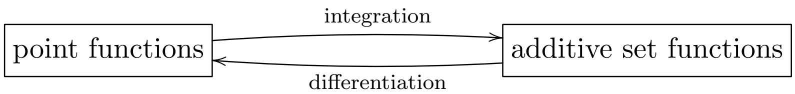
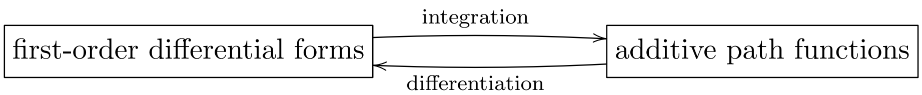
10 From path functions to differential forms

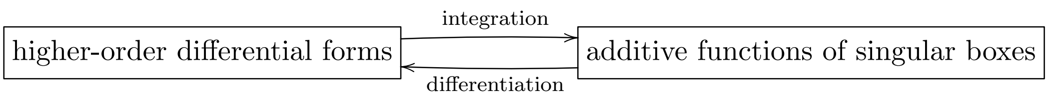
The relation



was treated in Sections 6 and 8. A similar relation



is treated here, and generalized:



. . . this chapter may seem rather abstract and artificial . . . the best procedure for the moment is simply to regard differential forms as completely new mathematical objects. . .
Corwin and Szczarba, p. 487

. . . a k-form ω is some sort of mapping
$\omega : \{k\text{-surfaces in } A\} \to \mathbb{R}$.
Shurman, p. 404.

10a Why path functions

Life is a path function. You begin life, you end life–that's not so interesting, right? But quality of life is a path function. It's the path that you take from the beginning to the end, the integral of that path, that's the special part. Christopher Edwards

By a *path* (in $\mathbb{R}^n$) we mean a piecewise continuously differentiable function $\gamma : [t_0, t_1] \to \mathbb{R}^n$ (real numbers $t_0 < t_1$ may depend on the path). Recall 1f1; there, γ was just continuous, but now we need the derivative. A path is called *closed* if $\gamma(t_0) = \gamma(t_1)$.

A path may describe the motion of a body (a car, aircraft, ship, submarine, planet, particle etc); $\gamma(t)$ is the position of the body at time t.

For a car, the fuel consumption is roughly proportional to the energy required to overcome resistance, namely, air resistance and rolling resistance. This energy is a function Ω of a path;

$$\Omega(\gamma) = \int_{t_0}^{t_1} |F(t)| v(t) \, \mathrm{d}t \,,$$

where $v(t) = |\gamma'(t)|$ is the speed of the car, and $F(t)$ is the resistance force. In a reasonable approximation,[1] the air resistance is of the form $c_2 v^2 + c_1 v$ (viscous and wind resistance), and the rolling resistance is a constant, c_0. Thus,

$$\Omega(\gamma) = \int_{t_0}^{t_1} \big(c_2 |\gamma'(t)|^2 + c_1 |\gamma'(t)| + c_0 \big) |\gamma'(t)| \, \mathrm{d}t \,.$$

For a planet or a particle resistance is usually negligible, but external fields (usually gravitational and/or electromagnetic) do a work (energy exchange)

$$\Omega(\gamma) = \int_{t_0}^{t_1} \langle F_\gamma(t), \gamma'(t) \rangle \, \mathrm{d}t$$

where $F_\gamma(t)$ is the force vector. Its dependence on γ is often of the form $F_\gamma(t) = F(\gamma(t))$ for a given vector field F; that is, $F : \mathbb{R}^n \to \mathbb{R}^n$.

And the most famous path function is, of course, the length,

$$\Omega(\gamma) = \int_{t_0}^{t_1} |\gamma'(t)| \, \mathrm{d}t \,.$$

10a1 Exercise. [2] Derive the energy conservation

$$\frac{1}{2} m |\gamma'(t_1)|^2 - \frac{1}{2} m |\gamma'(t_0)|^2 = \int_{t_0}^{t_1} \langle F_\gamma(t), \gamma'(t) \rangle \, \mathrm{d}t$$

from the Newton's second law of motion

$$m \gamma''(t) = F_\gamma(t) \,.$$

[1] Wikipedia, "Fuel economy in automobiles" and "Drag (physics)".

[2] Shifrin, Sect. 8.3.

10b Some properties of path functions

Path functions may be roughly classified according to presence or absence of the following properties.

ADDITIVITY: for every path $\gamma : [t_0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma|_{[t_0,t]}) + \Omega(\gamma|_{[t,t_1]}) = \Omega(\gamma) \quad \text{for all } t \in (t_0, t_1)\,. \tag{10b1}$$

All path functions mentioned in Sect. 10a are additive.

STATIONARITY: for every path $\gamma : [t_0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma(\cdot - s)) = \Omega(\gamma) \quad \text{for all } s \in \mathbb{R}\,; \tag{10b2}$$

here $\gamma(\cdot - s)$ is the time shifted path $t \mapsto \gamma(t - s)$ for $t \in [t_0 + s, t_1 + s]$.

Non-examples: for an aircraft, a night flight may differ in fuel consumption from a similar day flight; for a particle, external field sources may change in time.

For a stationary Ω we may restrict ourselves to the case $t_0 = 0$.

