 1}}$. Because of (105) and (106), $T_a$ and $T_{a'}$ are in $T_{2^k - p}$, and, therefore, have a point in common. According to (102):

$$T_a \subseteq T_{a'} = T_{2^k - (p - 1)}, \quad (107)$$

which proves (104) for $p + 1$. This concludes the proof of the Proposition.
**Proof of Theorem 8.0.1** Suppose the following induction hypothesis:
For all natural \( k \), let \( T_a, T_b \) be any two subtrees of \( \{ T_z \} \), each with the sum of edge lengths greater than \( \frac{3}{2^k} \), then \( T_a, T_b \) have a point in common

\[
\forall k \in \mathbb{N}, k \geq 2, \ T_a \cap T_b \neq \emptyset \ \forall a, b \in \left[ \frac{3}{2^k}, 1 \right].
\]  

(108)

The result clearly holds for \( k = 2 \), since the sum of the lengths of \( T_a \) and \( T_b \) is more than 1. Assume the validity for \( k \) and prove it for \( k + 1 \). We consider three subcases:

I: \( 3/2^k \leq a < b \).

Follows from the induction hypothesis.

II: \( 3/2^{k+1} \leq a < 3/2^k \leq b \).

From Subcase I, \( T_{3/2^k} \subseteq T_b \). Therefore, for this Subcase it is sufficient to consider only instances where \( b = 3/2^k \). These instances are included in Subcase III.

III: \( 3/2^{k+1} \leq a < b \leq 3/2^k \)

From the above, since \( 1/2^k < 3/2^{k+1} \leq a < b \leq 3/2^k \), we apply Proposition 8.0.2 with \( p = 2^k - 3 \) and have \( T_a \subseteq T_{3/2^k} \) and \( T_b \subseteq T_{3/2^k} \). The sum of the lengths of \( T_a \) and \( T_b \) is greater than or equal to the length of \( T_{3/2^k} \). Therefore, \( T_a \) and \( T_b \) have a nonempty intersection, which implies that \( T_a \subseteq T_b \).

**Proof of Theorem 6.3.1** Consider a mapping \( T_t : [0, 1] \rightarrow A(T) \), which assigns to each \( t \) a subtree in \( A(T) \). \( T_t \) is an optimal solution for the tactical subtree problem (66) with total edge lengths \( t \), in the case of multiple optimal solutions, we choose the subtree which is the lexicographic minimum of distances to the leaves of \( T \). Our goal is to show that the conditions of Theorem 8.0.1 hold.

Assuming that two subtrees \( T_{i_1} \) and \( T_{i_2} \), \( t_1 < t_2 \), have a point in common, say \( y \), and assume \( y \in e_{ij} \). Then we augment the set \( V \) by a node located at \( y \), relabeling the nodes and calling the node at \( y \) \( v_1 \), and giving it a weight \( w_1 = 0 \). We refer to the node \( v_1 \) as the root of both subtrees. We obtain a new tree \( T^* = (V^*, E^*) \), where \( V^* = V + v_1 \) and \( E^* = E - e_{ij} + e_{i1} + e_{j1} \). This augmentation doesn’t change the optimal solutions of the problem, since the added node has zero weight. If \( y \) is already a node of \( T \), we call this node \( v_1 \) and skip the augmentation.

Next, we formulate a rooted subtree problem (67) on the tree \( T^* \). Since both \( T_{i_1} \) and \( T_{i_2} \) are optimal solutions to the tactical subtree problem, and \( v_1 \in T_{i_1} \), \( v_1 \in T_{i_2} \), \( T_{i_1}, T_{i_2} \) are optimal solutions to the rooted (with root \( v_1 \)) subtree problem (67) with parameters \( t = t_1 \) and \( t = t_2 \) correspondingly.

And since the lexicographic minimum of distances to leaves over all optimal solutions obtained at \( T_{i_1} \), \( T_{i_2} \), they can be represented as \( \pi_L(t_1), \pi_L(t_2) \) in terms of Theorem 6.4.3. Then \( \pi_L(t_1) \preceq \pi_L(t_2) \), or \( A(T_{i_1}) \subseteq A(T_{i_2}) \).

Applying Theorem 8.0.1 we obtain the required nestedness property:

\[
A(\tilde{T}_{t_1}) \subseteq A(\tilde{T}_{t_2}) \ \forall t_1 \leq t_2.
\]
Theorem 8.0.3 Let $T = (V, E)$ be a tree with edge lengths $\{l_j\}$, and $\sum_{j: e_j \in E} l_j = 1$. Denote by $T_t$ the optimal solution to the tactical COM location problem with extensive facility length $t$. Denote by $A(T_t)$ the mapping of $T_t$ into $\mathbb{R}^2$. Let $v_0 = \bigcap_{0 < t \leq 1} A(T_t)$, then $v_0$ is an optimal solution to the point tactical COM location problem.

