Solutions to selected exercises

2f2 Exercise. Let $U \subset \mathbb{R}^n$ be an open set, $f \in C^1(U)$, and $K \subset U$ is compact. Prove that $f\big|_K$ is a Lipschitz function; that is, $\exists M \quad \forall x, y \in K \quad |f(x) - f(y)| \le M|x - y|$.[1]

Solution. Assume the contrary: $\forall M \; \exists x, y \in K \quad |f(x)-f(y)| > M|x-y|$. Applying this to $M = 1, 2, 3, \dots$ we get $x_n, y_n \in K$ such that $|f(x_n)-f(y_n)| > n|x_n - y_n|$ for $n = 1, 2, \dots$

The function f, being continuous on the compact set K, is bounded on K; therefore $|x_n - y_n| < \frac{1}{n}|f(x_n) - f(y_n)| \to 0$ as $n \to \infty$.

By compactness, there exists a convergent subsequence; we take $n_1 < n_2 < \dots$ and $a \in K$ such that $x_{n_k} \to a$ as $k \to \infty$. Also $y_{n_k} \to a$, since $|y_{n_k} - a| \le |y_{n_k} - x_{n_k}| + |x_{n_k} - a| \to 0$.

We take a bounded convex neighborhood V of a (say, a small ball) such that $\overline{V} \subset U$. The mapping ∇f, being continuous on the compact set $\overline{V}$, is bounded on V by some M. By 2f1, $|f(x) - f(y)| \le M|x - y|$ for all $x, y \in V$.

For k large enough we have $x_{n_k}, y_{n_k} \in V$, thus $|f(x_{n_k})-f(y_{n_k})| \le M|x_{n_k} - y_{n_k}|$ in contradiction to $|f(x_{n_k}) - f(y_{n_k})| > n_k|x_{n_k} - y_{n_k}|$, $n_k \to \infty$. □

3b10 Exercise. Prove invariance of domain in dimension one.[2]

Solution. A set $U \subset \mathbb{R}$ is open; $f : U \to \mathbb{R}$ is continuous and one-to-one. We have to prove that f is an open map, that is, $f(U_1)$ is open for every open $U_1 \subset U$. It means: $f(U_1)$ is a neighborhood of $f(x)$ for every $x \in U_1$.

Given $x \in U_1$, we take $a < b$ such that $x \in (a, b) \subset [a, b] \subset U_1$. Consider the case $f(a) < f(b)$ (the other case $f(a) > f(b)$ is similar, and $f(a) \ne f(b)$ since f is one-to-one).

It is sufficient to prove that $f(x) \in \big(f(a), f(b)\big)$, since $f\big((a, b)\big) \supset \big(f(a), f(b)\big)$ by the intermediate value theorem, and $f(U_1) \supset f\big((a, b)\big)$.

Assume the contrary: $f(x) < f(a)$ or $f(x) > f(b)$ (we know that $f(x) \ne f(a)$, $f(x) \ne f(b)$ since f is one-to-one). Consider the case $f(x) < f(a)$ (the other case $f(x) > f(b)$ is similar).

Take y such that $f(x) < y < f(a) < f(b)$. By the intermediate value theorem (again), $y = f(x_1)$ for some $x_1 \in (a, x)$; and similarly, $y = f(x_2)$ for some $x_2 \in (x, b)$. Thus $x_1 < x_2$ but $f(x_1) = f(x_2)$ in contradiction to the fact that f is one-to-one. □

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[1]Hint: note that U need not be convex; assuming $f(x_n) - f(y_n) \ge n|x_n - y_n|$ take a convergent subsequence...

[2]Hint: recall 3b4.

3b12 Exercise. Taking Prop. 3b9 for granted, prove the following claim:

Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable near x_0, Df is continuous at x_0, and the operator $(Df)_{x_0}$ is one-to-one. Then there exists a bounded open neighborhood U of x_0 such that $f|_{\overline{U}}$ is a homeomorphism $\overline{U} \to f(\overline{U})$.[1]

Solution. The one-to-one linear operator $(Df)_{x_0}$ maps $\mathbb{R}^n$ onto an n-dimensional subspace $E = (Df)_{x_0}(\mathbb{R}^n)$ of $\mathbb{R}^m$ (and therefore $n \le m$). We take an orthonormal basis $(e_1, \dots, e_n)$ of E and define a linear operator $T : \mathbb{R}^m \to \mathbb{R}^n$ by $T(y) = \big(\langle y, e_1\rangle, \dots, \langle y, e_n\rangle\big)$.

Claim: the operator $T \circ (Df)_{x_0}$ (from $\mathbb{R}^n$ to itself) is one-to-one (therefore invertible). Proof: if $x \in \mathbb{R}^n$, $x \ne 0$, then $(Df)_{x_0}(x) = y \in E$, $y \ne 0$; thus, $0 \ne |y|^2 = \langle y, e_1\rangle^2 + \dots + \langle y, e_n\rangle^2 = |T(y)|^2 = |T \circ (Df)_{x_0}(x)|^2$.

A mapping $g = T \circ f$ (from $\mathbb{R}^n$ to itself) is differentiable near x_0, Dg is continuous at x_0, and the operator $(Dg)_{x_0} = T \circ (Df)_{x_0}$ is invertible. Proposition 3b9 gives a bounded open neighborhood U_0 of x_0 such that $g|_{\overline{U_0}}$ is a homeomorphism $\overline{U_0} \to g(\overline{U_0})$.

On the other hand, there exists an open neighborhood U_1 of x_0 such that f is (differentiable, therefore) continuous on $\overline{U_1}$. The set $U = U_0 \cap U_1$ is a bounded open neighborhood of x_0. It remains to prove that $f|_{\overline{U}}$ is a homeomorphism $\overline{U} \to f(\overline{U})$.

We know that $g|_{\overline{U}}$ is a homeomorphism $\overline{U} \to g(\overline{U})$, since $\overline{U} \subset \overline{U_0}$. Thus, $g|_{\overline{U}}$ is one-to-one; it follows that $f|_{\overline{U}}$ is one-to-one, since $x_1 \ne x_2 \implies g(x_1) \ne g(x_2) \implies T(f(x_1)) \ne T(f(x_2)) \implies f(x_1) \ne f(x_2)$ for $x_1, x_2 \in \overline{U}$.

We know that f is continuous on $\overline{U} \subset \overline{U_1}$, that is, $x_n \to x \implies f(x_n) \to f(x)$ for $x, x_1, x_2, \dots \in \overline{U}$. It remains to prove that $f(x_n) \to f(x) \implies x_n \to x$. We have $f(x_n) \to f(x) \implies T(f(x_n)) \to T(f(x)) \implies g(x_n) \to g(x) \implies x_n \to x$, since $g|_{\overline{U}}$ is a homeomorphism. $\square$

[1] Hint: the operator $T \circ (Df)_{x_0}$ is one-to-one for some linear $T : \mathbb{R}^m \to \mathbb{R}^n$.