SYMMETRY AND ANTISYMMETRY (FOR STATIONARY Ω ONLY): for every path $\gamma : [0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma_{-1}) = \Omega(\gamma)\,; \qquad \text{symmetry; or} \tag{10b3}$$

$$\Omega(\gamma_{-1}) = -\Omega(\gamma)\,; \qquad \text{antisymmetry} \tag{10b4}$$

here the inverse path $\gamma_{-1} : t \mapsto \gamma(t_1 - t)$ for $t \in [0, t_1]$.

Every stationary path function Ω is the sum of its symmetric part $\gamma \mapsto \big(\Omega(\gamma) + \Omega(\gamma_{-1})\big)/2$ and antisymmetric part $\gamma \mapsto \big(\Omega(\gamma) - \Omega(\gamma_{-1})\big)/2$; and if Ω is additive then its symmetric part and antisymmetric part are also additive (think, why).

NO WAITING CHARGE:

$$\gamma(\cdot) = \text{const (that is, } \gamma'(\cdot) = 0) \quad \text{implies} \quad \Omega(\gamma) = 0\,. \tag{10b5}$$

PARAMETRIZATION INVARIANCE:

$$\Omega(\gamma \circ \varphi) = \Omega(\gamma) \tag{10b6}$$

whenever $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is a path and $\varphi : [s_0, s_1] \to [t_0, t_1]$ an increasing diffeomorphism. In this case the path $\gamma \circ \varphi : [s_0, s_1] \to \mathbb{R}^n$ is called *equivalent* to γ. Recall Sect. 1f: there, φ was just an increasing bijection (therefore, homeomorphism), but now we need a diffeomorphism.

Clearly, parametrization invariance implies stationarity.

10b7 Exercise. Consider path functions of the form

$$\Omega : \gamma \mapsto \int_{t_0}^{t_1} f\big(t, \gamma(t), \gamma'(t)\big)\, \mathrm{d}t \tag{10b8}$$

for arbitrary continuous functions $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

(a) For each of the properties defined above give a sufficient condition in terms of f.

(b) Are your conditions necessary?

10b9 Exercise. [1] Determine the work $\int \langle F(\gamma(t)), \gamma'(t)\rangle \, \mathrm{d}t$ done on a particle moving along γ in $\mathbb{R}^3$ through the force field $F(x, y, z) = (1, -x, z)$, where γ is

(a) the line segment from $(0, 0, 0)$ to $(1, 2, 1)$;

(b) the unit circle in the plane $z = 1$ with center $(0, 0, 1)$ beginning and ending at $(1, 0, 1)$ and starting toward $(0, 1, 1)$.

10b10 Exercise. [2] The same for $F(x, y, z) = (x^2, y^2, z^2)$ and $\gamma(t) = (\cos t, \sin t, at)$, $t \in [0, t_1]$ (the arc of helix).

The following property holds for a very restricted but very important class of path functions.

Given paths $\gamma, \gamma_1, \gamma_2, \dots : [t_0, t_1] \to \mathbb{R}^n$, we define convergence, $\gamma_k \to \gamma$, as follows:

$$\begin{gathered} \forall t \in [t_0, t_1] \;\; \gamma_k(t) \to \gamma(t) \,, \\ \exists L \; \forall k \;\; \gamma_k \in \operatorname{Lip}(L) \,, \end{gathered} \tag{10b11}$$

The condition $\gamma_k \in \operatorname{Lip}(L)$ is equivalent to $\forall t \;\; |\gamma'(t)| \le L$ (with one-sided derivatives when needed). Note that this convergence is stronger than the uniform convergence.

Continuity:

$$\gamma_k \to \gamma \quad \text{implies} \quad \Omega(\gamma_k) \to \Omega(\gamma)\,. \tag{10b12}$$

Significantly, the length is a discontinuous path function. A counterexample: $\gamma_k(t) = \big(t, \frac{1}{k}\sin kt\big)$ (or just $\gamma_k(t) = \frac{1}{k}\sin kt$).

All path functions mentioned in Sect. 10a become continuous if one stipulates convergence in C^1 for paths, that is, $\max_t |\gamma_k'(t) - \gamma'(t)| \to 0$. But we do not!

---

[1] Corwin, Szczarba Sect. 13.3.

[2] Hubbard, Sect. 6.5.

10c First-order differential forms emerge

10c1 Definition. Let Ω be a stationary additive path function, and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a continuous function. We say that f is the *derivative* of Ω (symbolically, $f = D\Omega$) if

$$\Omega(\gamma) = \int_{t_0}^{t_1} f(\gamma(t), \gamma'(t))\, \mathrm{d}t \tag{10c2}$$

for every path γ.

Such f is unique (if exists), since

$$f(\gamma(t), \gamma'(t)) = \frac{\mathrm{d}}{\mathrm{d}t} \Omega\big(\gamma|_{[t_0,t]}\big) = \lim_{\varepsilon \to 0+} \frac{1}{\varepsilon} \Omega\big(\gamma|_{[t,t+\varepsilon]}\big) \,.$$

If such f exists, we say that Ω is continuously differentiable (or that $D\Omega$ exists), and denote $f(x, h)$ by $(D_h\Omega)_x$.[1]

10c3 Proposition. If a stationary additive path function Ω is continuous and $D\Omega$ exists then for every x the function $h \mapsto (D_h\Omega)_x$ is affine (that is, the function $h \mapsto (D_h\Omega)_x - (D_0\Omega)_x$ is linear).