Proof Assume WLOG that $v_0 \in V$ and $w_0 = 0$. Formulate the rooted tactical COM location problem taking $v_0$ as a root. Denote by $\hat{T}_t$ the solution of the (67) (rooted tactical COM location problem with extensive facility length $t$). Note, that since $v_0 \in A(T_t)$ for all $0 < t \leq 1$, $\hat{T}_t$ is also the solution of the unrooted problem, i.e., $\hat{T}_t = T_t$. Let the function $g^*(t) : (0, 1] \to \mathbb{R}$ be defined as the objective function value of $\hat{T}_t$.

According to the LP formulation of the tactical rooted COM problem (70) and from Lemma 10.1.1, $g^*(t)$ is continuous, monotone, piecewise linear, non-increasing function of $t$. Define $g^*(0) = \lim_{t \to 0^+} g^*(t)$, the existence of the limit follows from piecewise linearity of $g^*(t)$. Because the continuity of the solution of the parametrized LP problem, $g^*(0)$ is the optimal objective value of the point tactical rooted COM location problem. We will prove that $g^*(0)$ is the optimal objective value for point tactical (unrooted) COM location problem.

Suppose by contradiction that $v_0$ is not the optimal solution of the tactical (unrooted) COM location problem. Denote by $\hat{v}_0$ the optimal solution to the point tactical (unrooted) COM location problem nearest to $v_0$. Denote by $F(v_0)$ and $F(\hat{v}_0)$ the optimal objective values of $v_0$ and $\hat{v}_0$ respectively. Since $v_0$ is not an optimal solution, the following holds:

$$F(v_0) - F(\hat{v}_0) = \Delta > 0$$
$$L(P(v_0, \hat{v}_0)) = \delta > 0$$

(109)

Since $F(v_0) = g^*(0)$, and $g^*(t)$ is continuous, there exists $\alpha > 0$ such that $g^*(\alpha) > g^*(0) - \Delta$. So, there exists $\hat{T}_\alpha = T_\alpha$ with optimal objective value $F(T_\alpha) = g^*(\alpha) > g^*(0) - \Delta$. By taking $\alpha$ small enough we assure that $v_0 \notin T_\alpha$.

Denote by $\hat{T}_\alpha$ an arbitrary subtree with total length $\alpha$ rooted at $\hat{v}_0$. Denote by $F(\hat{T}_\alpha)$ the objective value of $\hat{T}_\alpha$ in tactical (unrooted) COM location problem. It is clear that $F(\hat{T}_\alpha) < F(\hat{v}_0)$, therefore

$$F(\hat{T}_\alpha) < F(T_\alpha).$$

(110)

Equation (110) contradicts the optimality of $T_\alpha$.

This finishes the proof of the main nestedness Theorem 6.3.1.  

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9 References


10 Appendices

10.1 Continuity of the optimal solution objective value of a parametrized LP problem

Lemma 10.1.1 Given a parametrized LP problem:

Minimize $CT\bar{z}$
subject to:
$AT\bar{z} \leq D,$
$E^T\bar{z} = G,$
$M^T\bar{z} \leq R(t),$  
$\bar{z} \geq 0,$
$\bar{z} \in \mathbb{R}^n,$  $C \in \mathbb{R}^n,$  $D \in \mathbb{R}^d,$  $A \in \mathbb{R}^{n \times d},$  $G \in \mathbb{R}^g,$  $E \in \mathbb{R}^{n \times g},$  $M \in \mathbb{R}^{n \times m},$
$R(t) = (r_1(t), r_2(t), \ldots, r_m(t)), \ r_i(t) : T \rightarrow \mathbb{R}, \ i = 1, \ldots, m$
where $T$ is a closed connected interval on $\mathbb{R}$.

Denote by $H(t)$ the optimal objective value of (111). If (111) is feasible and bounded, and all functions $r_1(t), \ldots, r_m(t)$ are continuous on $T$, then $H(t)$ is a continuous function on $T$. Moreover, if $r_1(t), \ldots, r_m(t)$ are concave, then $H(t)$ is convex. Also if the functions $r_1(t), \ldots, r_m(t)$ are linear, then $H(t)$ is piecewise linear.

