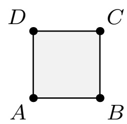
14 Higher order forms; divergence theorem

Boundary and derivative are generalized to 3*-chains and* 2*-forms, and higher. Stokes' theorem and divergence theorem are generalized accordingly.*

14a Forms of order three

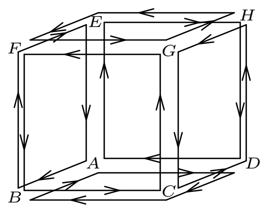
Similarly to the boundary of a singular 2-box, defined in Sect. 11d as

$$\Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} = \Gamma|_{AB} + \Gamma|_{BC} - \Gamma|_{DC} - \Gamma|_{AD} \, ,$$



we define the boundary of a singular 3-box as follows:[1]

(14a1)
$$\begin{aligned} &\Gamma|_{ADCB} + \Gamma|_{EFGH} + \Gamma|_{ABFE} + \\ &\quad + \Gamma|_{DHGC} + \Gamma|_{AEHD} + \Gamma|_{BCGF} = \\ &= - \Gamma|_{ABCD} + \Gamma|_{EFGH} - \Gamma|_{AEFB} + \\ &\quad + \Gamma|_{DHGC} - \Gamma|_{ADHE} + \Gamma|_{BCGF} \, . \end{aligned}$$



Similarly to (11d1),

(14a2)
$$\partial(\partial\Gamma) = 0 \quad \text{for a singular 3-box } \Gamma \, .$$

14a3 Exercise. Similarly to Sect. 11d, find

$$\lim_{\varepsilon\to 0+} \frac{1}{\varepsilon^3} \int_{\partial\Gamma_\varepsilon} \omega$$

where $\Gamma_\varepsilon : [0,1]^3 \to \mathbb{R}^n$, $\Gamma_\varepsilon(u_1,u_2,u_3) = x + \varepsilon u_1 h_1 + \varepsilon u_2 h_2 + \varepsilon u_3 h_3$, and ω is an arbitrary 2-form (of class C^1) on $\mathbb{R}^n$.

Answer: $\big(D_{h_1}\omega(\cdot,h_2,h_3)\big)_x + \big(D_{h_2}\omega(\cdot,h_3,h_1)\big)_x + \big(D_{h_3}\omega(\cdot,h_1,h_2)\big)_x$.

We proceed similarly to Def. 11d2.

[1] Here we rely on our geometric intuition; for a formal approach see Sect. 14c.

14a4 Definition. The *exterior derivative* of a 2-form ω of class C^1 is a 3-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, h_2, h_3) = D_{h_1}\omega(\cdot, h_2, h_3) + D_{h_2}\omega(\cdot, h_3, h_1) + D_{h_3}\omega(\cdot, h_1, h_2)\,.$$

Wedge product was defined in Sect. 11e for two 1-forms. Now we extend it.

14a5 Definition. (a) Let L_1, L_2 be linear forms on $\mathbb{R}^n$. Their *wedge product* $L_1 \wedge L_2$ is an antisymmetric bilinear form $L^{(2)}$ on $\mathbb{R}^n$ defined by

$$L^{(2)}(a,b) = L_1(a)L_2(b) - L_1(b)L_2(a) \quad \text{for all } a, b \in \mathbb{R}^n\,.$$

(b) Let $L^{(1)}$ be a linear form on $\mathbb{R}^n$, and $L^{(2)}$ an antisymmetric bilinear form on $\mathbb{R}^n$. Their wedge product $L^{(1)} \wedge L^{(2)} = L^{(2)} \wedge L^{(1)}$ is an antisymmetric trilinear form $L^{(3)}$ on $\mathbb{R}^n$ defined by

$$L^{(3)}(a,b,c) = L^{(1)}(a)L^{(2)}(b,c) + L^{(1)}(b)L^{(2)}(c,a) + L^{(1)}(c)L^{(2)}(a,b)$$

for all $a, b, c \in \mathbb{R}^n$.

(Check the antisymmetry.) This definition is suggested by determinants, as follows.

A trilinear form L on $\mathbb{R}^n$ is generally $L(a,b,c) = \sum_{i,j,k} c_{i,j,k} a_i b_j c_k$. If L is antisymmetric then

$$L = \sum_{i<j<k} c_{i,j,k} L_{i,j,k} \qquad \text{where} \quad L_{i,j,k}(a,b,c) = \begin{vmatrix} a_i & b_i & c_i \\ a_j & b_j & c_j \\ a_k & b_k & c_k \end{vmatrix}$$

(think, why). Introducing also L_i and $L_{i,j}$ by

$$L_i(a) = a_i\,, \quad L_{i,j}(a,b) = \begin{vmatrix} a_i & b_i \\ a_j & b_j \end{vmatrix}$$

we observe that $L_i \wedge L_j = L_{i,j}$ and $L_i \wedge L_{j,k} = L_{i,j,k}$ (think, why). Thus, $(L_i \wedge L_j) \wedge L_k = L_i \wedge (L_j \wedge L_k)$ (since $L_{k,i,j} = L_{i,j,k}$). Associativity follows by taking linear combinations:

$$(L_1 \wedge L_2) \wedge L_3 = L_1 \wedge (L_2 \wedge L_3) \quad \text{for all linear forms } L_1, L_2, L_3 \text{ on } \mathbb{R}^n.$$