10c4 Lemma. The following two conditions on a function $f : \mathbb{R}^n \to \mathbb{R}$ are equivalent:

(a) $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$ for all $x, y \in \mathbb{R}^n$ and $\theta \in (0, 1)$;

(b) the function $x \mapsto f(x) - f(0)$ is linear.

Proof. We define $g(x) = f(x) - f(0)$.

(b)$\Longrightarrow$(a): $f(\theta x+(1-\theta)y)-f(0) = g(\theta x+(1-\theta)y) = \theta g(x)+(1-\theta)g(y) = \theta(f(x) - f(0)) + (1 - \theta)(f(y) - f(0)) = \theta f(x) + (1 - \theta)f(y) - f(0)$.

(a)$\Longrightarrow$(b): for every $\lambda > 0$ we have

$$\frac{\lambda}{\lambda+1} g(x) + \frac{1}{\lambda+1} g(-\lambda x) = \frac{\lambda}{\lambda+1} f(x) + \frac{1}{\lambda+1} f(-\lambda x) - f(0) = \\ = f\Big(\frac{\lambda}{\lambda+1} x + \frac{1}{\lambda+1}(-\lambda x)\Big) - f(0) = 0 \,,$$

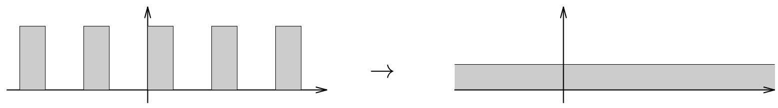
therefore $g(-\lambda x) = -\lambda g(x)$. Using this relation twice we get $g(\lambda x) = \lambda g(x)$ for all $\lambda \in \mathbb{R}$, $x \in \mathbb{R}^n$. Now,

$$g(x+y) = 2g\Big(\frac{x+y}{2}\Big) = 2f\Big(\frac{x+y}{2}\Big) - 2f(0) = 2\frac{f(x)+f(y)}{2} - 2f(0) = g(x)+g(y) \,.$$

$\square$

[1] The same condition may be imposed on an arbitrary path function, and then it may be called "additivity, stationarity and continuous differentiability".

10c5 Lemma. Let $\theta \in (0,1)$ and $T_k = \uplus_{i=-\infty}^{\infty} [\frac{i}{k}, \frac{i+\theta}{k}]$. Then $\int_{T_k} f \to \theta \int_{\mathbb{R}} f$ (as $k \to \infty$) for every Riemann integrable $f : \mathbb{R} \to \mathbb{R}$.



Proof. The claim holds when f is the indicator of an interval, since in this case $| \int_{T_k} f - \theta \int_{\mathbb{R}} f| \le \frac{\theta(1-\theta)}{k}$. By linearity the claim holds for all step functions. By sandwich, it holds for all integrable functions. □

Proof of Prop. 10c3. First,

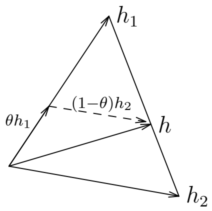
$$\text{(10c6)} \qquad \gamma_k \to \gamma \quad \text{implies} \quad \int_{t_0}^{t_1} f\big(\gamma(t), \gamma_k'(t)\big)\, \mathrm{d}t \to \int_{t_0}^{t_1} f\big(\gamma(t), \gamma'(t)\big)\, \mathrm{d}t\,,$$

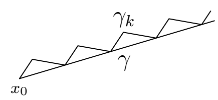
since $\Omega(\gamma_k) \to \Omega(\gamma)$ by continuity of Ω, and $\sup_t |f\big(\gamma(t), \gamma_k'(t)\big) - f\big(\gamma_k(t), \gamma_k'(t)\big)| \to 0$ due to uniform continuity of f on bounded sets. By 10c4 it is sufficient to prove that

$$(D_h\Omega)_{x_0} = \theta (D_{h_1}\Omega)_{x_0} + (1-\theta)(D_{h_2}\Omega)_{x_0}$$

whenever $h = \theta h_1 + (1-\theta)h_2$, $\theta \in (0,1)$, and $x_0 \in \mathbb{R}^n$. We construct paths $\gamma, \gamma_k : [0, t_1] \to \mathbb{R}^n$ such that

$$\gamma(0) = \gamma_k(0) = x_0\,,$$
$$\gamma'(t) = h \quad \text{for all } t \in (0, t_1)\,,$$
$$\gamma_k'(t) = \begin{cases} h_1 & \text{for } t \in (0,t_1) \cap T_k^\circ\,, \\ h_2 & \text{for } t \in (0,t_1) \setminus T_k\,, \end{cases}$$





T_k being as in Lemma 10c5.

