12 Low dimensions, vector fields

In three (or two) dimensions, differential forms correspond to vector fields, and exterior derivative corresponds to operations of vector calculus.

12a Vector fields in three dimensions

Bad news: a 5-form in $\mathbb{R}^{10}$ is specified by $\binom{10}{5} = 252$ functions $\mathbb{R}^{10} \to \mathbb{R}$; surely not easy to visualize!

Good news: we are especially interested in three dimensions ($n = 3$), and in this case $\binom{n}{k} \le n$ for all k. In other words: the cases $k = 0, 1, n-1, n$ are relatively simple, and exhaust all cases in three dimensions.

For an arbitrary 2-form ω on $\mathbb{R}^3$ we have

$$\omega = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j \, ;$$

$$\omega(x,h,k) = \sum_{i<j} f_{i,j}(x) \begin{vmatrix} h_i & k_i \\ h_j & k_j \end{vmatrix} = \begin{vmatrix} f_{2,3}(x) & h_1 & k_1 \\ -f_{1,3}(x) & h_2 & k_2 \\ f_{1,2}(x) & h_3 & k_3 \end{vmatrix} = \det\bigl(H(x), h, k\bigr)$$

where

$$H(x) = \bigl(f_{2,3}(x), -f_{1,3}(x), f_{1,2}(x)\bigr) = \bigl(f_{2,3}(x), f_{3,1}(x), f_{1,2}(x)\bigr) \, .$$

That is, we use a linear one-to-one correspondence between antisymmetric bilinear forms L on $\mathbb{R}^3$ and vectors $x \in \mathbb{R}^3$ given by

$$L(h,k) = \det(x,h,k) \, .$$

This duality (between L and x, or between ω and H), defined via determinant, may be called the determinant duality.

If H is dual to ω then fH is dual to $f\omega$, for arbitrary $f \in C^0(\mathbb{R}^3)$.

For every singular 2-box $\Gamma : B \to \mathbb{R}^3$,

$$\int_\Gamma \omega = \int_B \omega\bigl(\Gamma(u), (D_1\Gamma)_u, (D_2\Gamma)_u\bigr)\, \mathrm{d}u = \int_B \det\bigl(H(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u\bigr)\, \mathrm{d}u \, .$$

The latter integral is called the (total) *flux* of a vector field H through Γ (or across Γ).

The word "field" in "vector field" is not related to the algebraic notion of a field. Rather, it is related to the physical notion of a force field (gravitational, for example), or the velocity field of a moving matter (usually liquid or gas). Mathematically, a vector field formally is just a mapping $\mathbb{R}^3 \to \mathbb{R}^3$; less formally, a vector is attached to each point.[1] Rather easy to visualize!

If a vector field F on $\mathbb{R}^3$ is the velocity field of a fluid[2] (that is, $F(x)$ is the velocity of the fluid at the point x) then the flux of F through a surface is the amount[3] of fluid flowing through the surface (per unit time). If the fluid is flowing parallel to the surface then the flux is zero (since $F(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u$ are linearly dependent).

On the other hand, for an arbitrary 1-form ω on $\mathbb{R}^3$ we have

$$\omega = \sum_i f_i \, dx_i \, ;$$
$$\omega(x,h) = \sum_i f_i(x) h_i = \langle E(x), h \rangle$$

where

$$E(x) = \big(f_1(x), f_2(x), f_3(x) \big) \, .$$

Here we use a linear one-to-one correspondence between linear forms L on $\mathbb{R}^3$ and vectors $x \in \mathbb{R}^3$ given by

$$L(h) = \langle x, h \rangle \, .$$

This duality (between L and x, or between ω and E), defined via the Euclidean metric (namely, inner product), may be called the Euclidean duality.[4]

If E is dual to ω then fE is dual to $f\omega$, for arbitrary $f \in C^0(\mathbb{R}^3)$.

For every path $\gamma : [t_0, t_1] \to \mathbb{R}^3$,

$$\int_\gamma \omega = \int_{t_0}^{t_1} \omega\big(\gamma(t), \gamma'(t)\big) \, \mathrm{d}t = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t \, .$$

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[1] A vector field on an affine space is a mapping from this space to its difference space.

[2] See also mathinsight.