Proof The Dual Problem of (111) is:

Maximize $\left[D^T\bar{u} + G^T\bar{v} + \sum_{i=1}^{m} r_i(t)w_i\right]$  
subject to:
$[A \ E \ M] \begin{bmatrix} \bar{u} \\ \bar{v} \\ w \end{bmatrix} \leq C,$
$\bar{u} \leq 0,$
$\bar{v} \leq 0,$
$w \leq 0,$
$w \in \mathbb{R}^m, \bar{u} \in \mathbb{R}^d, \bar{v} \in \mathbb{R}^g.$

Denote by $J(t)$ the optimal value of the objective function in (112). Because of feasibility and boundedness of (111) strong duality holds, and:

$J(t) = H(t), \ \forall t \in T.$  (113)

We replace the unconstrained vector $\bar{v}$ by two non-negative vectors $\bar{v}^1$ and $\bar{v}^2$, such that:

$\bar{v} = \bar{v}^1 - \bar{v}^2, \ \bar{v}^1 \geq 0, \ \bar{v}^2 \geq 0.$  (114)
The dual problem becomes:

Maximize \[ DTu + GTv^1 - GTv^2 + \sum_{i=1}^{m} r_i(t)w_i \]
subject to:

\[
\begin{bmatrix}
A & E & -E & M
\end{bmatrix}
\begin{bmatrix}
u \\
v^1 \\
v^2 \\
w
\end{bmatrix}
\leq C,
\]

\( u \leq 0, \)
\( w \leq 0, \)
\( v^1 \geq 0, \)
\( v^2 \geq 0, \)
\( w \in \mathbb{R}^m, u \in \mathbb{R}^d, v^1 \in \mathbb{R}^g, v^2 \in \mathbb{R}^g. \)

For each \( t \), there is an optimal solution of \( u, v^1, v^2, w \) that represents an extreme point of the polyhedral set:

\[
\begin{bmatrix}
A & E & -E & M
\end{bmatrix}
\begin{bmatrix}
u \\
v^1 \\
v^2 \\
w
\end{bmatrix}
\leq C,
\]

\( u \leq 0, \)
\( w \leq 0, \)
\( v^1 \geq 0, \)
\( v^2 \geq 0, \)
\( w \in \mathbb{R}^m, u \in \mathbb{R}^d, v^1 \in \mathbb{R}^g, v^2 \in \mathbb{R}^g. \)

Denote by \( \{ u(k), v^1(k), v^2(k), w(k) \}_{k=1}^{N'} \) the set of all extreme points of \( (115) \), where \( N' \) is the number of the extreme points. Note that the polyhedral set \( (115) \) has an extreme point, since no line can be included in this polyhedral set, \( (115) \), (all variables have sign constraints).

\( J(t) \) can be formulated as:

\[
J(t) = \max_k \{DTu(k) + GTv^1(k) - GTv^2(k) + \sum_{i=1}^{m} r_i(t)w_i(k) \}. \quad (116)
\]

Because of continuity of \( r_1(t), \ldots, r_m(t) \), \( J(t) \) is the maximum of a finite set of continuous functions, and therefore continuous itself.

If \( r_1(t), \ldots, r_m(t) \) are linear functions of \( t \), then \( J(t) \) is piecewise linear.

If all \( r_1(t), \ldots, r_m(t) \) are concave, \( \sum_{i=1}^{m} r_i(t)w_i(k) \) is the sum of convex functions (note that \( w_i(k) \leq 0 \) from \( (D) \)). Then \( J(t) \) is the maximum of convex functions, therefore, \( J(t) \) and \( H(t) \) are convex.
10.2 Monotonicity lemma

Lemma 10.2.1 Let \( f(x) : \mathbb{R} \to \mathbb{R} \) be a real valued function defined on a closed interval \( \hat{D} \subseteq \mathbb{R} \), and suppose that the following property holds:
\[
\forall x \exists \epsilon > 0 \text{ s.t. } f(x^-) \leq f(x) \leq f(x^+); \quad \forall x^- \in [x-\epsilon, x) \cap \hat{D}; \quad \forall x^+ \in (x, x+\epsilon] \cap \hat{D}
\]

then \( f(x) \) is a non-decreasing function of \( x \).