We have $\gamma_k(\frac{i}{k}) = \gamma(\frac{i}{k})$ (for integer i such that $\frac{i}{k} \in [0,t_1]$), since $\int_{i/k}^{(i+1)/k} \gamma_k'(t)\, \mathrm{d}t = \int_{i/k}^{(i+1)/k} \gamma'(t)\, \mathrm{d}t$; thus, $\sup_t |\gamma_k(t) - \gamma(t)| \le \theta |h_1|/k \to 0$; and $\gamma_k \in \operatorname{Lip}(\max(|h_1|, |h_2|)$. Thus, $\gamma_k \to \gamma$.

By (10c6),

$$\int_0^{t_1} f\big(x_0 + th, \gamma_k'(t)\big)\, \mathrm{d}t \to \int_0^{t_1} f(x_0 + th, h)\, \mathrm{d}t\,.$$

We have

$$\int_0^{t_1} f\big(x_0 + th, \gamma_k'(t)\big)\, \mathrm{d}t = \int_{[0,t_1]\cap T_k} f(x_0 + th, h_1)\, \mathrm{d}t + \int_{[0,t_1]\setminus T_k} f(x_0 + th, h_2)\, \mathrm{d}t\,.$$

By Lemma 10c5, in the limit $k \to \infty$ we get

$$\int_0^{t_1} f(x_0 + th, h)\,\mathrm{d}t = \theta \int_0^{t_1} f(x_0 + th, h_1)\,\mathrm{d}t + (1-\theta) \int_0^{t_1} f(x_0 + th, h_2)\,\mathrm{d}t\,.$$

We see that the continuous function

$$x \mapsto f(x,h) - \theta f(x,h_1) - (1-\theta) f(x,h_2)$$

has zero integral on every straight interval. It follows easily that this function vanishes everywhere. □

10c7 Exercise. Assume that an additive path function Ω is continuous, and satisfies

$$\Omega(\gamma) = F\big(|\gamma(t_1)|\big) - F\big(|\gamma(t_0)|\big)$$

(where F is a given function) in two cases: first, for all γ of the form $\gamma(t) = \varphi(t)x$ ("radial"), and second, for all γ such that $|\gamma(\cdot)| = \text{const}$ ("tangential"). Prove that the same formula holds for all γ.

10c8 Definition. A *first-order differential form* of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x,\cdot)$ is linear.

For brevity we say just "1-form".

Every 1-form ω leads to an additive path function Ω,

$$\Omega(\gamma) = \int_{t_0}^{t_1} \omega\big(\gamma(t), \gamma'(t)\big)\,\mathrm{d}t = \int_\gamma \omega\,; \tag{10c9}$$

note the convenient notation $\int_\gamma \omega$. This Ω satisfies the "no waiting charge" condition (10b5).

Now Proposition 10c3 may be reformulated: if an additive path function Ω is continuous and $D\Omega$ exists then

$$\forall \gamma \quad \Omega(\gamma) = \int_\gamma \omega + \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$$

for some 1-form ω of class $C^{(0)}$ and some continuous function $f : \mathbb{R}^n \to \mathbb{R}$.

Indeed, $f(x) = (D_0\Omega)_x$ and $\omega(x,h) = (D_h\Omega)_x - (D_0\Omega)_x$.

10c10 Exercise. Prove that the symmetric part of Ω is $\gamma \mapsto \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$ and the antisymmetric part is $\gamma \mapsto \int_\gamma \omega$.

Note that the symmetric part (if not identically zero) violates the "no waiting charge" condition (10b5), while the antisymmetric part satisfies this condition.

10c11 Exercise. The path function $\gamma \mapsto \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$ is continuous for arbitrary continuous $f : \mathbb{R}^n \to \mathbb{R}$.

Prove it.

Is the path function $\gamma \mapsto \int_\gamma \omega$ continuous for arbitrary 1-form ω? We'll return to this question later.

Traditionally one denotes the coordinates $h_1, \dots, h_n$ of the vector h by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$ and writes

$$\omega = f_1(x_1, \dots, x_n)\mathrm{d}x_1 + \dots + f_n(x_1, \dots, x_n)\mathrm{d}x_n$$

rather than

$$\omega(x_1, \dots, x_n; \mathrm{d}x_1, \dots, \mathrm{d}x_n) = f_1(x_1, \dots, x_n)\mathrm{d}x_1 + \dots + f_n(x_1, \dots, x_n)\mathrm{d}x_n \,.$$

In this notation,

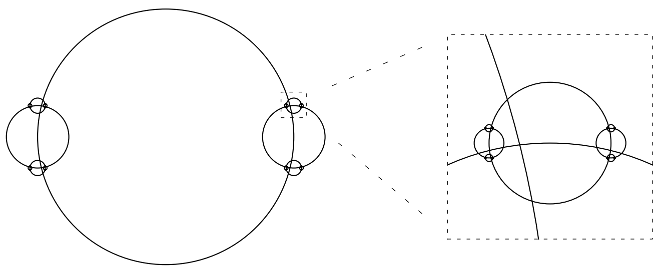
$$\int_\gamma \big(f_1(x)\mathrm{d}x_1 + \dots + f_n(x)\mathrm{d}x_n\big) = \int_{t_0}^{t_1} \big(f_1(\gamma(t))\mathrm{d}\gamma_1(t) + \dots + f_n(\gamma(t))\mathrm{d}\gamma_n(t)\big)$$

for $\gamma(t) = (\gamma_1(t), \dots, \gamma_n(t))$.