[3] The volume is meant, not the mass. However, these are proportional if the density (kg/m^3) of the matter is constant (which often holds for fluids).

[4] The notation "E" may be interpreted as "Euclidean", and "H" as "Hodge", since the well-known Hodge duality is (in particular) the relation between a 1-form ω_1 and a 2-form ω_2 on $\mathbb{R}^3$ such that the vector field E corresponding to ω_1 is equal to the vector field H corresponding to ω_2.

The latter integral is called the integral[1] of a vector field E along a path γ. In some sense it measures how much the vector field is aligned with the path.[2] (If E is orthogonal to γ then this integral vanishes.) If the path γ is closed then this integral is called *circulation* of E around γ; it indicates how much the vector field tends to circulate around γ.

12a1 Exercise. If $\omega = \omega_1 \wedge \omega_2$ is the exterior product of two 1-forms ω_1, ω_2 on $\mathbb{R}^3$, then

(a) $H = E_1 \times E_2$ is the cross product; here H is dual to ω, E_1 to ω_1, and E_2 to ω_2;

(b) $\omega_1\big(x, H(x)\big) = 0$ and $\omega_2\big(x, H(x)\big) = 0$.

Prove it.

If a 1-form ω on $\mathbb{R}^3$ is the exterior derivative dg of a 0-form $g \in C^1(\mathbb{R}^3)$, then clearly the vector field E dual to ω is the gradient,

$$E = \nabla g \, .$$

12a2 Exercise. Let ω_1 be a 1-form (of class C^1) on $\mathbb{R}^3$, $\omega_2 = d\omega_1$ its exterior derivative, E the vector field dual to ω_1, and H the vector field dual to ω_2. Then

$$H = \operatorname{curl} E \, ,$$

that is,

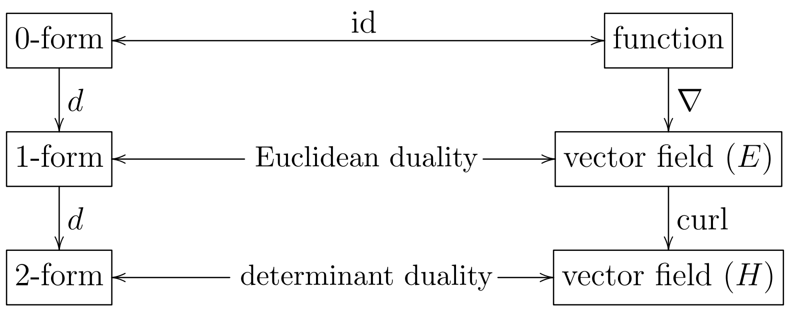
$$H_1 = D_2E_3 - D_3E_2 \, , \quad H_2 = D_3E_1 - D_1E_3 \, , \quad H_3 = D_1E_2 - D_2E_1 \, ;$$

here $H = (H_1, H_2, H_3)$ and $E = (E_1, E_2, E_3)$.

Prove it.

We summarize it by a commutative diagram [Sh:Ex.9.8.5]

(12a3)



By the way, the relation (11e4) $d(df) = 0$ becomes

(12a4) $$\operatorname{curl}(\nabla f) = 0 \, .$$

By Stokes' theorem 11d3, $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ for every singular 2-box[3] Γ in

[1] Also "line integral" or "flow integral".

[2] Nice formulation from mathinsight.

[3] The same holds for 2-chains, of course.

$\mathbb{R}^3$; in terms of the vector fields E, H dual to ω and $d\omega$ we get

$$\int_B \det\bigl(H(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u\bigr)\,\mathrm{d}u = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t$$

where $\gamma = \partial\Gamma$. This is the *"classical Stokes' theorem"* (also known as "Kelvin-Stokes theorem", "curl theorem" and "Stokes' formula"):

(12a5) the circulation of E around $\gamma = \partial\Gamma$
is equal to the flux of $H = \operatorname{curl} E$ through Γ

for every[1] vector field E (of class C^1) on $\mathbb{R}^3$ and every singular 2-box Γ in $\mathbb{R}^3$. In this sense, the curl is the circulation density, called also "vorticity" (and its flux is called also the net vorticity of E throughout Γ). A small paddle-wheel in the flow spins the fastest when its axle points in the direction of the curl vector, and in this case its angular speed is half the length of the curl vector.[2]

