6 Riemann integral

One-dimensional integrals are taken over intervals, while n-dimensional integrals are taken over more complicated sets in $\mathbb{R}^n$.

> It is frequently claimed that Lebesgue integration is as easy to teach as Riemann integration. This is probably true, but I have yet to be convinced that it is as easy to learn.
>
> T.W. Körner[1]

6a What is the problem

A quote:

> As already pointed out, many of the quantities of interest in continuum mechanics represent *extensive properties,* such as mass, momentum and energy. An extensive property assigns a value to *each part of the body.* From the mathematical point of view, an extensive property can be regarded as a *set function,* in the sense

[1] "A companion to analysis: A second first and first second course in analysis", AMS 2004. (See page 197.)

> that it assigns a value to each subset of a given set. Consider, for example, the case of the mass property. Given a material body, this property assigns to each subbody its mass. Other examples of extensive properties are: volume, electric charge, internal energy, linear momentum. *Intensive properties,* on the other hand, are represented by *fields,* assigning to *each point of the body* a definite value. Examples of intensive properties are: temperature, displacement, strain.
>
> As the example of mass clearly shows, very often the extensive properties of interest are *additive set functions,* namely, the value assigned to the union of two disjoint subsets is equal to the sum of the values assigned to each subset separately. Under suitable assumptions of continuity, it can be shown that an additive set function is expressible as the integral of a *density* function over the subset of interest. This density, measured in terms of property per unit size, is an ordinary pointwise function defined over the original set. In other words, the density associated with a continuous additive set function is an intensive property. Thus, for example, the mass density is a scalar field.
>
> Marcelo Epstein[1]

We need a mathematical theory of the correspondence between set functions $\mathbb{R}^n \supset E \mapsto F(E) \in \mathbb{R}$ and (ordinary) functions $\mathbb{R}^n \ni x \mapsto f(x) \in \mathbb{R}$ via integration, $F(E) = \int_E f$. The theory should address (in particular) the following questions.

* What are admissible sets E and functions f? (Arbitrary sets are as useless here as arbitrary functions.)
* What is meant by "disjoint"?
* What is meant by integral?
* What are the general properties of the integral?
* How to calculate the integral explicitly for given f and E ?

Postponing the last question to subsequent sections, we start now the integration theory based on two postulates. First,

$$\text{(6a1)} \qquad \operatorname{vol}(B) \inf_B f \le F(B) \le \operatorname{vol}(B) \sup_B f$$

whenever B is a box (to be defined). Second,

$$\text{(6a2)} \qquad F(B_1 \cup \dots \cup B_k) = F(B_1) + \dots + F(B_k)$$

[1] "The elements of continuum biomechanics", Wiley 2012. (See Sect. 2.2.1.)

whenever a box B is split into k boxes $B_1, \dots, B_k$.

For boxes the theory is similar to the one-dimensional Riemann integration. However, two problems need additional effort:

$*$ E need not be a box (it may be a ball, a cone, etc.);

$*$ rotation invariance should be proved.

These problems do not appear in dimension one; there an (ordinary) function $F : \mathbb{R} \to \mathbb{R}$ such that $F' = f$ leads to the set function $[s,t] \mapsto F(t) - F(s) = \int_s^t f$.

6b Dimension one: reminder

[Sh:6.1,6.2]

Interval: $I = [s,t] \subset \mathbb{R}$, where $-\infty < s < t < \infty$.
Its length: $\operatorname{length}(I) = t - s$.

A partition of I: $P = \{t_0, t_1, \dots, t_k\}$ where $s = t_0 < t_1 < \dots < t_k = t$. It divides I into k subintervals: $J_j = [t_{j-1}, t_j]$ for $j = 1, \dots, k$. Alternatively, $P = \{J_1, \dots, J_k\}$.[1] It is convenient to include $k = 1$ (the trivial partition).
Additivity of length: $\operatorname{length}(I) = \operatorname{length}(J_1) + \dots + \operatorname{length}(J_k) = \sum_{J \in P} \operatorname{length}(J)$.

A refinement P' of P: a partition $P' = \{t'_0, t'_1, \dots, t'_l\}$ such that $P \subset P'$.[2] Then, $\operatorname{length}(J) = \sum_{J' \subset J, J' \in P'} \operatorname{length}(J')$ for each $J \in P$ (indeed, these J' are a partition of J).

Common refinement $P_1 \vee P_2 = P_1 \cup P_2$ of two partitions.[3]

A bounded function $f : I \to \mathbb{R}$.

Lower and upper Darboux sums:

$$L(f,P) = \sum_{J \in P} \operatorname{length}(J) \inf_J f \, ; \quad U(f,P) = \sum_{J \in P} \operatorname{length}(J) \sup_J f \, .$$

Evident:

$$L(f,P) \le U(f,P) \, ; \quad \text{that is, } [L(f,P), U(f,P)] \ne \emptyset \, .$$

Easy to see:[4] if P' is a refinement of P then

$$L(f,P) \le L(f,P') \text{ and } U(f,P) \ge U(f,P') \, ; \quad \text{that is,}$$
$$[L(f,P'), U(f,P')] \subset [L(f,P), U(f,P)] \, .$$

[1] Not quite a partition; but the tiny overlap does not matter, since it does not break additivity of length.

[2] This notation is correct for sets of points, not of subintervals. It is better to write $P \prec P'$.

[3] Again, the notation $P_1 \cup P_2$ is correct for sets of points, not of subintervals.

[4] Additivity of length is used.

Not so evident:

$$L(f,P_1) \le U(f,P_2) \quad \text{for all } P_1, P_2\,;$$

proof: $L(f,P_1) \le L(f,P_1 \cup P_2) \le U(f,P_1 \cup P_2) \le U(f,P_2)$.

Lower and upper integrals:

$$\underline{\int_I} f = L\int_I f = \sup_P L(f,P)\,; \qquad \overline{\int_I} f = U\int_I f = \inf_P U(f,P)\,.$$

Evident: $\underline{\int_I} f \le \overline{\int_I} f$.

Integrability and integral (Riemann-Darboux):

$$\underline{\int_I} f = \overline{\int_I} f = \int_I f\,.$$

The same holds in a one-dimensional Euclidean affine space instead of $\mathbb{R}$. Accordingly, the integral (as well as the lower and upper integral) is invariant under translation: for every $r \in \mathbb{R}$,

$$\int_{[s,t]} f = \int_{[s+r,t+r]} g \quad \text{where } g(u) = f(u-r)\,,$$

and reflection:

$$\int_{[s,t]} f = \int_{[-t,-s]} g \quad \text{where } g(u) = f(-u)\,.$$

6b1 Exercise. If f and F satisfy (6a1) and (6a2) then $\underline{\int_B} f \le F(B) \le \overline{\int_B} f$, and therefore $F(B) = \int_B f$ if f is integrable.

Formulate it accurately, and prove.

6b2 Exercise. Let

$$\begin{aligned} f(x) &= 1\,, \quad g(x) = 0 \quad \text{for all rational } x\,, \\ f(x) &= 0\,, \quad g(x) = 1 \quad \text{for all irrational } x\,. \end{aligned}$$

Prove that

$$\underline{\int_I}(af+bg) = \min(a,b)\,\mathrm{length}(I)\,,$$
$$\overline{\int_I}(af+bg) = \max(a,b)\,\mathrm{length}(I)$$

for all $a, b \in \mathbb{R}$ and all intervals I.

6c Higher dimensions

[Sh:6.1,6.2]

Dimension two: a box is a rectangle $[s,t] \times [u,v] \subset \mathbb{R}^2$; its area is $(t-s)(v-u)$.

Dimension n: a box is $I_1 \times \dots \times I_n \subset \mathbb{R}^n$ where $I_1, \dots, I_n \subset \mathbb{R}$ are intervals (as in Sect. 6b). Its volume: $\operatorname{vol}(B) = \prod_{j=1}^n \operatorname{length}(I_j)$. *Note that all boxes are closed and bounded.*

A partition of B: the product P of one-dimensional partitions $P_1, \dots, P_n$ of the intervals $I_1, \dots, I_n$; it divides B into $k = k_1 \dots k_n$ subboxes of the form $C = J_1 \times \dots \times J_n$ where $J_1 \in P_1, \dots, J_n \in P_n$. It is convenient to write $P = P_1 \times \dots \times P_n$.