Proof Suppose by contradiction, that there are two points where the monotonicity property does not hold:
\[
\exists x_1, x_2 \in \hat{D}; \quad x_1 < x_2, \quad f(x_1) > f(x_2) \quad (118)
\]

We define the three recursive sequences \( \{u_k\}, \{l_k\}, \{y_k\} \) in the following way:
\[
u_1 = x_1, \quad u_n = \begin{cases} u_{n-1} & \text{if } f(y_{n-1}) > f(u_{n-1}) \\ y_{n-1} & \text{if } f(y_{n-1}) \leq f(u_{n-1}) \end{cases}, \quad (119)
\]
\[
l_1 = x_2, \quad l_n = \begin{cases} y_{n-1} & \text{if } f(y_{n-1}) > f(u_{n-1}) \\ l_{n-1} & \text{if } f(y_{n-1}) \leq f(u_{n-1}) \end{cases}, \quad (120)
\]
\[
y_1 = \frac{u_1 + l_1}{2}, \quad y_n = \frac{u_n + l_n}{2}. // (121)
\]

For every natural \( n \) we have the following situation:
\[
u_n < l_n; \quad f(u_n) > f(l_n); \quad l_n - u_n = \frac{x_2 - x_1}{2^n}.
\]

So
\[
\forall \epsilon > 0, \quad \exists N \quad \text{such that:}
\]
\[
|u_n - l_n| < \epsilon, \quad \forall n > N \quad \text{but } f(u_n) > f(l_n),
\]

which is a contradiction to (117).

10.3 Homogeneous solution of a system of linear equations

Lemma 10.3.1 Given the homogeneous linear system of equations:
\[
A\bar{z} = 0 \quad (122)
\]
\[
\bar{z} \in \mathbb{R}^n, \quad A \in \mathbb{R}^{(n-1)-n},
\]
\[
\text{rank}(A) = n - 1
\]

Denote by \( a^i = (a_{i,1}, a_{i,2}, \ldots, a_{i,n}) \) the \( i \)-th row of \( A \). Suppose all rows of \( A \) have the following structure:

- **either**
  \[
a_{i,j} = \begin{cases} 1 & j = k_i \\ 0 & j \neq k_i \end{cases}, \quad k_i \in \{1, 2, \ldots, n\}
\]

- **or**
  \[
a_{i,j} = \begin{cases} w_a & j = m_i \\ -w_b & j = n_i \\ 0 & j \neq m_i, j \neq n_i \end{cases}, \quad m_i, n_i \in \{1, 2, \ldots, n\}, \quad m_i \neq n_i, \quad w_a > 0, \quad w_b > 0
\]

(123)
Denote by $z^*$ a solution of the system (122) - (123), then all non-zero components of $z^*$ are of the same sign.

**Proof** Assume, WLOG, that there exists some index (column) $j_1$ such that $z^*_{j_1} > 0$. There are two possible situations:

- **Case $\hat{a}$**: Column $j_1$ in $A$ is the column of zeros.

- **Case $\hat{b}$**: There exists some row $i_1$ such that $a_{i_1,j_1} \neq 0$.

**Case $\hat{a}$**:
In this case the matrix $A$ in (122) and the linear equation in $A\vec{z} = 0$ can be decomposed (after some row and column permutations) in the following way:

\[
A\vec{z} = 0 \rightarrow [0 \ A_0] \begin{bmatrix} \frac{z_{j_1}}{\vec{z}_0} \end{bmatrix} = 0 \rightarrow A_0\vec{z}^0 = 0 \tag{124}
\]

Matrix $A_0$ has $n - 1$ columns and $n - 1$ rows, the rows of $A$ are linearly independent, $\text{rank}(A_0) = n - 1$, and $A_0$ is full rank. Denote by $\vec{z}^0$ a solution of $A_0\vec{z}^0 = 0$, since $A_2$ is full rank, $\vec{z}^{0*} = 0$.

$\vec{z}^*$, the homogeneous solution of (122), consists of the component $z^*_{j_1}$ augmented to the vector $\vec{z}^{0*}$.

**Case $\hat{b}$**:
Consider the row $i_1$, which has the non-zero entry $a_{i_1,j_1} \neq 0$. If $a_{i_1,j_1}$ is the only non-zero entry of row $i_1$, then $z^*_{j_1}$ would be zero in any homogeneous solution. Therefore, row $i_1$ must contain another non-zero entry, say $a_{i_1,j_2} \neq 0$, $j_2 \neq j_1$. Thus, from the construction of the matrix $A$, the row $i_1$ contains only two non-zero entries and they are of opposite signs. Therefore, there exists $j_2 \neq j_1$, such that $z^*_{j_2} > 0$.

Starting from one positive component in a homogeneous solution $\vec{z}^*$, we conclude that either all other components of $\vec{z}^*$ are zeros, or there is another non-zero component in $\vec{z}^*$.

We continue the proof by induction on the number of positive components in $\vec{z}^*$. Denote by $k$ the number of positive components in $\vec{z}^*$.