Wedge product of differential forms is defined pointwise:

$$(\omega_1 \wedge \omega_2)(x) = \omega_1(x) \wedge \omega_2(x)\,.$$

It follows that $(f\omega_1) \wedge (g\omega_2) = (fg)(\omega_1 \wedge \omega_2)$ for $f, g \in C^0(\mathbb{R}^n)$. Note that $\omega_2 \wedge \omega_1 = \pm\omega_1 \wedge \omega_2$; the sign is minus for two 1-forms, but plus for a 1-form and 2-form. By associativity, $\omega_1 \wedge \omega_2 \wedge \omega_3$ is well-defined for three 1-forms. In particular,

$$(dx_i \wedge dx_j \wedge dx_k)(x, h_1, h_2, h_3) = L_{i,j,k}(h_1, h_2, h_3)$$

is the 3×3 determinant.

A 2-form (of class C^1) is called *closed,* if its derivative is zero. The 2-form $dx_i \wedge dx_j$ is closed, since $(dx_i \wedge dx_j)(x, h, k)$ does not depend on x.

The following two exercises are similar to (11e4) and (11e5).

14a6 Exercise. Prove that

$$d(d\omega) = 0$$

for all 1-forms ω of class C^2 on $\mathbb{R}^n$.

Thus, all exact 2-forms of class C^1 are closed. By the way, the 2-form $dx_i \wedge dx_j$ is exact by 13b18, or just because $d(x_i\,dx_j) = dx_i \wedge dx_j$ by (11e6). Moreover,

(14a7) $$df \wedge dg \text{ is exact, therefore closed, for all } f, g \in C^1(\mathbb{R}^n)\,.$$

14a8 Exercise. Prove that

$$d(f\omega) = df \wedge \omega + f\,d\omega$$

for all $f \in C^1(\mathbb{R}^n)$ and all 2-forms ω of class C^1 on $\mathbb{R}^n$.

Therefore

(14a9) $$d(f\omega) = df \wedge \omega \quad \text{whenever } \omega \text{ is closed}\,.$$

In particular, $d(f\,dx_i \wedge dx_j) = df \wedge dx_i \wedge dx_j$ for all $f \in C^1(\mathbb{R}^n)$. Similarly to 11e7 we get the following definition equivalent to 14a4.

14a10 Definition. The *exterior derivative* of a 2-form ω of class C^1 is a 3-form $d\omega$ defined by

$$d\omega = \sum_{i<j} df_{i,j} \wedge dx_i \wedge dx_j \quad \text{for } \omega = \sum_{i<j} f_{i,j}\,dx_i \wedge dx_j\,.$$

We turn to change of variables, treated in Sect. 11f for 2-forms (and 1-forms, and 0-forms). Let $\varphi \in C^1(\mathbb{R}^\ell \to \mathbb{R}^n)$. Recall the pullback $\varphi^*\omega$ defined by 11f1 for all k-forms ω on $\mathbb{R}^n$. We generalize 11f5 and 11f6 as follows.

14a11 Exercise. Prove that

$$\varphi^*(\omega_1 \wedge \omega_2 \wedge \omega_3) = (\varphi^*\omega_1) \wedge (\varphi^*\omega_2) \wedge (\varphi^*\omega_3)$$

for all 1-forms $\omega_1, \omega_2, \omega_3$ on $\mathbb{R}^n$.[1]

14a12 Lemma. For every 2-form ω of class C^1 on $\mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^\ell \to \mathbb{R}^n)$,

$$\varphi^*(d\omega) = d(\varphi^*\omega)\,.$$

Proof. We have $\omega = \sum_{i<j} f_{i,j}\,dx_i \wedge dx_j$ and $d\omega = \sum_{i<j} df_{i,j} \wedge dx_i \wedge dx_j$. It is sufficient to prove that $\varphi^*(df_{i,j} \wedge dx_i \wedge dx_j) = d\big(\varphi^*(f_{i,j}\,dx_i \wedge dx_j)\big)$. We denote

$$g_{i,j} = \varphi^* f_{i,j}\,, \quad y_i = \varphi^* x_i\,, \quad y_j = \varphi^* x_j\,.$$

By 11f4, $\varphi^*(dx_i) = dy_i$, $\varphi^*(dx_j) = dy_j$ and $\varphi^*(df_{i,j}) = dg_{i,j}$. By 11f5, $\varphi^*(dx_i \wedge dx_j) = dy_i \wedge dy_j$. By 14a11, $\varphi^*(df_{i,j} \wedge dx_i \wedge dx_j) = dg_{i,j} \wedge dy_i \wedge dy_j$. On the other hand, $d\big(\varphi^*(f_{i,j}\,dx_i \wedge dx_j)\big) = d(g_{i,j}\,dy_i \wedge dy_j) = dg_{i,j} \wedge dy_i \wedge dy_j$ by (14a7), (14a9). □

14a13 Theorem. (*Stokes' theorem for $k = 3$*)

Let C be a 3-chain in $\mathbb{R}^n$, and ω a 2-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega\,.$$

Proof. It is sufficient to prove the equality $\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$ for every singular 3-box Γ. Similarly to 11g, using (11f2) we transform the needed equality into $\int_B \Gamma^*(d\omega) = \int_{\partial B} \Gamma^*\omega$. Similarly to 11g we may assume that Γ is of class C^2. Thus, 14a12 applies, and the needed equality becomes

$$\int_B d(\Gamma^*\omega) = \int_{\partial B} \Gamma^*\omega\,.$$

Similarly to 11g it remains to prove the equality $\int_B d\omega = \int_{\partial B} \omega$ for every 2-form ω of class C^1 on the cube $B = [0,1]^3 \subset \mathbb{R}^3$; we consider only $\omega = f(u_1, u_2, u_3)\,du_1 \wedge du_2$, since the other two cases are similar.