Additivity of volume:

$$\operatorname{vol}(B) = \sum_{C \in P} \operatorname{vol}(C)\,; \tag{6c1}$$

follows from the one-dimensional additivity:

$$\sum_{C \in P} \operatorname{vol}(C) = \sum_{J_1 \in P_1, \dots, J_n \in P_n} \operatorname{length} J_1 \dots \operatorname{length} J_n =$$
$$\Big(\sum_{J_1 \in P_1} \operatorname{length}(J_1) \Big) \dots \Big(\sum_{J_n \in P_n} \operatorname{length}(J_n) \Big) = \operatorname{length}(I_1) \dots \operatorname{length}(I_n) = \operatorname{vol}(B)\,.$$

A refinement of P: $P' = P'_1 \times \dots \times P'_n$ where each P'_j is a refinement of P_j. Symbolically, $P \prec P'$. If $P \prec P'$ then

$$\operatorname{vol}(C) = \sum_{C' \subset C, C' \in P'} \operatorname{vol}(C') \quad \text{for each } C \in P \tag{6c2}$$

(indeed, these C' are a partition of C).

Common refinement $P_1 \vee P_2$ of two partitions P_1, P_2 (just the product of one-dimensional common refinements).

The rest is completely similar to Sect. 6b (with boxes and volumes instead of intervals and lengths); it is reproduced here mostly for references.

A bounded function $f : B \to \mathbb{R}$.

Lower and upper Darboux sums:

$$L(f,P) = \sum_{C \in P} \operatorname{vol}(C) \inf_C f\,; \quad U(f,P) = \sum_{C \in P} \operatorname{vol}(C) \sup_C f\,. \tag{6c3}$$

Evident:

$$L(f,P) \le U(f,P)\,; \quad \text{that is, } [L(f,P), U(f,P)] \ne \emptyset\,.$$

Easy to see (using (6c2)): if P' is a refinement of P then

$$L(f,P) \le L(f,P') \text{ and } U(f,P) \ge U(f,P') \,; \quad \text{that is,}$$
$$[L(f,P'),U(f,P')] \subset [L(f,P),U(f,P)] \,.$$

Not so evident:

$$L(f,P_1) \le U(f,P_2) \quad \text{for all } P_1, P_2 \,;$$

proof: $L(f,P_1) \le L(f,P_1 \cup P_2) \le U(f,P_1 \cup P_2) \le U(f,P_2)$.

Lower and upper integrals:

(6c4) $$\int_{*B} f = \sup_P L(f,P) \,; \qquad \int_B^* f = \inf_P U(f,P) \,.$$

Evident:

(6c5) $$\int_{*B} f \le \int_B^* f \,.$$

Integrability and integral (Riemann-Darboux):

(6c6) $$\int_{*B} f = \int_B^* f = \int_B f \,.$$

The same holds in the product $S_1 \times \dots \times S_n$ of n one-dimensional Euclidean affine spaces instead of $\mathbb{R}^n$. Accordingly, the integral (as well as the lower and upper integral) is invariant under translation: for every $r \in \mathbb{R}^n$,

(6c7) $$\int_B f = \int_{B+r} g \quad \text{where } g(u) = f(u-r) \,,$$

and reflections (of some or all the coordinates). Permutations of coordinates are also unproblematic. However, for now we cannot integrate over an arbitrary n-dimensional Euclidean space, since rotation invariance of the integral is not proved yet.

6d Basic properties of integrals

[Sh:6.2]

The constant function $c\mathbb{1}(x) = c$ is integrable, and

(6d1) $$\int_B c\mathbb{1} = c \operatorname{vol}(B) \quad \text{for all } c \in \mathbb{R} \,.$$

(Do not bother to use (6c1); just take the trivial partition P and observe that $L(f,P) = U(f,P) = c \operatorname{vol}(B)$.)

A number of properties of integrals are proved according to a pattern

(6d2)
$$
\begin{tikzcd}
\sup\limits_C f \arrow[r] & U(f,P) \arrow[r] & {}^*\!\int_B f \arrow[dr] & \\
& & & \int_B f \,. \\
\inf\limits_C f \arrow[r] & L(f,P) \arrow[r] & {}_*\!\int_B f \arrow[ur] &
\end{tikzcd}
$$

It means: an evident property of $\sup_C f$ implies the corresponding property of $U(f,P)$ and then of ${}^*\!\int_B f$ (assuming only boundedness); similarly, from $\inf_C f$ to ${}_*\!\int_B f$; and finally, *assuming integrability,* the properties of ${}^*\!\int_B f$ and ${}_*\!\int_B f$ are combined into a property of $\int_B f$.

Monotonicity:

(6d3) $$\text{if } f(\cdot) \le g(\cdot) \text{ on } B \quad \text{then} \quad {}_*\!\int_B f \le {}_*\!\int_B g \,, \quad {}^*\!\int_B f \le {}^*\!\int_B g \,,$$

(6d4) $$\text{and for integrable } f, g, \quad \int_B f \le \int_B g \,.$$

(It can happen that ${}^*\!\int_B f > {}_*\!\int_B g$; find an example.)

Homogeneity:

(6d5) $${}_*\!\int_B cf = c \, {}_*\!\int_B f \,, \quad {}^*\!\int_B cf = c \, {}^*\!\int_B f \quad \text{for } c \ge 0 \,;$$

(6d6) $${}_*\!\int_B cf = c \, {}^*\!\int_B f \,, \quad {}^*\!\int_B cf = c \, {}_*\!\int_B f \quad \text{for } c \le 0 \,;$$

(6d7) $$\text{if } f \text{ is integrable then } cf \text{ is, and} \quad \int_B cf = c \int_B f \quad \text{for all } c \in \mathbb{R} \,.$$

(Sub-, super-) additivity:

(6d8) $${}^*\!\int_B (f+g) \le {}^*\!\int_B f + {}^*\!\int_B g \,;$$

(6d9) $${}_*\!\int_B (f+g) \ge {}_*\!\int_B f + {}_*\!\int_B g \,;$$

(6d10) $$\text{if } f, g \text{ are integrable then } f+g \text{ is, and} \quad \int_B (f+g) = \int_B f + \int_B g \,.$$

(It can happen that ${}^*\!\int_B (f+g) < {}^*\!\int_B f + {}^*\!\int_B g$; find an example.)

Combining properties (6d7) and (6d10) we get linearity (for integrable functions only):

(6d11) $$\int_B (c_1 f_1 + \dots + c_k f_k) = c_1 \int_B f_1 + \dots + c_k \int_B f_k$$

for $c_1, \dots, c_k \in \mathbb{R}$ and integrable $f_1, \dots, f_k$.

Translation invariance; see (6c7).

6d12 Exercise. Prove (6d3)–(6d11).

6d13 Exercise. Prove that the set of all integrable functions is closed under uniform convergence. In other words: let $f, f_n : B \to \mathbb{R}$, $\sup_B |f_n - f| \to 0$ as $n \to \infty$. If each f_n is integrable then f is integrable.[1]

6d14 Exercise. Prove that the set of all integrable functions is not closed under pointwise convergence. In other words: let $f, f_n : B \to \mathbb{R}$, $f_n(x) \to f(x)$ (as $n \to \infty$) for every $x \in B$. It can happen that each f_n is integrable but f is not integrable (even if f is bounded).[2]

The set of all integrable functions is closed under integral convergence in the following sense.

6d15 Proposition. Let $f, f_n : B \to \mathbb{R}$ be bounded functions such that

$$\int_B^* |f_n - f| \to 0 \quad \text{as } n \to \infty .$$

Then

$$\int_{*B} f_n \to \int_{*B} f \quad \text{and} \quad \int_B^* f_n \to \int_B^* f \quad \text{as } n \to \infty .$$

If each f_n is integrable then f is integrable and $\int_B f_n \to \int_B f$.