Relation (12a5) is only half-classical; differential forms are already replaced with vector fields, but singular boxes are not yet replaced with manifolds (curves and surfaces). The transition to manifolds needs much more effort than the transition to vector fields; wait for the second half of this course. For now, in order to write relations like (12a5) in symbols rather than words we introduce notation for integral along a path,

(12a6) $$\int_\gamma E = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t\,,$$

and through a singular 2-box,

(12a7) $$\int_\Gamma H = \int_B \det\bigl(H(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u\bigr)\,\mathrm{d}u\,;$$

now (12a5) becomes

(12a8) $$\int_{\partial\Gamma} E = \int_\Gamma \operatorname{curl} E\,.$$

No danger of confusion, since all one-dimensional integrals must be "along" (rather than "through"), while all two-dimensional integrals must be "through" (rather than "along").

[1] Since every E is dual to some ω.
[2] Shifrin p. 394.

12b Examples and exercises

12b1 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

$$\Big\langle \operatorname{curl} E\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big\rangle = 0 \quad \text{whenever } x^2+y^2+z^2=1,\ z>-0.9\,;$$

prove that $\int_{\gamma_z} E = 0$ for all $z \in (-0.9, 1)$, where $\gamma_z(t) = \begin{pmatrix} \sqrt{1-z^2}\cos t \\ \sqrt{1-z^2}\sin t \\ z \end{pmatrix}$ for $t \in [0, 2\pi]$.

12b2 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

$$\Big\langle \operatorname{curl} E\begin{pmatrix} x \\ y \\ z \end{pmatrix}, \begin{pmatrix} x \\ y \\ z \end{pmatrix} \Big\rangle = 0 \quad \text{whenever } x^2+y^2+z^2=1,\ -0.9<z<0.9\,;$$

prove that $\int_{\gamma_z} E$ does not depend on $z \in (-0.9, 0.9)$; here γ_z is the same as in 12b1.

12b3 Exercise. Consider a vector field

$$E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} -yf\big(\sqrt{x^2+y^2}\big) \\ xf\big(\sqrt{x^2+y^2}\big) \\ 0 \end{pmatrix}, \quad \text{that is,} \quad E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} rf(r)\cos(\theta+\frac{\pi}{2}) \\ rf(r)\sin(\theta+\frac{\pi}{2}) \\ 0 \end{pmatrix}$$

for a function $f : [0, \infty) \to \mathbb{R}$ of class C^1.

(a) Check that E is of class C^1, and

$$\operatorname{curl} E\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \sqrt{x^2+y^2}f'\big(\sqrt{x^2+y^2}\big)+2f\big(\sqrt{x^2+y^2}\big) \end{pmatrix}, \quad \text{that is,}$$
$$\operatorname{curl} E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ rf'(r)+2f(r) \end{pmatrix}.$$

(b) Given $\varepsilon > 0$, construct f such that

$$\begin{aligned} rf'(r) + 2f(r) > 0 \quad & \text{for } r \in (0, \varepsilon)\,, \\ rf'(r) + 2f(r) = 0 \quad & \text{for } r \in [\varepsilon, \infty)\,. \end{aligned}$$

(c) Conclude that $\int_{\gamma_z} E$ in 12b2 need not vanish.

12b4 Exercise. Let a vector field E (of class C^1) on $\mathbb{R}^3$ satisfy

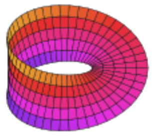
$$\operatorname{curl} E\begin{pmatrix} r\cos\theta \\ r\sin\theta \\ 1/r \end{pmatrix} = 0 \quad \text{for all } r>0,\ \theta\in[0,2\pi]\,,$$

and in addition, $|E(x,y,z)| = o\big(\sqrt{x^2+y^2+z^2}\big)$ as $x^2+y^2+z^2 \to \infty$. Prove that $\int_{\gamma_r} E = 0$ for all $r > 0$; here $\gamma_r(t) = \begin{pmatrix} r\cos t \\ r\sin t \\ 1/r \end{pmatrix}$ for $t \in [0, 2\pi]$.