We have $d\omega = df \wedge du_1 \wedge du_2 = \big(\frac{\partial f}{\partial u_1}\,du_1 + \frac{\partial f}{\partial u_2}\,du_2 + \frac{\partial f}{\partial u_3}\,du_3\big) \wedge du_1 \wedge du_2 = \frac{\partial f}{\partial u_3}\,du_1 \wedge du_2 \wedge du_3$, thus

$$\int_B d\omega = \int_{[0,1]^3} \frac{\partial f}{\partial u_3}\,du_1 du_2 du_3 = \iint_{[0,1]^2} du_1 du_2 \int_0^1 du_3\,\frac{\partial f}{\partial u_3} = \\ = \iint du_1 du_2 \big(f(u_1,u_2,1) - f(u_1,u_2,0)\big)\,,$$

which is equal to $\int_{\partial B} \omega$ (see (14a1)). □

[1] Hint: similar to 11f5; use the 3×3 determinant $L_{i,j,k}$.

14a14 Corollary.

$$C_1 \sim C_2 \quad \text{implies} \quad \partial C_1 \sim \partial C_2$$

for arbitrary 3-chains C_1, C_2 in $\mathbb{R}^n$. (Similar to 11h1.)

14a15 Exercise. [1] Check that

$$(y\,dx + x\,dy) \wedge (x\,dx \wedge dz + y\,dy \wedge dz) = (y^2 - x^2)\,dx \wedge dy \wedge dz\,.$$

14a16 Exercise. [2] Check that

$$d(x\,dy \wedge dz + y\,dz \wedge dx + z\,dx \wedge dy) = 3\,dx \wedge dy \wedge dz\,.$$

14a17 Exercise. [3] Prove that

$$d(\omega_1 \wedge \omega_2) = (d\omega_1) \wedge \omega_2 - \omega_1 \wedge d\omega_2$$

for arbitrary 1-forms ω_1, ω_2 on $\mathbb{R}^n$.

Thus, if ω_1 and ω_2 are closed 1-forms then $\omega_1 \wedge \omega_2$ is a closed 2-form. (Compare it with 13b18.)

14a18 Exercise. [4] Prove a generalization of the formula for integration by parts,

$$\int_C f\,d\omega = \int_{\partial C} f\omega - \int_C df \wedge \omega$$

for arbitrary 2-form ω (of class C^1) on $\mathbb{R}^n$, function $f \in C^1(\mathbb{R}^n)$, and 3-chain C in $\mathbb{R}^n$.

14b Divergence theorem in three dimensions

A 2-form ω on $\mathbb{R}^3$ corresponds to a vector field H (recall Sect. 12a), namely,

$$\omega(x, h_1, h_2) = \det(H(x), h_1, h_2)\,,$$

$$H(x) = \big(f_{2,3}(x), f_{3,1}(x), f_{1,2}(x)\big) \quad \text{for } \omega = \underbrace{f_{1,2}}_{H_3}\,dx_1 \wedge dx_2 + \underbrace{f_{2,3}}_{H_1}\,dx_2 \wedge dx_3 + \underbrace{f_{3,1}}_{H_2}\,dx_3 \wedge dx_1\,.$$

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[1] Sjamaar, p. 19.

[2] Shurman, p. 423.

[3] Shurman, Th. 9.8.2 shows that in general the sign depends on the order of ω_1.

[4] Shurman, Ex. 9.14.3.

14b1 Exercise. Let a vector field E correspond to a 1-form ω_1, and a vector field H correspond to a 2-form ω_2. Prove that

$$\omega_1 \wedge \omega_2 = \langle E, H \rangle \, dx_1 \wedge dx_2 \wedge dx_3 \, .$$

For every singular 2-box $\Gamma : B \to \mathbb{R}^3$,

$$\int_\Gamma \omega = \int_B \det\big(H(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u\big) \, \mathrm{d}u = \int_\Gamma H$$

(recall (12a7)) is the flux of H through Γ. This relation extends by linearity to 2-chains; in particular, $\int_{\partial\Gamma} \omega = \int_{\partial\Gamma} H$ is the flux of H through the boundary of a singular 3-box Γ.

The derivative $d\omega$ (assuming that ω is of class C^1), being a 3-form on $\mathbb{R}^3$, is