Proof. Denote $\varepsilon_n = {}^*\!\!\int_B |f_n - f|$; $\varepsilon_n \to 0$. We have $f - f_n \le |f_n - f|$, thus ${}^*\!\!\int_B (f - f_n) \le \varepsilon_n$. Similarly, ${}^*\!\!\int_B (f_n - f) \le \varepsilon_n$, that is, ${}_*\!\!\int_B (f - f_n) \ge -\varepsilon_n$. We get

$$-\varepsilon_n \le \int_{*B} (f - f_n) \le \int_B^* (f - f_n) \le \varepsilon_n .$$

Similarly,

$$-\varepsilon_n \le \int_{*B} (f_n - f) \le \int_B^* (f_n - f) \le \varepsilon_n .$$

Taking into account that $f = f_n + (f - f_n)$ we get

$$\int_B^* f \le \int_B^* f_n + \int_B^* (f - f_n) \le \int_B^* f_n + \varepsilon_n ,$$

[1]The indicator of the Cantor set is integrable, but not the uniform limit of continuous functions, nor of step functions.

[2]Hint: try $\sum_k \mathbb{1}_{\{x_n\}}$ for a sequence $(x_n)_n$ dense in B.

and similarly ${}^*\!\!\int_B f_n \le {}^*\!\!\int_B f + \varepsilon_n$. Doing the same for the lower integral we get
$$\Big| {}_*\!\!\int_B f_n - {}_*\!\!\int_B f \Big| \le \varepsilon_n \quad \text{and} \quad \Big| {}^*\!\!\int_B f_n - {}^*\!\!\int_B f \Big| \le \varepsilon_n \, .$$
□

6d16 Exercise. For bounded $f, g : B \to \mathbb{R}$ prove that

(a) ${}^*\!\!\int_B |fg| \le \frac12 \big({}^*\!\!\int_B f^2 + {}^*\!\!\int_B g^2 \big)$;

(b) ${}^*\!\!\int_B |fg| \le \min_{c>0} \frac12 \big(c\,{}^*\!\!\int_B f^2 + \frac1c {}^*\!\!\int_B g^2 \big) = \sqrt{{}^*\!\!\int_B f^2} \sqrt{{}^*\!\!\int_B g^2}$.

6d17 Exercise. (a) For f, g as in 6b2 prove that
$$\begin{aligned}
{}_*\!\!\int_I (af+b)(cg+d) &= \big(\min(ad, bc) + bd \big) \operatorname{length}(I) \, , \\
{}_*\!\!\int_I (af+b)^2 &= \min\big((a+b)^2, b^2 \big) \operatorname{length}(I) \, , \\
{}_*\!\!\int_I (cg+d)^2 &= \min\big((c+d)^2, d^2 \big) \operatorname{length}(I)
\end{aligned}$$
for all $a, b, c, d \in \mathbb{R}$ and all intervals I.

(b) Prove existence of bounded $f, g : I \to \mathbb{R}$ such that ${}_*\!\!\int_I |fg| > \sqrt{{}_*\!\!\int_I f^2} \sqrt{{}_*\!\!\int_I g^2}$.

6d18 Exercise. For given $s_1, \dots, s_n > 0$ define $T : \mathbb{R}^n \to \mathbb{R}^n$ by $T(t_1, \dots, t_n) = (s_1 t_1, \dots, s_n t_n)$. Prove that
$$s_1 \dots s_n \, {}_*\!\!\int_{T^{-1}(B)} f \circ T = {}_*\!\!\int_B f \, , \quad s_1 \dots s_n \, {}^*\!\!\int_{T^{-1}(B)} f \circ T = {}^*\!\!\int_B f$$
for bounded $f : B \to \mathbb{R}$, and if f is integrable on B then $f \circ T$ is integrable on $T^{-1}(B)$ and
$$s_1 \dots s_n \int_{T^{-1}(B)} f \circ T = \int_B f \, .$$

6e Escaping the box

First, dimension one. Let $-\infty < r < s < t < u < \infty$, and $f : [r, u] \to \mathbb{R}$ a bounded function that vanishes outside (s, t). Then
$$ {}_*\!\!\int_{[r,u]} f = {}_*\!\!\int_{[s,t]} f \, , \quad {}^*\!\!\int_{[r,u]} f = {}^*\!\!\int_{[s,t]} f \, .$$

Proof. For every partition $P_1 = \{t_0, \dots, t_k\}$ of $[s,t]$ there exists a partition $P_2 = \{r, t_0, \dots, t_k, u\}$ of $[r,u]$ such that $U(f,P_2) = U(f,P_1)$; therefore $\int^*_{[r,u]} f \le \int^*_{[s,t]} f$.

On the other hand, let P_2 be a partition of $[r,u]$. If s and t are partition points of P_2 then the "restriction" of P_2 to $[s,t]$ is a partition P_1 of $[s,t]$ such that $U(f,P_1) = U(f,P_2)$. Otherwise, adding s and t to P_2 we get $P_2 \prec P_2'$ and then P_1 such that $U(f,P_1) = U(f,P_2') \le U(f,P_2)$. In all cases we get $\int^*_{[s,t]} f \le \int^*_{[r,u]} f$. Therefore the upper integrals are equal.

For the lower integrals we may use a similar argument; or alternatively, take the upper integrals of $(-f)$. □

Dimension n. (By B° we denote the interior of B.) Let two boxes $B_1, B_2 \subset \mathbb{R}^n$ satisfy $B_1 \subset B_2^\circ$, and $f : B_2 \to \mathbb{R}$ be a bounded function that vanishes outside B_1°. Then

(6e1) $$\int_{*B_2} f = \int_{*B_1} f \,, \qquad \int^*_{B_2} f = \int^*_{B_1} f \,.$$

Proof. We apply the one-dimensional argument to each coordinate, and consider the product of one-dimensional partitions. □

6e2 Exercise. Let $f : B_2 \to \mathbb{R}$ be a bounded function such that $f(\cdot) \ge 0$ outside $\mathrm{Int}(B_1)$. Prove that

$$\int_{*B_2} f \ge \int_{*B_1} f \,, \qquad \int^*_{B_2} f \ge \int^*_{B_1} f \,.$$

6e3 Definition. A function $f : \mathbb{R}^n \to \mathbb{R}$ has *bounded support,* if the set $\{x : f(x) \ne 0\}$ is bounded.

6e4 Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a bounded function with bounded support. Then

$$\int_{*\mathbb{R}^n} f = \int_{*B} f \,, \qquad \int^*_{\mathbb{R}^n} f = \int^*_{B} f$$

where $B \subset \mathbb{R}^n$ is an arbitrary box such that $\{x : f(x) \ne 0\} \subset B^\circ$.

By (6e1), these integrals do not depend on B. Indeed, for arbitrary B_1, B_2 there exists B such that $B_1 \subset B^\circ$ and $B_2 \subset B^\circ$.

Box-free counterparts of (6d3)–(6d11) follow readily. They hold for all bounded functions with bounded support $f, g : \mathbb{R}^n \to \mathbb{R}$ (note that $f + g$ also is such function). All integrals are taken over $\mathbb{R}^n$.

Monotonicity:

$$\text{if } f(\cdot) \le g(\cdot) \text{ everywhere} \quad \text{then} \quad \int_* f \le \int_* g \,, \quad \int^* f \le \int^* g \,, \tag{6e5}$$

$$\text{and for integrable } f, g, \quad \int f \le \int g \,. \tag{6e6}$$

Homogeneity:

$$\int_* cf = c \int_* f \,, \quad \int^* cf = c \int^* f \quad \text{for } c \ge 0 \,; \tag{6e7}$$

$$\int_* cf = c \int^* f \,, \quad \int^* cf = c \int_* f \quad \text{for } c \le 0 \,; \tag{6e8}$$

$$\text{if } f \text{ is integrable then } cf \text{ is, and} \quad \int cf = c \int f \quad \text{for all } c \in \mathbb{R} \,. \tag{6e9}$$

(Sub-, super-) additivity:

$$\int^* (f+g) \le \int^* f + \int^* g \,; \tag{6e10}$$

$$\int_* (f+g) \ge \int_* f + \int_* g \,; \tag{6e11}$$

$$\text{if } f, g \text{ are integrable then } f+g \text{ is, and} \quad \int (f+g) = \int f + \int g \,. \tag{6e12}$$

Linearity: for $c_1, \dots, c_k \in \mathbb{R}$ and integrable $f_1, \dots, f_k$,

$$\int (c_1 f_1 + \dots + c_k f_k) = c_1 \int f_1 + \dots + c_k \int f_k \,. \tag{6e13}$$

Translation invariance:

$$\int f = \int g \quad \text{where } g(u) = f(u-r) \,.$$

6e14 Exercise. For $s_1, \dots, s_n$, T as in 6d18 and integrable $f : \mathbb{R}^n \to \mathbb{R}$ prove that $f \circ T$ is integrable and

$$s_1 \dots s_n \int f \circ T = \int f \,.$$

6f Volume as Jordan measure

[Sh:6.5]

The indicator function $1\!\!1_E$ of a bounded set $E \subset \mathbb{R}^n$ evidently is a bounded function with bounded support.