10c12 Exercise. Prove that the path function $\gamma \mapsto \int_\gamma \omega$ is parametrization invariant.

A *curve* is often defined as an equivalence class of paths. Then, by 10c12, a 1-form may be integrated over a curve. But be warned: such "curve" need not be piecewise smooth (since $\gamma'(\cdot)$ may vanish on an infinite set) even if paths are $C^{(1)}$. On the picture below you see what may happen to the set $\gamma([t_0, t_1])$ for $\gamma \in C^{(1)}$.

(10c13)



10c14 Exercise. [1] Prove that the following pairs of paths are equivalent:
(a) $\gamma_1(t) = (\sin t, \cos t)$, $t \in [0, 2\pi]$; $\gamma_2(t) = (-\cos t, \sin t)$, $t \in [\frac{\pi}{2}, \frac{5\pi}{2}]$;
(b) $\gamma_1(t) = (2\cos t, 2\sin t)$, $t \in [0, \frac{\pi}{2}]$; $\gamma_2(t) = \big(\frac{2-2t^2}{1+t^2}, \frac{4t}{1+t^2}\big)$, $t \in [0, 1]$.

10c15 Exercise. [2] Compute $\int_\gamma \omega$ for $\omega(x, y) = x\,\mathrm{d}x - y\,\mathrm{d}y$ over the following paths:
(a) $\gamma(t) = (\cos \pi t, \sin \pi t)$, $t \in [0, 1]$;
(b) $\gamma(t) = (1 - t, 0)$, $t \in [0, 2]$;
(c) $\gamma(t) = (1 - t, 1 - |1 - t|)$, $t \in [0, 2]$.

10c16 Exercise. [3] The same for $\omega(x, y, z) = yz\,\mathrm{d}x + xz\,\mathrm{d}y + xy\,\mathrm{d}z$ and
(a) $\gamma(t) = (\cos 2\pi t, \sin 2\pi t, 2t)$, $t \in [0, 3]$;
(b) $\gamma(t) = (1, 0, t)$, $t \in [0, 6]$.

10c17 Exercise. [4] The same for $\omega(x, y) = y\,\mathrm{d}x + xy\,\mathrm{d}y$ and a closed curve that traverses the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ once in the "counterclockwise" direction.

10c18 Exercise. [5] Integrate the 1-form $y\,\mathrm{d}x$ on $\mathbb{R}^3$ along the intersection of the unit sphere and the plane $x + y + z = 0$, oriented counterclockwise as viewed from high above the xy-plane.[6]

10d Example: winding number

Every point $(x, y) \in \mathbb{R}^2 \setminus \{(0, 0)\}$ is $(r\cos\theta, r\sin\theta)$ for $r = \sqrt{x^2 + y^2}$ and some θ, but θ is not unique. We note that

$$\begin{gathered}\mathbb{R}^2 \setminus \{(0,0)\} = U_1 \cup U_2 \cup U_3 \cup U_4\,,\\ U_1 = \{(x,y) : x > 0\}\,, \quad U_2 = \{(x,y) : y > 0\}\,,\\ U_3 = \{(x,y) : x < 0\}\,, \quad U_4 = \{(x,y) : y < 0\}\end{gathered}$$

and define functions $\theta_i : U_i \to \mathbb{R}$ for $i = 1, 2, 3, 4$ by

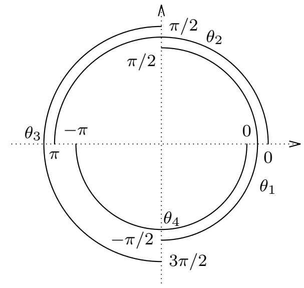
$$\begin{gathered}\theta_1(x,y) = \arcsin \frac{y}{\sqrt{x^2+y^2}}\,, \quad \theta_2(x,y) = \arccos \frac{x}{\sqrt{x^2+y^2}}\,,\\ \theta_3(x,y) = \pi - \arcsin \frac{y}{\sqrt{x^2+y^2}}\,, \quad \theta_4(x,y) = -\arccos \frac{x}{\sqrt{x^2+y^2}}\end{gathered}$$

---

[1] Corwin, Szczarba Sect. 13.1.
[2] Devinatz, Sect. 9.1.
[3] Devinatz, Sect. 9.1.
[4] Devinatz, Sect. 9.1.
[5] Shifrin, Sect. 8.3.
[6] Hint: find an orthonormal basis for the plane.

(see also Sect. 4a), then

$$\theta_1 = \theta_2 \text{ on } U_1 \cap U_2 \,, \quad \theta_2 = \theta_3 \text{ on } U_2 \cap U_3 \,,$$
$$\theta_3 = \theta_4 + 2\pi \text{ on } U_3 \cap U_4 \,, \quad \theta_4 = \theta_1 \text{ on } U_4 \cap U_1 \,.$$



They conform only up to a constant; but their derivatives (or gradients) do conform,

$$D\theta_i = D\theta_j \text{ on } U_i \cap U_j \,.$$

A calculation gives

$$\forall (x,y) \in U_i \quad \nabla \theta_i(x,y) = \frac{1}{x^2+y^2}(-y,x) \,,$$

that is, for all $x = (x_1, x_2) \in U_i$, $h = (h_1, h_2) \in \mathbb{R}^2$,

$$(D_h \theta_i)_x = \frac{\det(x,h)}{|x|^2} = \frac{1}{x_1^2 + x_2^2} \begin{vmatrix} x_1 & h_1 \\ x_2 & h_2 \end{vmatrix} \,.$$