Consider the following induction hypothesis:

There are $k$ positive components in $\vec{z}^*$, at indices $\{j_1, \ldots, j_k\}$. There are $k - 1$ rows of $A$, at indices $\{i_1, \ldots, i_{k-1}\}$, each one of them has zeros at indices outside $\{j_1, \ldots, j_k\}$:

\[
\exists j_1, \ldots, j_k \in \{1, \ldots, n\} \quad \text{s.t.} \quad z^*_j > 0, \quad \forall j \in \{j_1, \ldots, j_k\}
\]

\[
\exists i_1, \ldots, i_{k-1} \in \{1, \ldots, n-1\} \quad \text{s.t.} \quad a_{i_s,j_t} = 0, \quad \forall i_s \in \{i_1, \ldots, i_{k-1}\}, \quad \forall j_t \notin \{j_1, \ldots, j_k\} . \tag{125}
\]

Then one of the following holds:
1) \( k = n. \)

2) All other \( n - k \) components of \( z^* \) are zeros:

\[
z^*_{js} = 0, \quad \forall js \notin \{j_1, \ldots, j_k\} \quad (126)
\]

3) There exists an additional index, \( k + 1 \), which is associated with a positive component of \( z^* \), say \( j_{k+1} \notin \{j_1, \ldots, j_k\} \), and there is an additional row of \( A \), say \( k \), which has zeros on indices outside \( \{j_1, \ldots, j_{k+1}\} \):

\[
\exists j_1, \ldots, j_{k+1} \in \{1, \ldots, n\} \quad s.t. \quad z^*_j > 0, \quad \forall j \in \{j_1, \ldots, j_{k+1}\}
\]

\[
\exists i_1, \ldots, i_k \in \{1, \ldots, n - 1\} \quad s.t. \quad a_{i_s,j_t} = 0, \quad \forall i_s \in \{i_1, \ldots, i_k\}, \quad \forall j_t \notin \{j_1, \ldots, j_{k+1}\}. \quad (127)
\]

End of the induction hypothesis.

For \( k = 2 \)

Consider the columns \( j_1, j_2 \) of \( A \). There are two possibilities:

a) All entries in columns which are not in row \( i_1 \) are zeros:

\[
a_{i_s,j_t} = 0 \quad \forall j_t \in \{j_1, j_2\}, \quad \forall i_s \neq i_1. \quad (128)
\]

b) One of those columns has a non-zero entry outside row \( i_1 \). Denote this column by \( j' \) and the corresponding row by \( i_2 \):

\[
\exists j' \in \{j_1, j_2\}, \exists i_2 \neq i_1, \text{ such that: } a_{i_2,j'} \neq 0. \quad (129)
\]

Case a):

In this case the matrix \( A \) in (122) and the linear equations \( A\bar{z} = 0 \) can be decomposed (after some row and column permutations) in the following way:

\[
A\bar{z} = 0 \rightarrow \begin{bmatrix} a_{j_1} & a_{j_2} & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} z_{j_1} \\ z_{j_2} \\ \bar{z}^2 \end{bmatrix} = 0 \rightarrow a_{j_1}z_{j_1} + a_{j_2}z_{j_2} = 0, \quad A_2\bar{z}^2 = 0 \quad (130)
\]

\( A_2 \in \mathbb{R}^{(n-2)\times(n-2)} \), and \( \bar{z}^2 \in \mathbb{R}^{n-2} \).

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The matrix $A_2$ has $n-2$ columns and $n-2$ rows, and the rows of $A$ are linearly independent, $\text{rank}(A_2) = n-2$ and $A_2$ is full rank. Denote by $\tilde{\pi}^{2*}$ a solution of $A_2\tilde{\pi}^{2*} = 0$, since $A_2$ is full rank, $\tilde{\pi}^{2*} = 0$. $\tilde{\pi}^*$, the homogeneous solution of (122), consists of the components $z_{j_1}^*, z_{j_2}^*$ augmented to the vector $\pi^{2*}$. Thus, we obtained Case 2) of the induction hypothesis.

Case b): Consider row $i_2$. It has a non zero component at column $j'$. One of the following holds:

b.1) All components of the $i_2$ row, (except $a_{i_2,j'}$) are zeros.

b.2) The $i_2$ row has one additional non-zero component at column $\tilde{j}$ and $\tilde{j} \in \{j_1, j_2\}$.

b.3) The $i_2$ row has one additional non-zero component at column $\tilde{j}$ and $\tilde{j} \notin \{j_1, j_2\}$.

Case b.1):

In this case the $i_2$ row has only one non-zero component at column $j'$, which means that $z_{j'}^* = 0$. Since $j'$ is one of the indices $j_1, j_2$, it contradicts the induction hypothesis. Therefore Case b.1 is impossible.