6f1 Definition. Let $E \subset \mathbb{R}^n$ be a bounded set. Its *inner Jordan measure* $v_*(E)$ and *outer Jordan measure* $v^*(E)$ are

$$v_*(E) = {}_*\!\!\int_{\mathbb{R}^n} 1\!\!1_E \,, \quad v^*(E) = {}^*\!\!\int_{\mathbb{R}^n} 1\!\!1_E \,.$$

If they are equal (that is, if $1\!\!1_E$ is integrable) then E is *Jordan measurable*,[1] and its Jordan measure[2] is

$$v(E) = \int_{\mathbb{R}^n} 1\!\!1_E \,.$$

Monotonicity (follows from (6e5)):

$$E_1 \subset E_2 \quad \text{implies} \quad v_*(E_1) \le v_*(E_2) \,, \; v^*(E_1) \le v^*(E_2) \,. \tag{6f2}$$

(Sub-, super-) additivity (follows from (6e10), (6e11), (6e12) and (6e5)):

$$v^*(E_1 \cup E_2) \le v^*(E_1) + v^*(E_2) \,, \tag{6f3}$$

$$v_*(E_1 \uplus E_2) \ge v_*(E_1) + v_*(E_2) \,; \tag{6f4}$$

(6f5) if E_1, E_2 are Jordan measurable then $E_1 \uplus E_2$ is, and

$$v(E_1 \uplus E_2) = v(E_1) + v(E_2) \,.$$

Here "$\uplus$" stands for disjoint union; that is, $A \uplus B$ is just $A \cup B$ but only if $A \cap B = \emptyset$ (otherwise undefined). Thus, disjointedness is assumed in (6f4), (6f5), and implies $1\!\!1_{E_1 \uplus E_2} = 1\!\!1_{E_1} + 1\!\!1_{E_2}$.

Later[3] we'll see that Jordan measurability of E and F implies Jordan measurability of $E \cap F$, $E \cup F$ and $E \setminus F$.

Translation invariance: for every $r \in \mathbb{R}^n$,

$$v_*(E + r) = v_*(E) \,, \quad v^*(E + r) = v^*(E) \,. \tag{6f6}$$

6f7 Proposition. Every box $B \subset \mathbb{R}^n$ is Jordan measurable, and $v(B) = \mathrm{vol}(B)$.

---

[1] Or just a Jordan set.

[2] Or the n-dimensional volume, or Jordan content, or Peano-Jordan measure, etc.

[3] See 6j4, 6k6.

6f8 Lemma. For every box $B \subset \mathbb{R}^n$ and every $\varepsilon > 0$ there exist boxes B_1, B_2 such that $B_1 \subset B^\circ$, $B \subset B_2^\circ$, and $\operatorname{vol}(B_1) \ge \operatorname{vol}(B) - \varepsilon$, $\operatorname{vol}(B_2) \le \operatorname{vol}(B) + \varepsilon$.

Proof. Given $B = [s_1, t_1] \times \dots \times [s_n, t_n]$ we introduce $B_\delta = [s_1 - \delta, t_1 + \delta] \times \dots \times [s_n - \delta, t_n + \delta]$, then $\operatorname{vol}(B_\delta) = (t_1 - s_1 + 2\delta) \dots (t_n - s_n + 2\delta) \to \operatorname{vol}(B)$ as $\delta \to 0$. We take $B_2 = B_\delta$ for $\delta > 0$ small enough, and $B_1 = B_{-\delta}$ for $\delta > 0$ small enough. $\square$

Proof of Prop. 6f7. [1] Due to 6e2, $v_*(E) \ge {}_*\!\int_B 1\!\!1_E$. Taking $E = B$ and using (6d1) we get

$$v_*(B) \ge \operatorname{vol}(B) \,.$$

It is sufficient to prove that $v^*(B) \le \operatorname{vol}(B)$. We cannot take $f = 1\!\!1_B$ in 6e4, but we can take $f = 1\!\!1_{B^\circ}$, getting ${}^*\!\!\int_{\mathbb{R}^n} 1\!\!1_{B^\circ} = {}^*\!\!\int_B 1\!\!1_{B^\circ} \le \int_B 1\!\!1 = \operatorname{vol}(B)$, that is,

$$v^*(B^\circ) \le \operatorname{vol}(B) \,.$$

We apply it to B_2 such that $B \subset B_2^\circ$ and $\operatorname{vol}(B_2) \le \operatorname{vol}(B) + \varepsilon$ (such B_2 exists by 6f8), getting

$$v^*(B) \le v^*(B_2^\circ) \le \operatorname{vol}(B_2) \le \operatorname{vol}(B) + \varepsilon$$

for arbitrary $\varepsilon > 0$; therefore $v^*(B) \le \operatorname{vol}(B)$. $\square$

6f9 Exercise. For every box $B \subset \mathbb{R}^n$ its interior B° is Jordan measurable, and $v(B^\circ) = \operatorname{vol}(B)$.

Prove it.

6f10 Lemma. For every box $B \subset \mathbb{R}^n$ its boundary $\partial B = B \setminus B^\circ$ is Jordan measurable, and $v(\partial B) = 0$.

Proof. The linear combination $1\!\!1_B - 1\!\!1_{B^\circ} = 1\!\!1_{\partial B}$ of integrable functions is integrable, and $v(\partial B) = \int 1\!\!1_{\partial B} = \int 1\!\!1_B - \int 1\!\!1_{B^\circ} = \operatorname{vol}(B) - \operatorname{vol}(B) = 0$. $\square$

We define a *set of volume zero* as a bounded set $E \subset \mathbb{R}^n$ such that $v^*(E) = 0$. Equivalently: a Jordan measurable set such that $v(E) = 0$. Due to (6f2), (6f3),

(6f11) if $E \subset F$ and F is of volume zero then E is;

(6f12) if $E_1, \dots, E_k$ are of volume zero then $E_1 \cup \dots \cup E_k$ is.

---

[1] The proof is a bit tricky because the subboxes of a partition may overlap on the boundary. Some authors escape this trick by using only intervals of the form $[s, t)$ (or alternatively, only $(s, t]$) from the beginning. However, this proof is a moderate price for the classical definition; and the boundary overlap will never bother us again.

6f13 Exercise. Prove that

(a) the inner Jordan measure of a closed n-dimensional ball of radius r is $\ge \big(\frac{2r}{\sqrt{n}}\big)^n$;

(b) every nonempty open set has a non-null inner Jordan measure;

(c) the inner Jordan measure of an open n-dimensional ball of radius r is $> \big(\frac{2r}{\sqrt{n}}\big)^n$;

(d) every set of volume zero has empty interior.

6f14 Exercise. For $s_1, \dots, s_n$ and T as in 6d18, 6e14 prove that

$$v_*(T(E)) = s_1 \dots s_n v_*(E) \, , \quad v^*(T(E)) = s_1 \dots s_n v^*(E)$$

for every bounded E, and if E is Jordan measurable then $T(E)$ is Jordan measurable and $v(T(E)) = s_1 \dots s_n v(E)$. In particular, $v(sE) = s^n v(E)$.

6g Sandwiching by step functions

"Sets of volume zero are small enough that they don't interfere with integration" [Sh:p.272].