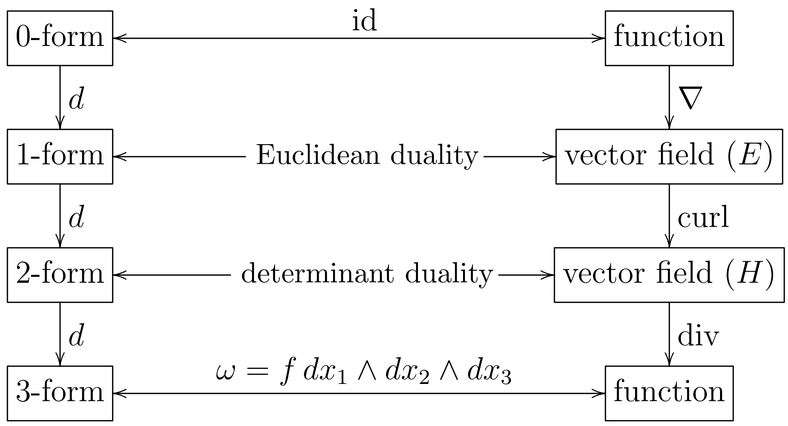
$$d\omega = f \, dx_1 \wedge dx_2 \wedge dx_3$$

for some $f \in C^0(\mathbb{R}^3)$. Taking into account that $d(H_3 \, dx_1 \wedge dx_2) = D_3 H_3 \, dx_1 \wedge dx_2 \wedge dx_3$ we get

$$d\omega = (\operatorname{div} H) \, dx_1 \wedge dx_2 \wedge dx_3 \, ,$$
$$\operatorname{div} H = D_1 H_1 + D_2 H_2 + D_3 H_3 \, .$$

Now we finalize the diagram (12a3) (see also (12c9)),

(14b2)



14b3 Exercise. [1] Prove that

$$\operatorname{div}(fH) = \langle \nabla f, H \rangle + f \operatorname{div} H$$

for all vector fields H (of class C^1) on $\mathbb{R}^3$ and all functions $f \in C^1(\mathbb{R}^3)$.[2]

[1] Zorich, (14.18).

[2] Hint: 14a8 and 14b1.

14b4 Exercise. [1] Prove that

$$\operatorname{div}(E_1 \times E_2) = \langle \operatorname{curl} E_1, E_2 \rangle - \langle E_1, \operatorname{curl} E_2 \rangle$$

for all vector fields E_1, E_2 (of class C^1) on $\mathbb{R}^3$.[2]

Theorem 14a13 gives the three-dimensional divergence theorem (recall (12c8)):

$$\int_{\partial\Gamma} H = \int_{\Gamma} \operatorname{div} H \tag{14b5}$$

for every vector field H (of class C^1) on $\mathbb{R}^3$ and every singular 3-box Γ in $\mathbb{R}^3$. Here (as in 12c) by $\int_\Gamma f$ we mean $\int_\Gamma f \, dx_1 \wedge dx_2 \wedge dx_3$.

If $\Gamma : B \to \mathbb{R}^3$ is such that $\Gamma|_{B^\circ}$ is a diffeomorphism between B° and an open set $G = \Gamma(B^\circ) \subset \mathbb{R}^3$ then

$$\int_\Gamma f(x) \, dx_1 \wedge dx_2 \wedge dx_3 = \pm \int_G f$$

(a similar fact in two dimensions was noted in Sect. 12c, before (12c6)). Assuming that $\det \mathrm{d}\Gamma > 0$ we get $\int_\Gamma (\operatorname{div} H) \, dx_1 \wedge dx_2 \wedge dx_3 = \int_G \operatorname{div} H$, and so,

$$\int_{\partial\Gamma} H = \int_{G} \operatorname{div} H \tag{14b6}$$

similarly to (12c6), (12c8).

In particular, spherical coordinates suggest a singular 3-box Γ_R that represents a ball of radius R,

$$\begin{gathered} \Gamma_R : [0,R] \times [0,\pi] \times [0, 2\pi] \to \mathbb{R}^3 \,, \\ \Gamma_R(r,\theta,\varphi) = (r \sin\theta \cos\varphi, r \sin\theta \sin\varphi, r\cos\theta) \,. \end{gathered} \tag{14b7}$$

14b8 Exercise. Prove that

$$\int_{\Gamma_R} f(x) \, dx_1 \wedge dx_2 \wedge dx_3 = \int_{B_R} f$$

for every $f \in C^0(B_R)$.[3]

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[1] Zorich, (14.19).

[2] Hint: 14a17 and 14b1.

[3] Hint: the determinant is equal to $r^2 \sin\theta$.

Rotation invariance follows (recall 6m4):

$$\Gamma_R \sim T \circ \Gamma_R$$

for every linear isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$.[1] By 14a14 it follows that

$$\partial\Gamma_R \sim T \circ \partial\Gamma_R \tag{14b9}$$

since generally $T \circ \partial\Gamma = \partial(T \circ \Gamma)$ (think, why).

14b10 Exercise. (a) Consider a radial vector field F on $\mathbb{R}^3$,

$$F(x) = f(|x|)x \,, \quad f \in C^0[0,\infty)$$

(like 13c3). Check that[2]

$$\int_{\partial\Gamma_R} F = 4\pi R^3 f(R) = 4\pi R^2 \cdot f(R)R$$

(the area of the sphere times the length of the vector).

(b) More generally, consider $F(x) = f(x)x$, $f \in C^0(\mathbb{R}^3)$; check that

$$\int_{\partial\Gamma_R} F = R\int_0^\pi d\theta \int_0^{2\pi} d\varphi \cdot R^2 \sin\theta \cdot f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)\,.$$

Postponing integration on surfaces in general, for now we define the integral of a function over the sphere ∂B_R (the boundary of the ball $B_R = \{x : |x| \le R\} \subset \mathbb{R}^3$) by

(14b11)
$$\int_{\partial B_R} f = \int_0^\pi d\theta \int_0^{2\pi} d\varphi \cdot R^2 \sin\theta \cdot f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)$$

for arbitrary continuous function f on the sphere. Note that

$$\int_{\partial B_R} 1 = 4\pi R^2\,; \quad \int_{\partial\Gamma_R} f(x)x = R\int_{\partial B_R} f\,.$$

Now we may define the *mean value* of f on the sphere as $\frac{1}{4\pi R^2}\int_{\partial B_R} f$. This could not be done via Riemann integral (proper or improper), since the sphere is a set of volume zero.