A Möbius strip may be defined as such a surface:

$$\left\{ \begin{pmatrix} (R+rs\cos\frac{\theta}{2})\cos\theta \\ (R+rs\cos\frac{\theta}{2})\sin\theta \\ rs\sin\frac{\theta}{2} \end{pmatrix} : s \in [-1,1],\, \theta \in [0,2\pi] \right\},$$

for given $R > r > 0$.[1] Its boundary is a curve $\{\gamma(t) : t \in [0,4\pi]\}$,



$$\gamma(t) = \begin{pmatrix} (R+r\cos\frac{t}{2})\cos t \\ (R+r\cos\frac{t}{2})\sin t \\ r\sin\frac{t}{2} \end{pmatrix}.$$

12b5 Exercise. For γ as above and E of 12b3,

(a) check that

$$\langle E(\gamma(t)), \gamma'(t) \rangle = (R + r\cos\tfrac{t}{2}) f(R + r\cos\tfrac{t}{2});$$

(b) choose f such that $\operatorname{curl} E = 0$ on the Möbius strip, but $\int_\gamma E > 0$;

(c) does it contradict (12a5), (12a8)? Explain.

12c Vector fields in two dimensions

Dimension 2 is even more special than dimension 3, since the two cases $k = 1$ and $k = n - 1$ coincide for $n = 2$.

A linear form L on $\mathbb{R}^2$ can be represented by a vector in both ways:

$$L(h) = c_1 h_1 + c_2 h_2 = \det(x, h) = \langle y, h \rangle,$$
$$x = (c_2, -c_1), \quad y = (c_1, c_2);$$

the rotation by $\pi/2$ turns x into y.

Thus, a 1-form ω on $\mathbb{R}^2$ corresponds to two vector fields:

$$\omega = f_1\,dx_1 + f_2\,dx_2,$$
$$\omega(x,h) = \det(H(x), h) = \langle E(x), h \rangle,$$
$$H = (f_2, -f_1), \quad E = (f_1, f_2);$$

the rotation by $\pi/2$ turns $H(x)$ into $E(x)$.

[1] Images from Wikipedia.
Compare the Möbius strip with the torus (discussed in Sect. 8b),

$$\left\{ \begin{pmatrix} (R+r\cos\varphi)\cos\theta \\ (R+r\cos\varphi)\sin\theta \\ r\sin\varphi \end{pmatrix} : \varphi \in [0,2\pi],\, \theta \in [0,2\pi] \right\}.$$

Accordingly,

$$\int_\gamma \omega = \int_{t_0}^{t_1} \omega\bigl(\gamma(t), \gamma'(t)\bigr)\,\mathrm{d}t = \int_{t_0}^{t_1} \det\bigl(H(\gamma(t)), \gamma'(t)\bigr)\,\mathrm{d}t$$

is the (total) flux of a vector field H through a path $\gamma : [t_0, t_1] \to \mathbb{R}^2$. As before, if H is parallel to γ then the flux is zero.

On the other hand,

$$\int_\gamma \omega = \int_{t_0}^{t_1} \omega\bigl(\gamma(t), \gamma'(t)\bigr)\,\mathrm{d}t = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t$$

is the integral of a vector field E along a path γ. As before, if E is orthogonal to γ then this integral is zero.

In contrast to Sect. 12a, now we have two types of one-dimensional integrals, and must bother to avoid confusion. We introduce such notation:

$$\text{(12c1)} \qquad \int_{\text{along}\,\gamma} E = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t\,,$$

$$\text{(12c2)} \qquad \int_{\text{through}\,\gamma} H = \int_{t_0}^{t_1} \det\bigl(H(\gamma(t)), \gamma'(t)\bigr)\,\mathrm{d}t\,.$$

Let ω_1 be a 1-form (of class C^1) on $\mathbb{R}^2$ and $\omega_2 = d\omega_1$ its exterior derivative. Then

$$\begin{aligned} \omega_1 = f_1\,dx_1 + f_2\,dx_2 &= E_1\,dx_1 + E_2\,dx_2\,, \\ \omega_2 &= (\operatorname{curl} E)\,dx_1 \wedge dx_2\,, \\ \operatorname{curl} E &= D_1E_2 - D_2E_1\,. \end{aligned}$$