We introduce a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ by

$$\omega(x,h) = (D_h \theta_i)_x \quad \text{whenever } x \in U_i \,.$$

That is,

$$\omega(x_1, x_2; \mathrm{d}x_1, \mathrm{d}x_2) = \frac{1}{x_1^2 + x_2^2} \begin{vmatrix} x_1 & \mathrm{d}x_1 \\ x_2 & \mathrm{d}x_2 \end{vmatrix} \,; \qquad \omega = \frac{-y\mathrm{d}x + x\mathrm{d}y}{x^2+y^2} \,.$$

It is easy to guess that $\int_\gamma \omega$ is the angle of rotation (around the origin), and therefore

$$\int_\gamma \omega \in 2\pi\mathbb{Z} \quad \text{for all closed paths } \gamma \text{ in } \mathbb{R}^2 \setminus \{0\} \,.$$

Here is a way to the proof.

10d1 Exercise. (a) If $\gamma : [t_0, t_1] \to U_i$ then $\int_\gamma \omega = \theta_i(\gamma(t_1)) - \theta_i(\gamma(t_0))$;

(b) for every $\gamma : [t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ there exists a partition $t_0 < s_1 < \dots < s_k < t_1$ of $[t_0, t_1]$ and $i_0, \dots, i_k \in \{1,2,3,4\}$ such that $\gamma([t_0, s_1]) \subset U_{i_0}$, $\gamma([s_1, s_2]) \subset U_{i_1}$, …, $\gamma([s_{k-1}, s_k]) \subset U_{i_{k-1}}$, $\gamma([s_k, t_1]) \subset U_{i_k}$;[1]

[1]Hint: continuity of γ is enough, differentiability does not help.

(c) every $\gamma : [t_0, t_1] \to \mathbb{R}^2 \setminus \{0\}$ satisfies $\theta_{i_1}(\gamma(t_1)) - \theta_{i_0}(\gamma(t_0)) - \int_\gamma \omega \in 2\pi\mathbb{Z}$ whenever $\gamma(t_0) \in U_{i_0}$, $\gamma(t_1) \in U_{i_1}$;

(d) if $\gamma(t_0) = \gamma(t_1)$ then $\int_\gamma \omega \in 2\pi\mathbb{Z}$.

Prove it.

The integer $\frac{1}{2\pi} \int_\gamma \omega$ is called the *winding number* (or index) of a close path γ on $\mathbb{R}^2 \setminus \{0\}$ around 0. The winding number of γ around another point $x_0 \in \mathbb{R}^2 \setminus \gamma([t_0, t_1])$ may be defined as the winding number of the shifted path $t \mapsto \gamma(t) - x_0$ around 0. This is an integer-valued continuous function of x_0 defined on the open set $\mathbb{R}^2 \setminus \gamma([t_0, t_1])$; therefore it is constant on each connected component of this open set (recall Sect. 1f). The proof of the continuity is simple: if $x_k \to x_0$ then

$$\int_{t_0}^{t_1} \omega(\gamma(t) - x_k, \gamma'(t))\mathrm{d}t \to \int_{t_0}^{t_1} \omega(\gamma(t) - x_0, \gamma'(t))\mathrm{d}t$$

since $\omega(x, h) = \frac{\det(x,h)}{|x|^2}$ is continuous in x (for a given h), uniformly outside a neighborhood of 0.

It would be interesting to integrate over all $x_0 \in \mathbb{R}^2$ the winding number around x_0. This could give us a formula for calculating the area of a planar domain via integral over the boundary of this domain. The function $x \mapsto \frac{\det(x,h)}{|x|^2}$ is unbounded (near 0), with unbounded support, which leads to an improper integral. It converges near 0, but diverges on infinity (try polar coordinates). Thus, the right choice of exhaustion is important. It is futile to nullify $\omega(x, h)$ for large x, but it is wise to integrate $\omega(\gamma(t) - x_0, \gamma'(t))$ over not too large x_0. It appears that[1]

$$\int_{|x_0| \le R} \omega(x - x_0, h) \to \pi \det(x, h) \quad \text{as } R \to \infty\,;$$

thus, the integrated winding number is $\frac{1}{2} \int_{t_0}^{t_1} \det\big(\gamma(t), \gamma'(t)\big)\mathrm{d}t$, the half of the integral over γ of the 1-form $(-y\mathrm{d}x + x\mathrm{d}y)$. We'll return to this form later.

10d2 Exercise. [2] Compute $\int_\gamma \omega$ for $\omega(x, y) = \frac{-y\,\mathrm{d}x + x\,\mathrm{d}y}{2}$ and γ that bounds the triangle with vertices $(0, 0), (a, 0), (b, c)$ $(a, b, c > 0)$ and traverses its boundary once in the "counterclockwise" direction.