[1] In spherical coordinates this is easy to see for rotations about the z axis, but problematic for other axes.

[2] Hint: only one (out of six) face of the boundary contributes; calculate the 3×3 determinant and integrate it.

14b12 Exercise. Prove that

$$\int_{B_R} f = \int_0^R \mathrm{d}r \int_{\partial B_r} f$$

for all $f \in C^0(B_R)$.[1]

Therefore

$$\int_{\partial B_R} f = \frac{\mathrm{d}}{\mathrm{d}R} \int_{B_R} f\,;$$

rotation invariance follows:

$$\int_{\partial B_R} f = \int_{\partial B_R} T \circ f$$

for every linear isometry $T : \mathbb{R}^3 \to \mathbb{R}^3$.[2] (Compare it with (14b9).)

Similarly to Sect. 12d (before (12d4),

$$\operatorname{div} \nabla f = \Delta f\,,$$

$$\Delta = D_1 D_1 + D_2 D_2 + D_3 D_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}$$

is the *Laplacian.* Functions $f \in C^2(\mathbb{R}^3)$ such that $\Delta f = 0$ are called *harmonic.*

Similarly to (12d4) we'll prove the *mean value property* of a harmonic function u on $\mathbb{R}^3$:

$$u(0) = \frac{1}{4\pi R^2} \int_{\partial B_R} u\,; \quad u(x) = \frac{1}{4\pi R^2} \int_{\partial B_R} u(x + \cdot)\,. \tag{14b13}$$

To this end we need Green formulas (again).

Applying (14b5) to $H = \nabla u$ we get *the first Green formula* (recall (12d5))

$$\int_{\partial\Gamma} \nabla u = \int_\Gamma \Delta u \quad \text{for all } u \in C^2(\mathbb{R}^3)\,. \tag{14b14}$$

Exercise 12d6 holds in all dimensions (with the same proof):

(a) $\operatorname{div}(fH) = f \operatorname{div} H + \langle \nabla f, H \rangle$ for all $f \in C^1(\mathbb{R}^3)$ and $H \in C^1(\mathbb{R}^3 \to \mathbb{R}^3)$;

(b) $\operatorname{div}(f\nabla g) = f\Delta g + \langle \nabla f, \nabla g \rangle$ for all $f \in C^1(\mathbb{R}^3)$ and $g \in C^2(\mathbb{R}^3)$;

(c) $f\Delta g - g\Delta f = \operatorname{div}(f\nabla g - g\nabla f)$ for all $f, g \in C^2(\mathbb{R}^3)$.

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[1] Hint: first, replace B_R with Γ_R.

[2] Again, in spherical coordinates this is easy to see for rotations about the z axis, but problematic for other axes.

Similarly to (12d7), (12d8) we get *the second Green formula*

(14b15)
$$\int_{\partial\Gamma} u\nabla v = \int_\Gamma \bigl(u\Delta v + \langle \nabla u, \nabla v\rangle\bigr) \quad \text{for all } u \in C^1(\mathbb{R}^3) \text{ and } v \in C^2(\mathbb{R}^3)\,,$$

and *the third Green formula*

(14b16)
$$\int_{\partial\Gamma} (u\nabla v - v\nabla u) = \int_\Gamma (u\Delta v - v\Delta u) \quad \text{for all } u, v \in C^2(\mathbb{R}^3)\,.$$

14b17 Exercise. Similarly to Γ_R of (14b7) introduce a singular 3-box Γ_{R_1,R_2} that represents the spherical shell $\{x : R_1 \le |x| \le R_2\} \subset \mathbb{R}^3$ (given $0 < R_1 < R_2 < \infty$) and check that

$$\partial\Gamma_{R_1,R_2} \sim \partial\Gamma_{R_2} - \partial\Gamma_{R_1}\,.$$

Here is a three-dimensional counterpart of 12d9.

14b18 Exercise. (a) Let u and v be harmonic functions on a spherical shell $\{x \in \mathbb{R}^3 : a < |x| < b\}$; prove that $\int_{\partial\Gamma_R}(u\nabla v - v\nabla u)$ does not depend on $R \in (a,b)$.

(b) In particular, taking $v(z) = 1/|z|$, prove that[1]

$$\int_{\partial\Gamma_R} u\nabla v = -\frac{1}{R^2}\int_{\partial B_R} u\,;$$
$$\int_{\partial\Gamma_R} v\nabla u = \frac{1}{R}\int_{\partial\Gamma_R} \nabla u\,.$$

(c) Assuming in addition that u is harmonic on the ball $\{x \in \mathbb{R}^3 : |x| < b\}$ prove that $\frac{1}{R^2}\int_{\partial B_R} u$ does not depend on $R \in (0,b)$ and is equal to $4\pi u(0)$, which proves the first equality of (14b13); the second follows by shift.

14b19 Exercise. *(Maximum principle for harmonic functions)*

Let u be a harmonic function on a connected open set $G \subset \mathbb{R}^3$. If $\sup_{x\in G} u(x) = u(x_0)$ for some $x_0 \in G$ then u is constant.