By Stokes' theorem 11d3, $\int_\Gamma d\omega_1 = \int_{\partial\Gamma} \omega_1$ for every singular 2-box Γ in $\mathbb{R}^2$; that is,

$$\int_\Gamma (\operatorname{curl} E)\,dx_1 \wedge dx_2 = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\,\mathrm{d}t$$

where $\gamma = \partial\Gamma$. This is Green's theorem: [Sh:9.16]

(12c3) the circulation of E around $\gamma = \partial\Gamma$
is equal to the integral of $\operatorname{curl} E$ over Γ

for every vector field E (of class C^1) on $\mathbb{R}^2$ and every singular 2-box Γ in $\mathbb{R}^2$. As before, the curl is the circulation density (or "vorticity").

In order to write it in symbols rather than words we introduce a notation for a two-dimensional integral:

(12c4) $$\int_\Gamma f = \int_\Gamma f \, dx_1 \wedge dx_2 \, ;$$

the right-hand side is the integral of a 2-form defined long ago; the left-hand side is just a short notation for it; similarly, in Sect. 6 we wrote $\int_E f$ rather than $\int_E f(x) \, \mathrm{d}x$. No danger of confusion, since we have no other integral of a *function* over a *singular* 2-box. Now (12c3) becomes

(12c5) $$\int_{\text{along } \partial\Gamma} E = \int_\Gamma \operatorname{curl} E \, .$$

If $\Gamma : B \to \mathbb{R}^2$ is such that $\Gamma|_{B^\circ}$ is a diffeomorphism between B° and an open set $G = \Gamma(B^\circ) \subset \mathbb{R}^2$ then

$$\int_\Gamma f(x) \, dx_1 \wedge dx_2 = \int_B f\big(\Gamma(u)\big) \det\big((D_1\Gamma)_u, (D_2\Gamma)_u\big) \, \mathrm{d}u = \\ = \int_{B^\circ} (f \circ \Gamma) \det D\Gamma = \pm \int_G f$$

by Theorem 9f5, for every $f \in C^0(\mathbb{R}^2)$; the "$\pm$" is the sign of $\det D\Gamma$ on B° (it is constant, since $\det D\Gamma$ does not vanish on the connected set B°). Assuming that the determinant is positive we get $\int_\Gamma (\operatorname{curl} E) \, dx_1 \wedge dx_2 = \int_G \operatorname{curl} E$, and Green's theorem becomes

(12c6) $$\int_{\text{along } \partial\Gamma} E = \int_G \operatorname{curl} E \, .$$

12c7 Exercise. (a) Let E be a vector field on $\mathbb{R}^3$ such that the third coordinate is dummy, that is,[1]

$$E(x, y, z) = \big(E_1(x, y), E_2(x, y), 0\big) \, .$$

Then

$$(\operatorname{curl} E)(x, y, z) = \big(0, 0, \operatorname{curl} \tilde{E}(x, y)\big) \, ,$$

where $\tilde{E} : (x, y) \mapsto \big(E_1(x, y), E_2(x, y)\big)$ is the corresponding vector field on $\mathbb{R}^2$.

(b) Let $\Gamma : B \to \mathbb{R}^3$ be a singular 2-box such that the third coordinate is dummy, that is,

$$\Gamma(u) = \big(\Gamma_1(u), \Gamma_2(u), 0\big) \, .$$

[1] Recall 12b3.

Then

the flux of $\operatorname{curl} E$ through Γ is equal to the integral of $\operatorname{curl} \tilde{E}$ over $\tilde{\Gamma}$

where $\tilde{\Gamma} : u \mapsto \big(\Gamma_1(u), \Gamma_2(u)\big)$ is the corresponding singular 2-box on $\mathbb{R}^2$.
Prove it.