Case b.2):

In this case the matrix $A$ in (122) and the linear equations $A\tilde{\pi} = 0$ can be decomposed (after some row and column permutations) in the following way (assuming WLOG $j' = j_1, \tilde{j} = j_2$):

$$
A\tilde{\pi} = 0 \rightarrow \begin{bmatrix} a_{j_1, j} & a_{j_2, j} & 0 \\ a_{j_1, \tilde{j}} & a_{j_2, \tilde{j}} & 0 \\ 0 & 0 & A_2 \end{bmatrix} \begin{bmatrix} z_{j_1} \\ z_{j_2} \\ \tilde{\pi} \end{bmatrix} = 0 \rightarrow \begin{bmatrix} a_{j_1, j_1}z_{j_1} + a_{j_2, j_2}z_{j_2} = 0, \\ a_{j_1, \tilde{j}}z_{j_1} + a_{j_2, \tilde{j}}z_{j_2} = 0, \\ A_2\tilde{\pi}^2 = 0 \end{bmatrix}
$$

Since all rows of $A$ are linearly independent, the solution of the above linear system implies $z_{j_1}^* = z_{j_2}^* = 0$. This contradicts the induction hypothesis, and, therefore, Case b.2 is impossible.

Case b.3):

In this case the row $i_2$ has only two non-zero components: $a_{i_2,j'}$ and $a_{i_2,\tilde{j}}$. Since, from the construction of $A$, the entries $a_{i_2,j'}, a_{i_2,\tilde{j}}$ must have opposite signs and $z_{j'}^* > 0$, therefore, $z_{\tilde{j}}^* > 0$. Denoting $\tilde{j}$ by $j_3$ we obtain Case 3) in the induction hypothesis.

For general $k < n$

Consider columns $j_1, \ldots, j_k$, of $A$. There are two possibilities:

a) All entries in columns $j_1, \ldots, j_k$ which are not in rows $\{i_1, \ldots, i_{k-1}\}$, are zeros:

$$
a_{i_1,j_t} = 0 \ \forall j_t \in \{j_1, \ldots, j_k\}, \forall i_s \notin \{i_1, \ldots, i_{k-1}\}. \quad (132)
$$

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b) One of those columns has a non-zero entry outside rows $i_1, \ldots, i_{k-1}$. Let $j'$ be such a column, and $i_k$ be the corresponding row:

$$\exists j' \in \{j_1, \ldots, j_k\}, i_k \notin \{i_1, \ldots, i_{k-1}\}, \text{ such that: } a_{i_k, j'} \neq 0. \quad (133)$$

Case a):

In this case the matrix $A$ in (122) and the linear equation in $Az = 0$ can be decomposed (after some row and column permutations) in the following way:

$$A\bar{z} = 0 \rightarrow \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \begin{bmatrix} \bar{z}^1 \\ \bar{z}^2 \end{bmatrix} = 0 \rightarrow A_1\bar{z}^1 = 0, \quad A_2\bar{z}^2 = 0 \quad (134)$$

where $A_1 \in \mathbb{R}^{(k-1)\times k}$, $A_2 \in \mathbb{R}^{(n-k)\times (n-k)}$, $\bar{z}^1 \in \mathbb{R}^k$, $\bar{z}^2 \in \mathbb{R}^{n-k}$.

Where $A_1$ consists of entries of $A$ in rows $i_1, \ldots, i_{k-1}$, and columns $j_1, \ldots, j_k$, and $\bar{z}^1$ consists of $j_1, \ldots, j_k$ components of $\bar{z}$. The variable $\bar{z}^2$ corresponds to the matrix $A_2$. Consider the rank of $A$, (recall that all rows of $A$ are independent):

$$\text{rank}(A) = \text{rank}(A_1) + \text{rank}(A_2) \rightarrow \text{rank}(A_2) = \text{rank}(A) - \text{rank}(A_1). \quad (135)$$

Matrix $A_1$ has $k$ columns and $k - 1$ rows, hence $\text{rank}(A_1) = k - 1$. Therefore, $\text{rank}(A_2) = n - k$ and $A_2$ is full rank. Denote by $\bar{z}^{2\ast}$ a solution of $A_2\bar{z}^2 = 0$, since $A_2$ is full rank, $\bar{z}^{2\ast} = 0$. $\bar{z}^\ast$, the homogeneous solution of (122), consists of the components $z_{j_1}^{\ast}, \ldots, z_{j_k}^{\ast}$ assembled with vector $\bar{z}^{2\ast}$. Thus, we obtained Case 2) of the induction hypothesis.