6g1 Lemma. If bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support differ only on a set of volume zero then ${}_*\!\int f = {}_*\!\int g$ and ${}^*\!\int f = {}^*\!\int g$.

Proof. We take M such that $|f(\cdot)| \le M$ and $|g(\cdot)| \le M$, note that $|f(\cdot) - g(\cdot)| \le 2M \mathbb{1}_E$ where $E = \{x : f(x) \ne g(x)\}$ is a set of volume zero, and get by (6e5), (6e7)

$$\int^* |f - g| \le 2M \int^* \mathbb{1}_E = 0 \, .$$

Taking into account that $g \le f + (g - f) \le f + |f - g|$ we get by (6e10) ${}^*\!\int g \le {}^*\!\int f$. Similarly, ${}^*\!\int f \le {}^*\!\int g$; thus ${}^*\!\int f = {}^*\!\int g$. Applying it to $(-f), (-g)$ we get ${}_*\!\int f = {}_*\!\int g$. □

Thus we may safely ignore values of integrands on sets of volume zero (as far as they are bounded). Likewise we may ignore sets of volume zero when dealing with Jordan measure.

Now we may add "outside a set of volume zero" to (6e5)–(6e13), like this:

Monotonicity: if $f(\cdot) \le g(\cdot)$ outside a set of volume zero then

$$\int_* f \le \int_* g \, , \quad \int^* f \le \int^* g \, , \tag{6g2}$$

$$\text{and for integrable } f, g, \quad \int f \le \int g \, . \tag{6g3}$$

Linearity: for $c_1,\dots,c_k \in \mathbb{R}$ and integrable $f_1,\dots,f_k$, if $f = c_1 f_1 + \dots + c_k f_k$ outside a set of volume zero then

$$\int f = c_1 \int f_1 + \dots + c_k \int f_k \,. \tag{6g4}$$

Given a partition P of a box B, we may ignore the values of a bounded function f on the union $\cup_{C\in P}\partial C$ of boundaries. Moreover, f need not be defined on $\cup_{C\in P}\partial C$. The function f_P^+ defined on $B \setminus \cup_{C\in P}\partial C$ by

$$f_P^+(x) = \sup_{C^\circ} f \quad \text{for } x \in C^\circ \,,\ C \in P$$

is integrable, and

$$\overline{\int_B} f \le \int_B f_P^+ = \sum_{C\in P} \Big(\sup_{C^\circ} f \Big) \operatorname{vol}(C) \,.$$

We modify Darboux sums accordingly (recall (6c3)),

$$L_0(f,P) = \sum_{C\in P} \operatorname{vol}(C) \inf_{C^\circ} f \,, \quad U_0(f,P) = \sum_{C\in P} \operatorname{vol}(C) \sup_{C^\circ} f \,, \tag{6g5}$$

and still (recall (6c4)),

$$\underline{\int_B} f = \sup_P L_0(f,P) \,; \quad \overline{\int_B} f = \inf_P U_0(f,P) \tag{6g6}$$

for all bounded f.

Functions on $B \setminus \cup_{C\in P}\partial C$ that are constant on each C° are called (n-dimensional) *step functions.* Note that f_P^+ is the least step function h such that $h \ge f$ on the domain of h. We see that (for all bounded f)

$$\overline{\int_B} f = \inf_{h\ge f} \int_B h \,, \quad \underline{\int_B} f = \sup_{h \le f} \int_B h \tag{6g7}$$

where h runs over all step functions, and the inequalities $h \ge f$, $h \le f$ are required on the domain of h. Thus, f is integrable on B if and only if for every $\varepsilon > 0$ there exist step functions h_1, h_2 on B such that $h_1 \le f \le h_2$ and $\int_B h_2 - \int_B h_1 \le \varepsilon$ ("sandwich").

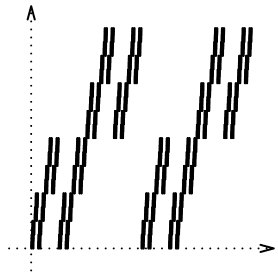
6g8 Exercise. Prove that every continuous function on a box is integrable.

6g9 Exercise. Find $\int_0^1 x \, \mathrm{d}x$ using only the theory of Sections 6b–6g. (That is, $\int_{[0,1]} f$ where $f(t) = t$.)

6g10 Exercise. Let $f : [0,1) \to [0,1)$ be defined via binary digits, by

$$f(x) = \sum_{k=1}^{\infty} \frac{\beta_{2k}(x)}{2^k} \quad \text{for } x = \sum_{k=1}^{\infty} \frac{\beta_k(x)}{2^k}\,, \ \beta_k(x) \in \{0,1\}\,, \ \liminf_k \beta_k(x) = 0\,.$$

Prove that f is integrable on $[0,1]$ and find $\int_{[0,1]} f$.[1]



6g11 Exercise. Let $f : [0,1) \times [0,1) \to [0,1)$ be defined by

$$f(x,y) = \sum_{k=1}^{\infty} \frac{\beta_k(x)}{2^{2k-1}} + \sum_{k=1}^{\infty} \frac{\beta_k(y)}{2^{2k}}$$

(where $\beta_k(\cdot)$ are as in 6g10). Prove that f is integrable on $[0,1] \times [0,1]$ and find $\int_{[0,1]\times[0,1]} f$.

6h The area under a graph

6h1 Proposition. Let $f : B \to [0,\infty)$ be an integrable function on a box $B \subset \mathbb{R}^n$, and

$$E = \{(x,t) : x \in B,\, 0 \le t \le f(x)\} \subset \mathbb{R}^{n+1}\,.$$

Then E is Jordan measurable (in $\mathbb{R}^{n+1}$), and

$$v(E) = \int_B f\,.$$

Proof. Let P be a partition of B. Consider sets

$$E_- = \bigcup_{C \in P} C \times [0, \inf_C f]\,, \quad E_+ = \bigcup_{C \in P} C \times [0, \sup_C f]\,.$$

We have $E_- \subset E \subset E_+$. The set E_+ is a finite union of boxes (in $\mathbb{R}^{n+1}$), disjoint up to sets of volume zero; by (6f5), E_+ is Jordan measurable, and

[1]Hint: split $[0,1]$ into 2^{2n} equal intervals and calculate lower and upper Darboux sums.

$v(E_+)$ is the sum of the volumes of these boxes; the same holds for E_-; namely,

$$v(E_-) = L(f,P)\,, \quad v(E_+) = U(f,P)\,.$$

The relation $E_- \subset E \subset E_+$ implies $v(E_-) \le v_*(E) \le v^*(E) \le v(E_+)$, thus $L(f,P) \le v_*(E) \le v^*(E) \le U(f,P)$, which implies ${}_*\!\int_B f \le v_*(E) \le v^*(E) \le {}^*\!\int_B f$. The rest is evident. $\square$

6h2 Exercise. For f and B as in 6h1, the graph

$$\Gamma = \{(x, f(x)) : x \in B\} \subset \mathbb{R}^{n+1}$$

is of volume zero.

Prove it.[1]

6h3 Exercise. Prove that

(a) the disk $\{x : |x| \le 1\} \subset \mathbb{R}^2$ is Jordan measurable;

(b) the ball $\{x : |x| \le 1\} \subset \mathbb{R}^n$ is Jordan measurable;

(c) for every $p > 0$ the set $E_p = \{(x_1,\dots,x_n) : |x_1|^p + \dots + |x_n|^p \le 1\} \subset \mathbb{R}^n$ is Jordan measurable;

(d) $v(E_p)$ is a strictly increasing function of p.

6h4 Exercise. For the balls $E_r = \{x : |x| \le r\} \subset \mathbb{R}^n$ prove that

(a) $v(E_r) = r^n v(E_1)$;

(b) $v(E_r) < \mathrm{e}^{-n(1-r)} v(E_1)$ for $r < 1$.

A wonder: in high dimension the volume of a ball concentrates near the sphere!