And what about the other vector field H (that corresponds to the same ω_1 via the other duality)? Can we reformulate the equality $\int_\Gamma d\omega_1 = \int_{\partial\Gamma} \omega_1$ in terms of H? Yes, easily. First, we know that $\int_{\partial\Gamma} \omega_1$ is not only the integral of E along γ but also the flux of H through γ. Second,

$$\operatorname{curl} E = D_1 E_2 - D_2 E_1 = \operatorname{div} H \,, \\ \operatorname{div} H = D_1 H_1 + D_2 H_2$$

(since $H_1 = E_2$ and $H_2 = -E_1$), the *divergence* of H. Thus,

$$\int_\Gamma (\operatorname{div} H)\, dx_1 \wedge dx_2 = \int_{t_0}^{t_1} \det\big(H(\gamma(t)), \gamma'(t)\big)\, \mathrm{d}t$$

where $\gamma = \partial\Gamma$. This is the two-dimensional divergence theorem: [Sh:9.16]

(12c8)
$$\int_{\text{through } \partial\Gamma} H = \int_\Gamma \operatorname{div} H$$

for every vector field H (of class C^1) on $\mathbb{R}^2$ and every singular 2-box Γ in $\mathbb{R}^2$.

If a vector field F is the velocity field of a flow, then every point x flows to another point y during a time t, and for small t we have

$$y = x + tF(x) + o(t)\,, \\ \frac{\partial y}{\partial x} = I + t(DF)_x + o(t)$$

(it is generally wrong to differentiate $o(t)$ this way, but for a smooth flow this can be justified); thus, the Jacobian

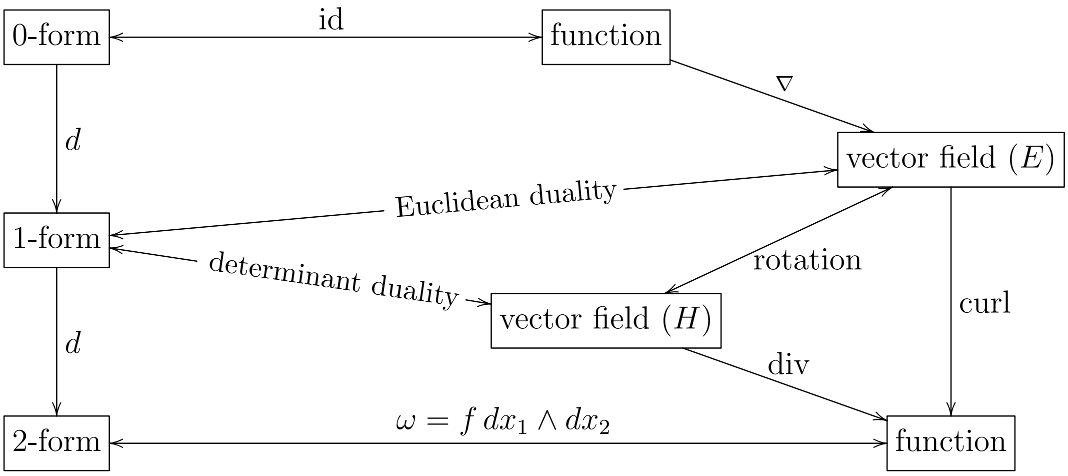
$$\det\Big(\frac{\partial y}{\partial x}\Big) = \begin{vmatrix} 1 + tD_1F_1 + o(t) & tD_2F_1 + o(t) \\ tD_1F_2 + o(t) & 1 + tD_2F_2 + o(t) \end{vmatrix} = 1 + t \operatorname{div} F + o(t)\,.$$

We see that a small drop of the flowing matter inflates if $\operatorname{div} F > 0$ and deflates if $\operatorname{div} F < 0$.

Divergence is often explained in terms of sources and sinks (of a moving matter). But be careful; the flux of a velocity field is the amount (per unit time) as long as “amount” means “volume”. If by “amount” you mean "mass", then you need the vector field of momentum, not velocity; multiply the velocity by the density of the matter. However, the problem disappears if the density is constant (which often holds for fluids).

We summarize it by another commutative diagram

(12c9)



12d Examples and exercises

12d1 Exercise. (a) Treating $\mathbb{R}^2$ as the complex plane $\mathbb{C}$,

$$\mathbb{R}^2 \ni (a,b) \longleftrightarrow a + b\mathrm{i} \in \mathbb{C}\,,$$

check that

$$\overline{z}w = \langle z, w\rangle + \mathrm{i}\det(z,w) \qquad \text{for } z,w \in \mathbb{C}\,.$$

(b) A mapping $f \in C^1(\mathbb{C} \to \mathbb{C})$ leads to 1-forms

$$\operatorname{Re}(f\,dz) : (z,dz) \mapsto \operatorname{Re}\big(f(z)\,dz\big) = \langle \overline{f(z)}, dz\rangle\,,$$
$$\operatorname{Im}(f\,dz) : (z,dz) \mapsto \operatorname{Im}\big(f(z)\,dz\big) = \det(\overline{f(z)}, dz)\,.$$

Writing $f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$ we get

$$\operatorname{Re}(f\,dz) = u\,dx - v\,dy\,, \quad \operatorname{Im}(f\,dz) = v\,dx + u\,dy\,.$$

Check it.