Case b):

Consider the row $i_k$. It has a non zero component at column $j'$. One of the following is true:

b.1) All components of the $i_k$ row, except $a_{i_k, j'}$ are zeros.

b.2) The $i_k$ row has one additional non zero component at column $\tilde{j}$ and $\tilde{j} \in \{j_1, \ldots, j_k\}$.

b.3) The $i_k$ row has one additional non zero component at column $\tilde{j}$ and $\tilde{j} \notin \{j_1, \ldots, j_k\}$.

Case b.1):

In this case the row $i_k$ has only one non-zero component at column $j'$, which means that $z_{j'}^{\ast} = 0$. Since $j'$ is one of the indices $j_1, \ldots, j_k$, it contradicts the induction hypothesis. Therefore case b.1 is impossible.
Case b.2):

In this case the matrix $A$ in (122) and the linear equation $A\mathbf{z} = 0$ can be decomposed (after some row and column permutations) in the following way:

\[
A\mathbf{z} = 0 \rightarrow \begin{bmatrix} A_3 & 0 \\ 0 & A_4 \end{bmatrix} \begin{bmatrix} \mathbf{z}_3 \\ \mathbf{z}_4 \end{bmatrix} = 0 \rightarrow A_3\mathbf{z}_3 = 0, \\
A_4\mathbf{z}_4 = 0
\]

$A_3 \in \mathbb{R}^{k \times k}$, $A_4 \in \mathbb{R}^{(n-k-1) \times (n-k)}$, $\mathbf{z}_3 \in \mathbb{R}^k$, $\mathbf{z}_4 \in \mathbb{R}^{n-k}$.

Where $A_3$ consists of entries of $A$ in rows $i_1, \ldots, i_k$ and columns $j_1, \ldots, j_k$ and $\mathbf{z}_3$ consists of $j_1, \ldots, j_k$ components of $\mathbf{z}$. The variable $\mathbf{z}_3$ corresponds to matrix $A_3$. Since all rows of $A$ are linearly independent and $A_3$ is a square matrix, $A_3$ of full rank. Therefore, $\mathbf{z}_3^T$, the solution of $A_3\mathbf{z}_3 = 0$, equals zero. It contradicts the induction hypothesis, because $\mathbf{z}_3^T$ is a permutation of $z_{j_1}^*, \ldots, z_{j_k}^*$. So the case b.2 is impossible.

Case b.3):

In this case the row $i_k$ has only two non-zero components: $a_{i_k,j}$ and $a_{i_k,j}$. Since, from the construction of $A$, the entries $a_{i_k,j}, a_{i_k,j}$ must have opposite signs and $z_{j}^* > 0$, therefore, $z_{j}^* > 0$. Denoting $j$ by $j_{k+1}$ we obtain the Case 3 of the induction hypothesis.

10.4 Example of the tactical COM (convex ordered median) location problem on the tree

Example 10.4.1 Consider the tree $T = (V, E)$ with $V = \{v_1, v_2, v_3, v_4, v_5\}$ and $E = \{(v_1, v_3), (v_1, v_4), (v_1, v_5), (v_2, v_3)\}$, and the length of each edge is 0.25. To each node the following weights are assigned: $w_1 = 100$, $w_2 = 3$, $w_3 = w_4 = 1$, $w_5 = 0$. We want to find the optimal subtree $T^*$ of length $t = 0.7$, that minimizes the COM function of weighted distances $\mathcal{F}(w_1d(T^*, v_1), \ldots, w_5d(T^*, v_5))$. The coefficients of the COM function are $\lambda_1 = \lambda_2 = 3$, $\lambda_3 = 1$, $\lambda_4 = \lambda_5 = 0$ (see (4)).

Solution:

Since $t > 0.5$ the optimal subtree $T^*$ has to contain the node $v_1$. Therefore the weighted distances $w_1\delta_1$ and $w_5\delta_5$ are zeros and the objective value depends only on $w_2\delta_2, w_3\delta_3, w_4\delta_4$ - see Fig. 13.

Since $\delta_2 + \delta_3 + \delta_4 = 1 - t = 0.3$ we will analyze the objective as a function of $\delta_3, \delta_4$ with $\delta_2 = 0.3 - \delta_3 - \delta_4$. The feasible values of $\delta_3, \delta_4$ are bounded by the following constraints:

\[
\begin{align*}
0 \leq \delta_3 \leq 0.25, \\
0 \leq \delta_4 \leq 0.25, \\
\delta_3 + \delta_4 \leq 0.3.
\end{align*}
\]