Prove it.[2]

The mean value may be taken on the ball rather than the sphere:

(14b20)
$$u(0) = \frac{3}{4\pi R^3}\int_{B_R} u\,; \quad u(x) = \frac{3}{4\pi R^3}\int_{B_R} u(x+\cdot)\,.$$

Proof: by 14b12 and (14b13),

$$\int_{B_R} u = \int_0^R \mathrm{d}r \int_{\partial B_r} u = \int_0^R 4\pi R^2 u(0)\,\mathrm{d}r = \frac{4\pi R^3}{3}u(0)\,.$$

[1]Hint: v is harmonic by 13c4.

[2]Hint: the set $\{x_0 : u(x_0) = \sup_{x\in G} u(x)\}$ is both open and closed in G.

14b21 Proposition. *(Liouville's theorem for harmonic functions, dimension three)*

Every harmonic function $\mathbb{R}^3 \to [0, \infty)$ is constant.

Proof. *(Nelson's short proof)*

For arbitrary $x, y \in \mathbb{R}^3$ and $R > 0$ we have

$$u(x) = \frac{3}{4\pi R^3} \int_{B_R} u(x + \cdot) \le \frac{3}{4\pi R^3} \int_{B_{R+|x-y|}} u(y + \cdot) = \Big(\frac{R + |x-y|}{R} \Big)^3 u(y) \,,$$

since the R-neighborhood of x is contained in the $(R+|x-y|)$-neighborhood of y. In the limit $R \to \infty$ we get $u(x) \le u(y)$; similarly, $u(y) \le u(x)$. □

14c Order four, and higher

In dimension four (and higher) we cannot rely on our geometric intuition as much as we did in (14a1); we need a formal approach to orientation.

We introduce three types of cubes:[1]

∗ a standard k-cube is the set $[-1, 1]^k$ in $\mathbb{R}^k$;

∗ a singular k-cube in $\mathbb{R}^n$ is a C^1 mapping $[-1, 1]^k \to \mathbb{R}^n$;

∗ a geometric k-cube in $\mathbb{R}^n$ is a set $X \subset \mathbb{R}^n$ isometric to $[-1, 1]^k$.

The group[2] G_k of all isometries[3] of the standard k-cube (to itself) consists of $2^k k!$ signed permutation matrices, like $\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix}$. The determinant of such matrix is ± 1.

Accordingly, for a given geometric k-cube in $\mathbb{R}^n$ there exist $2^k k!$ isometric mappings $[-1, 1]^k \to X$. If Γ_1 is such mapping then others are $\Gamma_1 \circ T$ for $T \in G_k$; that is, they are Γ_2 such that $\Gamma_1^{-1} \circ \Gamma_2 \in G_k$. All such mappings are singular k-cubes in $\mathbb{R}^n$, not all mutually equivalent; rather,

$$\begin{aligned} \Gamma_1 \sim \Gamma_2 \quad & \text{whenever} \quad \det(\Gamma_1^{-1} \circ \Gamma_2) = 1 \,, \\ \Gamma_1 \sim -\Gamma_2 \quad & \text{whenever} \quad \det(\Gamma_1^{-1} \circ \Gamma_2) = -1 \,. \end{aligned}$$

Thus, a geometric k-cube $X \subset \mathbb{R}^n$ leads to two equivalence classes of singular k-cubes; these two equivalence classes will be called the two *orientations* of X. A k-form cannot be integrated over X unless an orientation is chosen; for the other orientation the integral is the opposite number.

[1]This time, $[-1, 1]$ is technically more convenient than $[0, 1]$.

[2]The so-called hyperoctahedral group.

[3]Called also automorphisms or congruences.

The simplest case is, $k = 1$. A geometric 1-cube in $\mathbb{R}^n$ is a straight interval $X = \{x : |A-x|+|x-B| = 2\}$ for given $A, B \in \mathbb{R}^n$, $|A-B| = 2$. An isometry $\gamma : [-1, 1] \to X$ defined by $\gamma(t) = \frac{1-t}{2}A + \frac{1+t}{2}B$ is a path; denote it just AB. Accordingly, BA is the other isometry $[-1, 1] \to X$, $t \mapsto \frac{1-t}{2}B + \frac{1+t}{2}A$. Note that $(BA)(t) = (AB)(-t)$. Clearly, $BA \sim -AB$, that is, $\int_{BA} \omega = -\int_{AB} \omega$ for all 1-forms ω on $\mathbb{R}^n$.

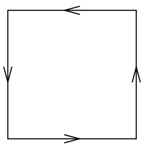
The next case is, $k = 2$. Let $X \subset \mathbb{R}^n$ be a geometric 2-cube. An isometry $\Gamma : [-1, 1]^2 \to X$ is a singular 2-cube; denote it by $ABCD$ where $A = \Gamma(-1, -1)$, $B = \Gamma(1, -1)$, $C = \Gamma(1, 1)$, $D = \Gamma(-1, 1)$; these are the vertices of X. There are 8 isometries: $ABCD$, $ADCB$, $BCDA$, $BADC$, $CDAB$, $CBAD$, $DABC$, $DCBA$; they result from $ABCD$ via elements of the group G_2. For $ABCD, BCDA, CDAB$ and $DABC$ the elements of the group are rotations by $0, \pi/2, \pi$ and $3\pi/2$, of Jacobian $+1$; for others, the elements of the group are reflections, of Jacobian -1. Thus,