(c) Rewrite the Cauchy-Riemann equations

$$u_x = v_y\,, \quad u_y = -v_x$$

as

$$d\operatorname{Re}(f\,dz) = 0\,, \quad d\operatorname{Im}(f\,dz) = 0\,.$$

(d) Treating $\overline{f}$ as a vector field $(x, y) \mapsto (u, -v)$ rewrite the Cauchy-Riemann equations as

$$\operatorname{curl} \overline{f} = 0 \,, \quad \operatorname{div} \overline{f} = 0 \,.$$

(e) If f is analytic, then

$$\int_\gamma \operatorname{Re}(f\,dz) = 0 \,, \quad \int_\gamma \operatorname{Im}(f\,dz) = 0$$

whenever γ is a boundary of a singular 2-box (or 2-chain) in $\mathbb{C}$. Prove it.

(f) Is (e) a necessary and sufficient condition for analyticity?

12d2 Exercise. Continuing 12d1, for an analytic f,

(a) introduce 1-forms $\operatorname{Re}(f\,d\overline{z})$, $\operatorname{Im}(f\,d\overline{z})$ and prove that[1]

$$d(f\,d\overline{z}) = -2if'\,dx \wedge dy$$

in the sense that real parts are equal and imaginary parts are equal, that is,

$$\begin{aligned} d\operatorname{Re}(f\,\overline{dz}) &= 2(\operatorname{Im} f')\,dx \wedge dy \,, \\ d\operatorname{Im}(f\,\overline{dz}) &= -2(\operatorname{Re} f')\,dx \wedge dy \,; \end{aligned}$$

here f' is the derivative as defined in complex analysis;

(b) prove that

$$\int_{\partial\Gamma} f\,d\overline{z} = -2\mathrm{i} \int_\Gamma f'\,dx \wedge dy$$

(in the sense that real and imaginary parts are equal) for every singular 2-box Γ in $\mathbb{C}$;

(c) deduce that

$$\int_0^{2\pi} f(\mathrm{e}^{\mathrm{i}t})\mathrm{e}^{-\mathrm{i}t}\,\mathrm{d}t = 2\int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta\, f'(r\mathrm{e}^{\mathrm{i}\theta}) \,;$$

calculate both sides separately for $f(z) = z^n$, $n = 0, 1, 2, \dots$.

12d3 Exercise. Continuing 12d1 and 12d2, for an analytic f, check that

(a) $df = f'\,dz$ in the sense that $d\operatorname{Re} f = \operatorname{Re}(f'\,dz)$, $d\operatorname{Im} f = \operatorname{Im}(f'\,dz)$; also, $\nabla\operatorname{Re} f = \overline{f'}$, $\nabla\operatorname{Im} f = \overline{if'}$;

(b) $\operatorname{div}\nabla\operatorname{Re} f = 0$, $\operatorname{div}\nabla\operatorname{Im} f = 0$.

[1] In some sense, $-2\mathrm{i}\,dx \wedge dy = dz \wedge \overline{dz}$.

Functions $f \in C^2(\mathbb{R}^2)$ such that $\operatorname{div} \nabla f = 0$ are called *harmonic.* By 12d3(b), the real (as well as imaginary) part of an analytic function is harmonic. The same applies to a function on an open subset of $\mathbb{R}^2$. For example, the four functions θ_i from Sect. 10d are harmonic on their domains U_i, since they are $z \mapsto \operatorname{Im} \log z$. Their (common) gradient vector field $E(z) = \frac{\mathrm{i}z}{|z|^2}$ has divergence zero (and curl zero[1]) on $\mathbb{C} \setminus \{0\}$. The function $z \mapsto \log |z|$ is harmonic on $\mathbb{C} \setminus \{0\}$, since it is $z \mapsto \operatorname{Re} \log z$. Its gradient vector field $H(z) = \frac{z}{|z|^2}$ also has divergence zero[2] (and curl zero) on $\mathbb{C} \setminus \{0\}$. Both vector fields are dual to the 1-form $\frac{1}{z}\, dz$ (that is, $\frac{-y\,dx + x\,dy}{x^2+y^2}$).