First, assume the weighted distance $w_3\delta_3$ has the smallest value among $\{w_2\delta_2, w_3\delta_3, w_4\delta_4\}$. In this case the objective function is:

\[
\mathcal{F}(w_2\delta_2, w_3\delta_3, w_4\delta_4) = \lambda_1 w_2\delta_2 + \lambda_2 w_4\delta_4 + \lambda_3 w_3\delta_3 = 9\delta_2 + 3\delta_4 + \delta_3 = 9(0.3 - \delta_3 - \delta_4) + 3\delta_4 + \delta_3 = 8(\frac{2.7}{8} - \frac{3}{4}\delta_4 - \delta_3)
\]

(138)
This happens if \( w_3\delta_3 \leq w_4\delta_4 \) and \( w_3\delta_3 \leq w_2\delta_2 \):

\[
\begin{align*}
  w_3\delta_3 & \leq w_2\delta_2 \\
  \delta_3 & \leq w_2(0.3 - \delta_3 - \delta_4) \\
  4\delta_3 + 3\delta_4 & \leq 0.9 \\
  \delta_3 + \frac{3}{4}\delta_4 & \leq 0.225
\end{align*}
\]  
(139)

The minimal value of \( \mathcal{F}(w_2\delta_2, w_3\delta_3, w_4\delta_4) \) obtained where \( \frac{3}{4}\delta_4 + \delta_3 \) reaches its upper limit, which is on the line \( \delta_3 + \frac{3}{4}\delta_4 = 0.225 \). Denote the optimal objective value in the region where the distance \( w_3\delta_3 \) is the smallest by \( \mathcal{F}^{3}_{\text{opt}} \). Its value is \( \mathcal{F}^{3}_{\text{opt}} = 0.9 \).

Assuming the node \( v_3 \) has the smallest weighted distance, the optimal objective value \( \mathcal{F}^{3}_{\text{opt}} = 0.9 \) is attained at \( \frac{3}{4}\delta_4 + \delta_3 = 0.225 \), \( \frac{0.9}{4} \leq \delta_4 \leq 0.25 \). The last inequality follows from the constraints \( w_3\delta_3 \leq w_4\delta_4 \) and \( \delta_4 \leq 0.25 \).

From symmetry, assuming the node \( v_4 \) has the smallest weighted distance, the optimal objective value \( \mathcal{F}^{4}_{\text{opt}} = 0.9 \) is attained at \( \frac{3}{4}\delta_3 + \delta_4 = 0.225 \), \( \frac{0.9}{4} \leq \delta_3 \leq 0.25 \).

Now, assume \( w_2\delta_4 \) has the smallest value among \( \{w_2\delta_2, w_3\delta_3, w_4\delta_4\} \). This happens in the region defined by :

\[
\begin{align*}
  \frac{3}{4}\delta_3 + \delta_4 & \geq 0.225 \\
  \frac{3}{4}\delta_4 + \delta_1 & \geq 0.225 \\
  0 \leq \delta_1 \leq 0.25 \\
  0 \leq \delta_4 \leq 0.25 \\
  \delta_3 + \delta_4 & \leq 0.3
\end{align*}
\]  
(140)

In this case the objective function is :

\[
\mathcal{F}^2 = \lambda_1 w_3\delta_3 + \lambda_2 w_4\delta_4 + \lambda_3 w_2\delta_2 = 3\delta_3 + 3\delta_4 + 3\delta_2 = 3(\delta_2 + \delta_4 + \delta_4).
\]  
(141)

Since \( \delta_2 + \delta_3 + \delta_4 = 1 - t = 0.3 \), \( \mathcal{F}^2 = 0.9 \). Meaning, any subtree, satisfying \( w_2\delta_2 \leq w_3\delta_3, \ w_2\delta_2 \leq w_4\delta_4 \), is optimal and has the objective value \( \mathcal{F}^2_{\text{opt}} = 0.9 \).

Summary:

The optimal solution of this example is any subtree satisfying (140) and the optimal objective value is 0.9.
קורקינט

בaming את המילים אל השיטה של קורקינט, נוכל למצוא מספר פעמים של קורקינט והזויות שלו. קורקינט הוא השיטהadelopedPor ב kellin שמתנה בין הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 בל. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינט וה🍃 ב γ^1 Blair. בקירוב המקובל, מתקיימה השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימה השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוה埕 ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימה השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, מתקיימת השיטה של הקורקינטוהcheng ב γ^1 Blair. בקירוב המקובל, M
תכונת השいろ מביעת מיקום על עץ עם פונקציות מצויות המורים והחתкова הקומור הסדור

ת karşısında נטיה המנהרה המשרד וצורת המסר Дан פוק

תכיר דנוב

הניצוד הקומור במחוז

פרות' אורי מוט

אינטגרטס של אביה