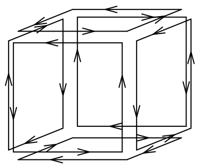
$$ABCD \sim BCDA \sim CDAB \sim DABC \quad \text{is one orientation of } X\,,$$
$$ADCB \sim BADC \sim CBAD \sim DCBA \quad \text{is the other orientation of } X\,.$$

The standard k-cube has $2k$ *hyperfaces*

$$\{(u_1, \dots, u_k) \in [-1, 1]^k : u_i = a\} \quad \text{for } i \in \{1, \dots, k\} \text{ and } a \in \{-1, 1\}\,;$$

each hyperface is a geometric $(k-1)$-cube. We want to define the boundary ∂X of the standard k-cube X as the sum $\sum_Y \tilde{Y}$ of its hyperfaces Y treated as singular $(k-1)$-cubes $\tilde{Y}$; to this end we have to choose orientations of these hyperfaces. We did it already for $k = 2, 3$.





In these two cases the chosen orientations are consistent in the following sense. For every hyperface Y and every $T \in G_k$ such that $\det T = +1$ (that is, $T(\tilde{X}) = \tilde{X}$),

$$T(\tilde{Y}) = \widetilde{T(Y)}\,.$$

This consistency is necessary for Stokes' theorem to hold, since $T(\tilde{X}) = \tilde{X}$ must imply $T(\partial\tilde{X}) = \partial\tilde{X}$ (recall 14a14).

Here is a special case of the consistency condition:

(14c1) $$\text{if } T(\tilde{X}) = \tilde{X} \text{ and } T(Y) = Y \text{ then } T(\tilde{Y}) = \tilde{Y}\,.$$

It is worth noting that such a condition fails for edges (rather than faces) of a 3-cube; here is a counterexample.

14c2 Example. Let $X = [-1,1]^3$, $Y = \{-1\}\times\{-1\}\times[-1,1]$ and $T(u_1,u_2,u_3) = (u_2,u_1,-u_3)$. Then T preserves Y and the orientation of X but does not preserve the orientation of Y.

Consider the hyperface $Y_0 = \{1\} \times [-1,1]^{k-1}$ of $[-1,1]^k$. If $T \in G_k$, $T(Y_0) = Y_0$, then

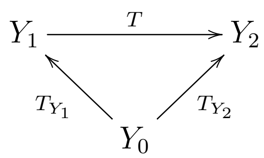
$$T = \left(\begin{array}{c|c} 1 & 0 \\ \hline 0 & T' \end{array}\right)$$

for some $T' \in G_{k-1}$. Thus, $\det T = \det T'$, which ensures (14c1) for Y_0.

Now we are in position to ensure the consistency condition in general. (This is somewhat similar to the proof of 13a1.) For each hyperface Y of $[-1,1]^k$ we choose $T_Y \in G_k$ such that $\det T_Y = +1$ and $T_Y(Y_0) = Y$. We choose an orientation of Y_0 and define

$$\tilde{Y} = T_Y(\tilde{Y}_0)$$

for all Y. Given hyperfaces Y_1, Y_2 and $T \in G_k$ such that $\det T = +1$ and $T(Y_1) = Y_2$, we have $(T_{Y_2}^{-1} \circ T \circ T_{Y_1})(Y_0) = Y_0$ and $\det(T_{Y_2}^{-1} \circ T \circ T_{Y_1}) = +1$.



Applying (14c1) to $T_{Y_2}^{-1} \circ T \circ T_{Y_1}$ and Y_0 we get $(T_{Y_2}^{-1} \circ T \circ T_{Y_1})(\tilde{Y}_0) = \tilde{Y}_0$; thus, $T\big(T_{Y_1}(\tilde{Y}_0)\big) = T_{Y_2}(\tilde{Y}_0)$, that is, $T(\tilde{Y}_1) = \tilde{Y}_2$. (Similarly to 13a1, the choice of T_Y does not really matter; think, why.)

Consistent orientations $\tilde{Y}$ are thus constructed in principle; but we need an explicit formula.

In terms of singular $(k-1)$-cubes

$$\Delta_{i,a} : [-1,1]^{k-1} \to [-1,1]^k \quad \text{for } i \in \{1,\dots,k\},\ a \in \{-1,+1\}\,,$$
$$\Delta_{i,a}(u_1,\dots,u_{k-1}) = (u_1,\dots,u_{i-1},a,u_i,\dots,u_{k-1})\,,$$

we have $\tilde{Y}_{i,a} \sim \pm\Delta_{i,a}$ where $Y_{i,a} = \{(u_1,\dots,u_k) \in [-1,1]^k : u_i = a\}$ are the hyperfaces. But what are the signs?

The sign for $Y_0 = Y_{1,+}$ is rather a matter of convention; let it be +1. That is, $\tilde{Y}_0 \sim \Delta_{1,+}$.[1] The mapping $T_{i,a} : (u_1,\dots,u_k) \mapsto (u_2,\dots,u_i,au_1,u_{i+1},\dots,u_k)$ satisfies[2]

$$T_{i,a}(Y_0) = Y_{i,a}\,; \quad T_{i,a} \circ \Delta_{1,+} = \Delta_{i,a}\,; \quad \det(T_{i,a}) = (-1)^{i-1}a\,.$$

[1]More formally, $\tilde{Y}_0 \ni \Delta_{1,+}$.