In general,

$$\operatorname{div} \nabla f = \operatorname{div}(D_1 f, D_2 f) = D_1 D_1 f + D_2 D_2 f = \Delta f \,,$$
$$\Delta = D_1 D_1 + D_2 D_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2}$$

is the so-called Laplace operator, or *Laplacian.* Thus, f is harmonic if and only if $\Delta f = 0$.

The mean value property[3] of a harmonic function u on $\mathbb{C}$:

$$u(z) = \frac{1}{2\pi} \int_0^{2\pi} u(z + r\mathrm{e}^{\mathrm{i}\theta})\, \mathrm{d}\theta \qquad \text{(12d4)}$$

for all $z \in \mathbb{C}$ and $r > 0$. The same holds for u harmonic on an open set provided that this open set contains the closed r-disk around z.

In order to prove the mean value property we need Green formulas.

Applying (12c8) to $H = \nabla u$ we get *the first Green formula*

$$\int_{\text{through}\,\partial\Gamma} \nabla u = \int_\Gamma \Delta u \quad \text{for all } u \in C^2(\mathbb{R}^2) \,. \qquad \text{(12d5)}$$

12d6 Exercise. Check that

(a) $\operatorname{div}(fH) = f \operatorname{div} H + \langle \nabla f, H \rangle$ for all $f \in C^1(\mathbb{R}^2)$ and $H \in C^1(\mathbb{R}^2 \to \mathbb{R}^2)$;

(b) $\operatorname{div}(f \nabla g) = f \Delta g + \langle \nabla f, \nabla g \rangle$ for all $f \in C^1(\mathbb{R}^2)$ and $g \in C^2(\mathbb{R}^2)$;

(c) $f \Delta g - g \Delta f = \operatorname{div}(f \nabla g - g \nabla f)$ for all $f, g \in C^2(\mathbb{R}^2)$.

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[1] Can you believe that this evidently rotating vector field has curl zero? Think about it! Also, recall 12b3.

[2] Can you believe that this evidently divergent vector field has divergence zero? Think about it!

[3] Not to be confused with the mean value theorem. . .

Now we get *the second Green formula*

(12d7)
$$\int_{\text{through }\partial\Gamma} u\nabla v = \int_\Gamma (u\Delta v + \langle \nabla u, \nabla v\rangle) \quad \text{for all } u \in C^1(\mathbb{R}^2) \text{ and } v \in C^2(\mathbb{R}^2)\,,$$

and *the third Green formula*

(12d8)
$$\int_{\text{through }\partial\Gamma} (u\nabla v - v\nabla u) = \int_\Gamma (u\Delta v - v\Delta u) \quad \text{for all } u, v \in C^2(\mathbb{R}^2)\,.$$

12d9 Exercise. (a) Let u and v be harmonic functions on an annulus $\{z \in \mathbb{C} : a < |z| < b\}$; prove that $\int_{\text{through }\gamma_r} (u\nabla v - v\nabla u)$ does not depend on $r \in (a,b)$; here $\gamma_r(t) = r\mathrm{e}^{\mathrm{i}t}$ for $t \in [0, 2\pi]$.

(b) In particular, taking $v(z) = \log|z|$, prove that

$$\int_{\text{through }\gamma_r} u\nabla v = \int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta\,;$$
$$\int_{\text{through }\gamma_r} v\nabla u = (\log r) \int_{\text{through }\gamma_r} \nabla u\,.$$

(c) Assuming in addition that u is harmonic on the disk $\{z \in \mathbb{C} : |z| < b\}$ prove that $\int_0^{2\pi} u(r\mathrm{e}^{\mathrm{i}\theta})\,\mathrm{d}\theta$ does not depend on $r \in (0, b)$ and is equal to $2\pi u(0)$, which proves (12d4).

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