6i Sandwiching by continuous functions

Here is the so-called *Lipschitz condition* (with constant L) on a function $f : \mathbb{R}^n \to \mathbb{R}$:

(6i1) $$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x, y\,.$$

One also says that f is Lipschitz continuous (with constant L), or L-Lipschitz, etc. Also, f is Lipschitz continuous if it satisfies the Lipschitz condition with some constant. Such functions are continuous (but the converse fails). The same holds for functions on boxes and other subsets of $\mathbb{R}^n$.

[1]Hint: maybe, $\Gamma \subset E_+ \setminus E_-$?

6i2 Proposition. For every bounded function f on a box B,

$$\underline{\int_B} f = \sup_{g \le f} \int_B g \,, \qquad \overline{\int_B} f = \inf_{g \ge f} \int_B g \,,$$

where g runs over all Lipschitz functions.

6i3 Exercise. Let f be a bounded function on a box $B \subset \mathbb{R}^n$, and $L \in (0,\infty)$. Then the function f_L^+ defined by

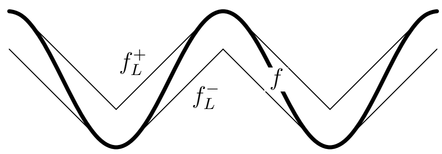
$$f_L^+(x) = \sup_{y \in B} \bigl(f(y) - L|x-y| \bigr) \quad \text{for } x \in B$$

is the least L-Lipschitz function satisfying $f_L^+ \ge f$.

Similarly, the function f_L^- defined by

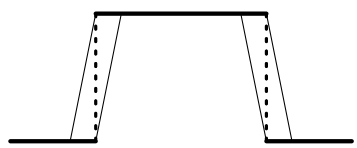
$$f_L^-(x) = \inf_{y \in B} \bigl(f(y) + L|x-y| \bigr) \quad \text{for } x \in B$$

is the greatest L-Lipschitz function satisfying $f_L^- \le f$.



6i4 Exercise.

$$(1\!\!1_E)_L^+(x) = \max\bigl(0, 1 - L \operatorname{dist}(x,E)\bigr) = \\ = 1 - \min\bigl(1, L \operatorname{dist}(x,E)\bigr) \,,$$
$$(1\!\!1_E)_L^-(x) = \min\bigl(1, L \operatorname{dist}(x, B \setminus E)\bigr)$$



for all $E \subset B$ and all $x \in B$. (Here $\operatorname{dist}(x,E) = \inf_{y \in E} |x-y|$ and $\operatorname{dist}(x,\emptyset) = +\infty$.)

Prove it.

6i5 Corollary. $(1\!\!1_E)_L^- = (1\!\!1_{E^\circ})_L^-$ and $(1\!\!1_E)_L^+ = (1\!\!1_{\overline{E}})_L^+$ for all bounded $E \subset B$.

Monotonicity (evident):

$$f \le g \quad \text{implies} \quad f_L^- \le g_L^- \text{ and } f_L^+ \le g_L^+ \,, \tag{6i6}$$
$$L_1 \le L_2 \quad \text{implies} \quad f_{L_1}^- \le f_{L_2}^- \text{ and } f_{L_1}^+ \ge f_{L_2}^+ \,. \tag{6i7}$$

6i8 Exercise. If $c \in \mathbb{R}$, boxes $B, C \subset \mathbb{R}^n$ satisfy $C \subset B$, and $f = c1\!\!1_C$, then

$$\int_B f_L^- \uparrow c \operatorname{vol}(C) \text{ and } \int_B f_L^+ \downarrow c \operatorname{vol}(C) \quad \text{as } L \to \infty \,.$$

Prove it.[1]

[1]Hint: use 6f8.

6i9 Lemma.

$$\int_B h_L^- \uparrow \int_B h \text{ and } \int_B h_L^+ \downarrow \int_B h \quad \text{as } L \to \infty$$

for every step function h on B.

Proof. We have (on the domain of h) $h = \sum_{C \in P} h(C) 1\!\!1_C$ for some partition P of B; here $h(C)$ is the (constant) value of h on C°. The function $\sum_{C \in P} (h(C) 1\!\!1_C)_L^+$ is NL-Lipschitz on B; here $N = \sum_{C \in P} 1$. This NL-Lipschitz function exceeds h, therefore it exceeds h_{NL}^+. Using (6i8),

$$\int_B h \le \int_B h_{NL}^+ \le \int_B \sum_{C \in P} (h(C) 1\!\!1_C)_L^+ = \\ \sum_{C \in P} \int_B (h(C) 1\!\!1_C)_L^+ \to \sum_{C \in P} \int_B h(C) 1\!\!1_C = \int_B h \,,$$

therefore $\int_B h_L^+ \downarrow \int_B h$. Similarly, $\int_B h_L^- \uparrow \int_B h$. $\square$

6i10 Lemma.

$$\int_B f_L^- \uparrow {}_*\!\!\int_B f \text{ and } \int_B f_L^+ \downarrow {}^*\!\!\int_B f \quad \text{as } L \to \infty$$

for every bounded function f on B.

Proof. On one hand, $\int_B f_L^+ = {}^*\!\!\int_B f_L^+ \ge {}^*\!\!\int_B f$ (by monotonicity).

On the other hand, if $h \ge f$ is a step function then $\lim_L \int_B f_L^+ \le \lim_L \int_B h_L^+ = \int_B h$; using (6g7), $\lim_L \int_B f_L^+ \le \inf_{h \ge f} \int_B h = {}^*\!\!\int_B f$.

Thus, $\lim_L \int_B f_L^+ = {}^*\!\!\int_B f$. Similarly, $\lim_L \int_B f_L^- = {}_*\!\!\int_B f$. $\square$

Thus, f is integrable on B if and only if for every $\varepsilon > 0$ there exist Lipschitz functions g_1, g_2 on B such that $g_1 \le f \le g_2$ and $\int_B g_2 - \int_B g_1 \le \varepsilon$ ("sandwich").

6i11 Exercise. Prove Prop. 6i2.

6i12 Exercise. A function f is integrable on B if and only if there exist Lipschitz functions f_n on B such that ${}^*\!\!\int_B |f_n - f| \to 0$.

Prove it.

6j Integral as additive set function

6j1 Lemma. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $\varphi(0) = 0$, and $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function. Then the function $\varphi \circ f : \mathbb{R}^n \to \mathbb{R}$ is integrable.

Proof. We take L such that φ is L-Lipschitz, and a box B such that f (as well as $\varphi \circ f$) vanishes outside B. Given $\varepsilon > 0$ we take a Lipschitz function g on B such that ${}^*\!\!\int_B |f - g| \le \varepsilon$. Then $\varphi \circ g$ is a Lipschitz function, and ${}^*\!\!\int_B |\varphi \circ f - \varphi \circ g| \le {}^*\!\!\int_B L|f - g| \le L\varepsilon$. $\square$

The same holds for a continuous (not just Lipschitz) φ, since every continuous function on a compact interval is the uniform limit of some Lipschitz functions.[1]

6j2 Exercise. Generalize 6j1 for $\varphi\big(f(\cdot), g(\cdot)\big)$ where $\varphi : \mathbb{R}^2 \to \mathbb{R}$ is a Lipschitz (or just continuous) function satisfying $\varphi(0, 0) = 0$.

6j3 Exercise. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable then $\min(f, g)$, $\max(f, g)$ and fg are integrable.

Prove it.

6j4 Exercise. If E, F are Jordan measurable then $E \cap F$, $E \cup F$ and $E \setminus F$ are Jordan measurable.

Prove it.

6j5 Definition. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function integrable on every box,[2] and $E \subset \mathbb{R}^n$ a Jordan measurable set; then

$$\int_E f = \int_{\mathbb{R}^n} f 1\!\!1_E \,.$$

Similarly to (6f5), but more generally, we have

$$\int_{E_1 \uplus E_2} f = \int_{E_1} f + \int_{E_2} f \tag{6j6}$$

whenever E_1, E_2 are Jordan measurable and disjoint.

[1] Linear interpolation...

[2] In other words, "locally integrable".