[2]Indeed, $(u_1,\dots,u_{k-1}) \overset{\Delta_{1,+}}{\mapsto} (1,u_1,\dots,u_{k-1}) \overset{T_{i,a}}{\mapsto} (u_1,\dots,u_{i-1},a,u_i,\dots,u_{k-1})$.

By the consistency condition, $T_{i,a}(\tilde{Y}_0) \sim \det(T_{i,a})\tilde{Y}_{i,a}$, that is, $\tilde{Y}_{i,a} \sim \det(T_{i,a})T_{i,a} \circ \Delta_{1,+} \sim (-1)^{i-1}a\Delta_{i,a}$.

14c3 Definition. The *boundary* of a singular k-cube $\Gamma : [-1,1]^k \to \mathbb{R}^n$ is a k-chain

$$\partial\Gamma = \sum_{i=1}^{k} \sum_{a=\pm 1} (-1)^{i-1} a(\Gamma \circ \Delta_{i,a}) \, .$$

14c4 Exercise. Check that the definitions used before for $k = 1, 2, 3$ conform to 14c3.

14c5 Exercise. Prove that $\partial(\partial\Gamma) = 0$ for all singular k-cubes Γ in $\mathbb{R}^n$.[1]

14c6 Exercise. Similarly to 14a3, find

$$\lim_{\varepsilon \to 0+} \frac{1}{(2\varepsilon)^k} \int_{\partial\Gamma_\varepsilon} \omega$$

where $\Gamma_\varepsilon : [-1,1]^k \to \mathbb{R}^n$, $\Gamma_\varepsilon(u_1,\dots,u_k) = x + \varepsilon u_1 h_1 + \dots + \varepsilon u_k h_k$, and ω is an arbitrary $(k-1)$-form (of class C^1) on $\mathbb{R}^n$.

Answer: $\sum_{i=1}^{k} (-1)^{i-1} \big(D_{h_i} \omega(\cdot, h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_k) \big)_x$.

14c7 Definition. The *exterior derivative* of a $(k-1)$-form ω of class C^1 is a k-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, \dots, h_k) = \sum_{i=1}^{k} (-1)^{i-1} D_{h_i} \omega(\cdot, h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_k) \, .$$

14c8 Theorem. (*Stokes' theorem*)

Let C be a k-chain in $\mathbb{R}^n$, and ω a $(k-1)$-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega \, .$$

I skip the proof. The general case is somewhat more technical than the case $k = 3$, but no new ideas appear in the proof. The equivalent definition 14a10 of exterior derivative becomes

$$d\omega = \sum_{i_1 < \dots < i_k} df_{i_1,\dots,i_k} \wedge dx_{i_1} \wedge \dots \wedge dx_{i_k} \quad \text{for } \omega = \sum_{i_1 < \dots < i_k} f_{i_1,\dots,i_k} dx_{i_1} \wedge \dots \wedge dx_{i_k} \, ;$$

the form $dx_{i_1} \wedge \dots \wedge dx_{i_k}$ is a determinant similar to $L_{i,j,k}$ of Sect. 14a. Still,

$$d(d\omega) = 0 \, .$$

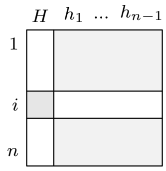
[1] Hint: you may use the idea of 14c2, if you like.

And still,

$$\varphi^*(d\omega) = d(\varphi^*\omega)\,.$$

Similarly to Sect. 14b, an $(n-1)$-form ω on $\mathbb{R}^n$ corresponds to a vector field H, namely,

$$\omega(x, h_1, \dots, h_{n-1}) = \det(H(x), h_1, \dots, h_{n-1})\,,$$
$$\omega = \sum_{i=1}^{n} (-1)^{n-1} H_i \, dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_n\,.$$



For every singular $(n-1)$-box $\Gamma : B \to \mathbb{R}^n$,

$$\int_\Gamma \omega = \int_B \det\big(H(\Gamma(u)), (D_1\Gamma)_u, \dots, (D_{n-1}\Gamma)_u\big) \, \mathrm{d}u = \int_\Gamma H$$

is the flux of H through Γ; and for an n-box Γ, $\int_{\partial\Gamma} \omega = \int_{\partial\Gamma} H$ is the flux of H through the boundary of Γ.

We have

$$d\omega = \sum_{i=1}^{n} (-1)^{n-1} dH_i \wedge dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge dx_n =$$
$$= \sum_{i=1}^{n} dx_1 \wedge \dots \wedge dx_{i-1} \wedge dH_i \wedge dx_{i+1} \wedge dx_n =$$
$$= \sum_{i=1}^{n} \frac{\partial H_i}{\partial x_i} dx_1 \wedge \dots \wedge dx_n = (\operatorname{div} H) \, dx_1 \wedge \dots \wedge dx_n\,,$$
$$\operatorname{div} H = D_1 H_1 + \dots + D_n H_n\,.$$

Thus, Th. 14c8 gives the n-dimensional divergence theorem (recall (14b1)):

$$\int_{\partial\Gamma} H = \int_\Gamma \operatorname{div} H \tag{14c9}$$

for every vector field H (of class C^1) on $\mathbb{R}^n$ and every singular n-box Γ in $\mathbb{R}^n$.

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