10e Higher-order differential forms

10e1 Definition. A *singular k-cube* in $\mathbb{R}^n$ is a mapping $\Gamma : [0, 1]^k \to \mathbb{R}^n$ of class $C^{(1)}$; that is, Γ is continuously differentiable on $(0, 1)^k$ and its derivative $D\Gamma$ extends by continuity to the boundary of the cube.

[1] Try to check it, if you are ambitious enough.

[2] Fleming, Sect. 6.4.

Similarly we may use any box in $\mathbb{R}^k$, not only $[0,1]^k$; then we have a singular k-box.

10e2 Example. A singular 2-box in $\mathbb{R}^2$: [Sh:Sect.9.13]

$$\Gamma(r,\theta) = (r\cos\theta, r\sin\theta) \quad \text{for } (r,\theta) \in [0,1]\times[0,2\pi]\,.$$

Note that this is not a homeomorphism.

10e3 Example. A singular 2-box in $\mathbb{R}^3$:

$$\Gamma(\varphi,\theta) = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta) \quad \text{for } (\varphi,\theta) \in [0,2\pi]\times[-\pi,\pi]\,.$$

Also, not a homeomorphism.

A singular 1-box is nothing but a path (continuously differentiable, not piecewise).

A singular 2-box may be thought of as a path in the space of paths. Even in two ways. Or, as a parametrized surface. But this "surface" may be rather strange (recall the one-dimensional example (10c13)).

A function Ω of a singular k-box is called *additive* if

$$\Omega(\Gamma) = \sum_{C\in P} \Omega(\Gamma|_C)$$

for every partition P of a box B (defined as in Sect. 6).

Similarly to (10c2) we consider Ω of the form

$$\Omega(\Gamma) = \int_B f\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\,\mathrm{d}u\,; \tag{10e4}$$

here $(D_1\Gamma)_x, \dots, (D_k\Gamma)_x \in \mathbb{R}^n$ are partial derivatives of Γ, and $f : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ is a continuous function.

Again, we wonder what can be said about f if Ω is continuous in the following sense:

$$\Gamma_i \to \Gamma \quad \text{implies} \quad \Omega(\Gamma_i) \to \Omega(\Gamma)\,, \tag{10e5}$$

where convergence of singular k-cubes (or boxes) $\Gamma, \Gamma_1, \Gamma_2, \dots : [0,1]^k \to \mathbb{R}^n$ is defined by

$$\begin{gathered} \forall u \in [0,1]^k \;\; \Gamma_i(u) \to \Gamma(u)\,, \\ \exists L \; \forall i \;\; \Gamma_i \in \mathrm{Lip}(L)\,, \end{gathered}$$

We consider first the case $k = 2$. Similarly to Prop. 10c3 we have the following.

10e6 Proposition. If Ω satisfies (10e4) and is continuous then for all $x, h_1 \in \mathbb{R}^n$ the function $h_2 \mapsto f(x, h_1, h_2)$ is affine.

Proof. Similarly to (10c6),
(10e7)

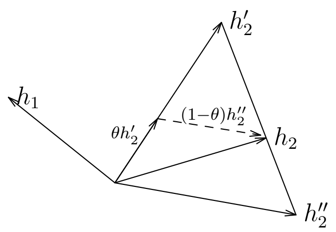
$$\Gamma_i \to \Gamma \quad \text{implies} \quad \int_B f\big(\Gamma(u), (D_1\Gamma_i)_u, (D_2\Gamma_i)_u\big)\,\mathrm{d}u \to \int_B f\big(\Gamma(u), (D_1\Gamma)_u, (D_2\Gamma)_u\big)\,\mathrm{d}u\,.$$

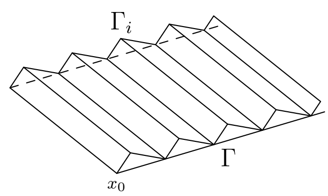
Again, by 10c4 it is sufficient to prove that

$$f(x_0, h_1, h_2) = \theta f(x_0, h_1, h_2') + (1-\theta) f(x_0, h_1, h_2'')$$

whenever $h_2 = \theta h_2' + (1-\theta) h_2''$, $\theta \in (0,1)$, and $x_0 \in \mathbb{R}^n$. Given a box $B = [0, U_1] \times [0, U_2] \subset \mathbb{R}^2$, we construct $\Gamma, \Gamma_i : B \to \mathbb{R}^n$ such that

$$\begin{gathered}\Gamma(0,0) = \Gamma_i(0,0) = x_0\,,\\ (D_1\Gamma)_u = (D_1\Gamma_i)_u = h_1 \quad \text{for all } u \in B^\circ\,,\\ (D_2\Gamma)_u = h_2 \quad \text{for all } u \in B^\circ\,,\\ (D_2\Gamma_i)_u = \begin{cases} h_2' & \text{for } u \in B^\circ \cap (\mathbb{R} \times T_i^\circ)\,,\\ h_2'' & \text{for } u \in B^\circ \setminus (\mathbb{R} \times T_i)\,,\end{cases}\end{gathered}$$