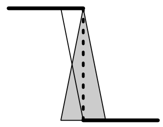
6k Miscellany

Jordan measure

Taking 6i5 into account we get

$$v_*(E) = v_*(E^\circ) \quad \text{and} \quad v^*(E) = v^*(\overline{E}) \tag{6k1}$$

for all bounded $E \subset \mathbb{R}^n$. (Now it is easy to find examples of bounded sets that are not Jordan measurable; their indicators are not integrable.)



6k2 Exercise. $(\mathbb{1}_E)^-_L + (\mathbb{1}_{\partial E})^+_L = (\mathbb{1}_E)^+_L$ for all $E \subset B$.
Prove it.[1]

6k3 Corollary. $v_*(E) + v^*(\partial E) = v^*(E)$ for all bounded $E \subset \mathbb{R}^n$.

6k4 Corollary. A bounded set is Jordan measurable if and only if its boundary is of volume zero.

Now you may compare 6h1 and 6h2 in the light of 6k4.

6k5 Exercise. $\partial(E \cap F) \subset \partial E \cup \partial F$, $\partial(E \cup F) \subset \partial E \cup \partial F$ and $\partial(E \setminus F) \subset \partial E \cup \partial F$ for all $E, F \subset \mathbb{R}^n$.

Prove it. And what about $\partial(E \cap F) \subset \partial E \cap \partial F$?

6k6 Exercise. Prove 6j4 once again, via 6k4, 6k5.

Now we can generalize (6f5):

$$v(E_1 \cup E_2) + v(E_1 \cap E_2) = v(E_1) + v(E_2) \tag{6k7}$$

for all Jordan measurable $E_1, E_2 \subset \mathbb{R}^n$. This follows from 6j4 and an evident equality

$$\mathbb{1}_{E_1 \cup E_2} + \mathbb{1}_{E_1 \cap E_2} = \mathbb{1}_{E_1} + \mathbb{1}_{E_2} \,.$$

Similarly to (6j6), the same holds for integrals.

It is less evident how to generalize (6k7) to $v(E_1 \cup E_2 \cup E_3)$. Denoting the complement $\mathbb{R}^n \setminus E$ by E^{c} and the indicator of the whole $\mathbb{R}^n$ by $\mathbb{1}$ we have

$$\begin{gathered}
\mathbb{1}_{E_1 \cup E_2 \cup E_3} = \mathbb{1} - \mathbb{1}_{(E_1 \cup E_2 \cup E_3)^{\mathrm{c}}} = \mathbb{1} - \mathbb{1}_{E_1^{\mathrm{c}} \cap E_2^{\mathrm{c}} \cap E_3^{\mathrm{c}}} = \mathbb{1} - \mathbb{1}_{E_1^{\mathrm{c}}} \mathbb{1}_{E_2^{\mathrm{c}}} \mathbb{1}_{E_3^{\mathrm{c}}} = \\
= \mathbb{1} - (\mathbb{1} - \mathbb{1}_{E_1})(\mathbb{1} - \mathbb{1}_{E_2})(\mathbb{1} - \mathbb{1}_{E_3}) = \\
= \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \mathbb{1}_{E_3} - \mathbb{1}_{E_1}\mathbb{1}_{E_2} - \mathbb{1}_{E_1}\mathbb{1}_{E_3} - \mathbb{1}_{E_2}\mathbb{1}_{E_3} + \mathbb{1}_{E_1}\mathbb{1}_{E_2}\mathbb{1}_{E_3} = \\
= \mathbb{1}_{E_1} + \mathbb{1}_{E_2} + \mathbb{1}_{E_3} - \mathbb{1}_{E_1 \cap E_2} - \mathbb{1}_{E_1 \cap E_3} - \mathbb{1}_{E_2 \cap E_3} + \mathbb{1}_{E_1 \cap E_2 \cap E_3} \,;
\end{gathered}$$

[1] Hint: use 6i4 and convexity of B.

thus, $v(E_1 \cup E_2 \cup E_3)$ equals

$$v(E_1)+v(E_2)+v(E_3)-v(E_1\cap E_2)-v(E_1\cap E_3)-v(E_2\cap E_3)+v(E_1\cap E_2\cap E_3)\,,$$

a special case of the inclusion-exclusion formula.

By 6h2, the graph of a Lipschitz function on a box (or a part of it) is of volume zero. Now consider a set of the form

$$Z_g = \{x : g(x) = 0\}$$

where $g : \mathbb{R}^n \to \mathbb{R}$ is a continuously differentiable function such that $\nabla g \ne 0$ on Z_g. By the implicit function theorem, Z_g is *locally* the graph of some continuously differentiable function of $n-1$ variables. It follows that every compact subset of Z_g is of volume zero (choose a finite subcovering of the open covering, and use (6f12)).

For example, the sphere $\{x \in \mathbb{R}^n : |x| = 1\}$ is of volume zero; and therefore the ball $\{x \in \mathbb{R}^n : |x| \le 1\}$ is Jordan measurable (as we know already, see 6h3).

Partitions of small mesh

We define the *mesh* of a partition P of a box B:

$$\operatorname{mesh}(P) = \max_{C\in P} \operatorname{diam}(C)\,,$$

where $\operatorname{diam}(C) = \max_{x,y\in C} |x-y| = \sqrt{l_1^2 + \dots + l_n^2}$ for $C = [t_1, t_1 + l_1] \times \dots \times [t_n, t_n + l_n]$.

6k8 Proposition. If f is integrable on B then

$$L(f,P) \to \int_B f \quad \text{and} \quad U(f,P) \to \int_B f \qquad \text{as } \operatorname{mesh}(P) \to 0\,.$$

That is,

$$\forall \varepsilon > 0\ \exists \delta > 0\ \forall P\ \Bigg(\operatorname{mesh}(P) \le \delta \implies \\ \Big| L(f,P) - \int_B f \Big| \le \varepsilon \ \wedge\ \Big| U(f,P) - \int_B f \Big| \le \varepsilon \Bigg)$$

(or equivalently, $U(f,P) - L(f,P) \le \varepsilon$).

6k9 Exercise. (a) If f is an L-Lipschitz function on a box B then

$$U(f,P) - L(f,P) \le L \operatorname{vol}(B) \operatorname{mesh}(P)$$

for every partition P of B; prove it.

(b) Prove Prop. 6k8.

6k10 Exercise. For every integrable $f : \mathbb{R}^n \to \mathbb{R}$,

$$\varepsilon^n \sum_{k_1,\dots,k_n \in \mathbb{Z}} f(\varepsilon k_1, \dots, \varepsilon k_n) \to \int f \quad \text{as } \varepsilon \to 0 \,.$$

Prove it.

Pixelated sandwich

Let us define[1] a closed δ-pixel as a box (cube) of the form $[\delta k_1, \delta k_1 + \delta] \times \dots \times [\delta k_n, \delta k_n + \delta]$ for $k_1, \dots, k_n \in \mathbb{Z}$, and a closed δ-pixelated set as a finite (maybe empty) union of δ-pixels. In addition, a non-closed δ-pixel is $[\delta k_1, \delta k_1 + \delta) \times \dots \times [\delta k_n, \delta k_n + \delta)$, and a non-closed δ-pixelated set is their finite union (necessarily disjoint).

6k11 Exercise. (a) For every $\varepsilon > 0$ and Jordan measurable $E \subset \mathbb{R}^n$, for all $\delta > 0$ small enough[2] there exist closed δ-pixelated sets E_-, E_+ such that $E_- \subset E \subset E_+$ and $v(E_+) - v(E_-) \le \varepsilon$.

(b) The same holds for non-closed δ-pixelated sets.

Prove it.

6l Uniqueness

We know that the set $\mathcal{J}(\mathbb{R}^n)$ of all Jordan measurable sets in $\mathbb{R}^n$ is translation invariant (as follows from (6f6)), and the Jordan measure v is a map $\mathcal{J}(\mathbb{R}^n) \to [0,\infty)$ satisfying additivity (6f5)

$$v(E_1 \uplus E_2) = v(E_1) + v(E_2)$$

and translation invariance (also follows from (6f6))

$$v(E + r) = v_(E) \,.$$

Surprisingly, these properties determine v uniquely up to a coefficient.

[1]Following Terence Tao.

[2]That is, $\delta \le \delta_{\varepsilon,E}$.

6l1 Proposition. If a map $w : \mathcal{J}(\mathbb{R}^n) \to [0,\infty)$ satisfies additivity and translation invariance then

$$\exists c \ge 0 \ \ \forall E \in \mathcal{J}(\mathbb{R}^n) \quad w(E) = cv(E)\,.$$

Proof. By translation invariance, w takes on the same value on all δ-pixels (for a given δ); here we use non-closed pixels. By additivity,

$$w\big([0,2\delta)^n\big) = 2^n w\big([0,\delta)^n\big)\,,$$

since a 2δ-pixel is the disjoint union of 2^n δ-pixels. Introducing

$$c = w\big([0,1)^n\big)$$

we get

$$w\big([0,2^{-k})^n\big) = 2^{-kn}c \quad \text{for } k = 0,1,2,\dots$$

Taking into account that $v\big([0,2^{-k})^n\big) = 2^{-kn}$ we conclude that $w(E) = cv(E)$ whenever E is a 2^{-k}-pixel and therefore, by additivity, whenever E is a 2^{-k}-pixelated set.

Additivity of w implies its monotonicity: $E \subset F$ implies $w(E) \le w(F)$ (since $w(F \setminus E) \ge 0$).

Given $\varepsilon > 0$ and a Jordan measurable E, 6k11 for k large enough gives 2^{-k}-pixelated sets E_-, E_+ such that $E_- \subset E \subset E_+$ and $v(E_+) - v(E_-) \le \varepsilon$. The interval $[w(E_-), w(E_+)] = [cv(E_-), cv(E_+)]$ contains both $w(E)$ and $cv(E)$. We see that $|w(E) - cv(E)| \le \varepsilon$ for all $\varepsilon > 0$. $\square$

6m Rotation invariance

6m1 Proposition. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry (that is, a linear operator satisfying $\forall x \ \ |T(x)| = |x|$). Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case

$$v\big(T(E)\big) = v(E)\,.$$

6m2 Lemma. $v^*(T(Q)) \le n^{n/2} v(Q)$ for every δ-pixel $Q \subset \mathbb{R}^n$.

Proof. Denoting by x the center of Q we have $|x - y| \le \frac12 \delta\sqrt{n}$, therefore $|T(x) - T(y)| \le \frac12 \delta\sqrt{n}$ for all $y \in Q$. In coordinates, $T(x) = (a_1,\dots,a_n)$, $T(y) = (b_1,\dots,b_n)$, we have $|a_k - b_k| \le \frac12 \delta\sqrt{n}$, therefore

$$T(Q) \subset [b_1 - \tfrac12\delta\sqrt{n}, b_1 + \tfrac12\delta\sqrt{n}] \times \dots \times [b_n - \tfrac12\delta\sqrt{n}, b_n + \tfrac12\delta\sqrt{n}]\,.$$

We see that $T(Q)$ is contained in a cube of volume $(\delta\sqrt{n})^n = n^{n/2} v(Q)$. $\square$

By additivity, $v^*(T(E)) \le n^{n/2} v(E)$ for every δ-pixelated set E. By 6k11 the same holds for every Jordan measurable E. In particular,

$$\text{if } E \text{ is of volume zero then } T(E) \text{ is.}$$

By 6k4, if E is Jordan measurable then $T(E)$ is; indeed, $\partial(T(E)) = T(\partial E)$ since T is a homeomorphism (recall 1d1). The same applies to T^{-1}, thus,

$$E \in \mathcal{J}(\mathbb{R}) \iff T(E) \in \mathcal{J}(\mathbb{R}) . \tag{6m3}$$

Proof of Prop. 6m1. We consider a map $w : \mathcal{J}(\mathbb{R}^n) \to [0, \infty)$ defined by

$$w(E) = v(T(E)) ;$$

it is well-defined due to (6m3), additive (indeed, v is additive, and $T(E_1 \uplus E_2) = T(E_1) \uplus T(E_2)$ since T is a bijection) and translation invariant (indeed, v is translation invariant, and $T(E+x) = T(E)+T(x)$ by linearity). Prop. 6l1 gives c such that $w(\cdot) = cv(\cdot)$. It remains to prove that $c = 1$. To this end we take the ball

$$E = \{x : |x| \le 1\} ;$$

it is Jordan measurable by 6h3, $T(E) = E$ (since T is isometric), thus

$$cv(E) = w(E) = v(T(E)) = v(E) ,$$

which implies $c = 1$ (indeed, $v(E) \ne 0$ by 6f13). □

Given an n-dimensional Euclidean vector space E, we choose a linear isometry $T : E \to \mathbb{R}^n$ and transfer the Jordan measure from $\mathbb{R}^n$ to E via T. That is, a set $A \subset E$ is Jordan measurable if $T(A) \subset \mathbb{R}^n$ is, and then $v(A) = v(T(A))$. This definition is correct by the argument used in Sect. 1d.[1] By translation invariance, the same holds for Euclidean affine spaces.

Jordan measure is well-defined on every Euclidean *fd* space, and preserved by affine isometries between these spaces.

6m4 Proposition. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded function with bounded support. Then

$$\underline{\int} f \circ T = \underline{\int} f \quad \text{and} \quad \overline{\int} f \circ T = \overline{\int} f .$$

Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case

$$\int f \circ T = \int f .$$

[1] Between 1d1 and 1d2.

Proof. First, if $h = 1\!\!1_B$ is the indicator of a box B then $h \circ T = 1\!\!1_{T^{-1}(B)}$ is integrable (since $T^{-1}(B)$ is Jordan measurable), and $\int h \circ T = v(T^{-1}(B)) = v(B) = \int h$.

Second, $\int h \circ T = \int h$ for all step functions (by linearity).

Third, by (6g7), for every $\varepsilon > 0$ there exists a step function h such that $h \ge f$ and $\int h \le {}^*\!\!\int f + \varepsilon$. We have $h \circ T \ge f \circ T$, thus, ${}^*\!\!\int f \circ T \le \int h \circ T = \int h \le {}^*\!\!\int f + \varepsilon$; it means that ${}^*\!\!\int f \circ T \le {}^*\!\!\int f$. The same holds for T^{-1}, thus ${}^*\!\!\int f \circ T = {}^*\!\!\int f$. Similarly, ${}_*\!\!\int f \circ T = {}_*\!\!\int f$. $\square$

Riemann integral is well-defined on every Euclidean ^{fd}space, and preserved by affine isometries between these spaces.

6n Linear transformation

6n1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case

$$v\big(T(E)\big) = |\det T| v(E) \,.$$

Also, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,

$$|\det T| \int_* f \circ T = \int_* f \quad \text{and} \quad |\det T| \int^* f \circ T = \int^* f \,.$$

Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case

$$|\det T| \int f \circ T = \int f \,.$$

Proof. The Singular Value Decomposition (1a2, 1c9) gives an orthonormal basis $(a_1, \dots, a_n)$ in $\mathbb{R}^n$ such that vectors $T(a_1), \dots, T(a_n)$ are orthogonal. Invertibility of T ensures that the numbers $s_k = |T(a_k)|$ do not vanish. Taking $b_k = (1/s_k)T(a_k)$ we get an orthonormal basis $(b_1, \dots, b_n)$ such that $T(a_1) = s_1 b_1, \dots, T(a_n) = s_n b_n$.

We have $s_1 \dots s_n = |\det T|$, since the singular values s_k are well-known to be square roots of the eigenvalues of T^*T (thus, $s_1 \dots s_n = \sqrt{\det(T^*T)} = \sqrt{(\det T)^2} = |\det T|$).

By the rotation invariance (Prop. 6m1) we may replace the usual basis in $\mathbb{R}^n$ with $(a_1, \dots, a_n)$ or $(b_1, \dots, b_n)$ leaving intact Jordan measure.[1] The (matrix of the) operator becomes diagonal: $T(x_1, \dots, x_n) = (s_1 x_1, \dots, s_n x_n)$. It remains to apply 6e14 and 6f14. $\square$

[1] That is, we downgrade the two copies of $\mathbb{R}^n$ into a pair of Euclidean vector spaces, choose new bases and upgrade back to two copies of $\mathbb{R}^n$.

Index