T_i being as in Lemma 10c5. These Γ_i are not singular boxes (since they are only piecewise $C^{(1)}$), but still, (10e7) applies to Γ_i, since there exist singular boxes $\tilde\Gamma_i$ such that $\tilde\Gamma_i \to \Gamma$ and

$$\bigg| \int_B f\big(\tilde\Gamma_i(u), (D_1\tilde\Gamma_i)_u, (D_2\tilde\Gamma_i)_u\big)\,\mathrm{d}u - \int_B f\big(\Gamma_i(u), (D_1\Gamma_i)_u, (D_2\Gamma_i)_u\big)\,\mathrm{d}u \bigg| \to 0\,.$$

Similarly to the proof of 10c3 we get

$$\int_0^{U_2} \int \mathrm{d}u_1\, f(x_0 + u_1h_1 + u_2h_2, h_1, (D_2\Gamma_i)_{x_0+u_1h_1+u_2h_2})\,\mathrm{d}u_2 \to$$
$$\to \theta \int_0^{U_2} \int \mathrm{d}u_1\, f(x_0+u_1h_1+u_2h_2, h_1, h_2')\,\mathrm{d}u_2 + (1-\theta) \int_0^{U_2} \int \mathrm{d}u_1\, f(x_0+u_1h_1+u_2h_2, h_1, h_2'')\,\mathrm{d}u_2\,.$$

We conclude that the continuous function

$$x \mapsto f(x, h_1, h_2) - \theta f(x, h_1, h_2') - (1-\theta) f(x, h_1, h_2'')$$

has zero integral on every parallelepiped, and therefore vanishes everywhere.
□

Assuming in addition that $\Gamma(\cdot) = \text{const}$ implies $\Omega(\Gamma) = 0$ we get $f(x, 0, 0) = 0$, but still, $f(x, h_1, 0)$ need not vanish. Here is an appropriate generalization of the "no waiting charge" condition (10b5):

(10e8) if $\Gamma(B)$ is contained in a $(k-1)$-dimensional affine subspace of $\mathbb{R}^n$ then $\Omega(\Gamma) = 0$.

Taking $\Gamma(u_1, u_2) = x_0 + u_1 h_1$ we see that (10e8) implies $f(x, h_1, 0) = 0$. Thus, for every x, $f(x, h_1, h_2)$ is linear in h_2 for each h_1; similarly it is linear in h_1 for each h_2; that is,

condition (10e8) implies that $f(x, \cdot, \cdot)$ is a bilinear form;

$$f(x, h_1, h_2) = \sum_{i,j=1}^{n} c_{i,j}(x)(h_1)_i (h_2)_j \,.$$

Further, taking $\Gamma(u_1, u_2) = x_0 + u_1 h + u_2 h$ we see that $f(x, h, h) = 0$ for all h (and x). It means that the bilinear form is antisymmetric,

$$f(x, h_2, h_1) = -f(x, h_1, h_2)\,;$$

indeed,

$$\underbrace{f(x, h_1+h_2, h_1+h_2)}_{=0} = \underbrace{f(x, h_1, h_1)}_{=0} + f(x, h_1, h_2) + f(x, h_2, h_1) + \underbrace{f(x, h_2, h_2)}_{=0}\,.$$

Generalization to $k = 3, 4, \dots$ is straightforward.

First, recall a notion from linear algebra: a (multililear) k-form[1] on $\mathbb{R}^n$ is a function $L : (\mathbb{R}^n)^k \to \mathbb{R}$ such that $L(x_1, \dots, x_k)$ is separately linear in each of the k variables $x_1, \dots, x_k \in \mathbb{R}^n$. Further, L is called antisymmetric[2] if it changes its sign under exchange of any pair of arguments.

10e9 Exercise. The following three conditions on a multililear k-form L on $\mathbb{R}^n$ are equivalent:

(a) L is antisymmetric;

(b) $L(x_1, \dots, x_k) = 0$ whenever $x_i = x_j$ for some $i \ne j$;

(c) $L(x_1, \dots, x_k) = 0$ whenever vectors $x_1, \dots, x_k$ are linearly dependent.

[1]Called also multililear form (or function) of degree (or order) k.

[2]Or "skew symmetric", or "alternating".

Now we generalize 10c8 and 10e6.

10e10 Definition. A *differential form* of order[1] k and of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x, \cdot, \dots, \cdot)$ is an antisymmetric multililear k-form on $\mathbb{R}^n$.

For brevity we say just "k-form".

10e11 Proposition. If a function Ω of a singular k-box in $\mathbb{R}^n$ is of the form (10e4), satisfies (10e5) and (10e8), then the function f from (10e4) is a k-form (of class C^0).

Similarly to (10c9) we define the integral of a k-form ω over a singular k-box Γ,

$$\int_\Gamma \omega = \int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\,\mathrm{d}u$$

(recall (10e4)) and observe that $\Gamma \mapsto \int_\Gamma \omega$ is an additive function of a singular box. Now, Prop. 10e11 gives a sufficient condition for Ω to be the integral of some ω.

10e12 Exercise. [2] Find $\int_\Gamma \omega$ where

$$\omega(x, h, k) = x_1 \begin{vmatrix} h_2 & k_2 \\ h_3 & k_3 \end{vmatrix} \quad \text{for } x, h, k \in \mathbb{R}^3 \,,$$

and $\Gamma(u, v) = (u^2, u + v, v^3)$ for $u, v \in [-1, 1]$.

[1] Or "degree".

[2] Hubbard, Sect. 6.2.

Index