2b9 Proposition. (Linearity of derivative) Let S be an affine space, V a vector space, $f,g : S \to V$, $a,b \in \mathbb{R}$, and $x_0 \in S$. If f,g are differentiable at x_0 then also $af+bg$ is, and

$$\big(D(af+bg)\big)_{x_0} = a(Df)_{x_0} + b(Dg)_{x_0} \,.$$

2b10 Proposition. (Product rule) Let S be an affine space, $f,g : S \to \mathbb{R}$, and $x_0 \in S$. If f,g are differentiable at x_0 then also fg (the pointwise product) is, and

$$\big(D(fg)\big)_{x_0} = f(x_0)(Dg)_{x_0} + g(x_0)(Df)_{x_0} \,.$$

2b12 Proposition. (Chain rule) Let S_1, S_2, S_3 be affine space s, $f : S_1 \to S_2$, $g : S_2 \to S_3$, and $x_0 \in S_1$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then $g \circ f$ is differentiable at x_0, and

$$\big(D(g\circ f)\big)_{x_0} = (Dg)_{f(x_0)} \circ (Df)_{x_0} \,.$$

2d1 Proposition. (Mean value) Assume that $x_0, h \in \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$ is differentiable at $x_0 + th$ for all $t \in (0,1)$, and continuous at x_0 and x_0+h. Then there exists $t \in (0,1)$ such that

$$f(x_0+h) - f(x_0) = (D_h f)_{x_0+th} \,.$$

2e1 Proposition. Assume that all partial derivatives of a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ exist *near* x_0 and are continuous *at* x_0. Then f is differentiable at x_0.

2f3 Lemma. Let a mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ be differentiable at x_0, and $f_1, \dots, f_m : \mathbb{R}^n \to \mathbb{R}$ be the coordinate functions of f (that is, $f(x) = \big(f_1(x), \dots, f_m(x)\big)$. Then the following two conditions are equivalent:

(a) vectors $\nabla f_1(x_0), \dots, \nabla f_m(x_0)$ are linearly independent;
(b) the linear operator $(Df)_{x_0}$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$.

$$f(x_0+h) = f(x_0) + D_h f(x_0) + \frac{1}{2!} D_h D_h f(x_0) + \dots + \frac{1}{k!} D_h^k f(x_0) + o(|h|^k) \,.$$

3b7 Proposition. Let $U \subset \mathbb{R}^n$ be open, and $f \in C^1(U \to \mathbb{R}^n)$. If the operator $(Df)_x$ is invertible for all $x \in U$ then f is open.

3b8 Lemma. Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ a continuous mapping, differentiable on U. If f is a homeomorphism $\overline{U} \to f(\overline{U})$ and the operator $(Df)_x$ is invertible for all $x \in U$ then $f|_U$ is open. (Here $\overline{U}$ is the closure of U.)

3b9 Proposition. Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable near x_0, Df is continuous at x_0, and the operator $(Df)_{x_0}$ is invertible. Then there exists a bounded open neighborhood U of x_0 such that $f|_{\overline{U}}$ is a homeomorphism $\overline{U} \to f(\overline{U})$, and f is differentiable on U, and the operator $(Df)_x$ is invertible for all $x \in U$.

3b11 Exercise. Let $U \subset \mathbb{R}^n$ be open, and $f \in C^1(U \to \mathbb{R}^m)$. If the operator $(Df)_x$ maps $\mathbb{R}^n$ *onto* $\mathbb{R}^m$ for all $x \in U$ then f is open.

3b12 Exercise. Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^m$ is differentiable near x_0, Df is continuous at x_0, and the operator $(Df)_{x_0}$ is one-to-one. Then there exists a bounded open neighborhood U of x_0 such that $f|_{\overline{U}}$ is a homeomorphism $\overline{U} \to f(\overline{U})$.

3d1 Lemma. Let $U \subset \mathbb{R}^n$ be open and bounded, $f : \overline{U} \to \mathbb{R}^n$ continuous. If f is a homeomorphism $\overline{U} \to f(\overline{U})$ with no regular boundary points then $f(U)$ is open.

3f1 Proposition. Assume that $f,g : \mathbb{R}^2 \to \mathbb{R}$ are continuously differentiable near a given point (x_0,y_0); vectors $\nabla f(x_0,y_0)$ and $\nabla g(x_0,y_0)$ are linearly independent; and $g(x_0,y_0) = 0$. Denote $z_0 = f(x_0,y_0)$. Then there exist $\varepsilon > 0$ and a path $\gamma : (z_0-\varepsilon, z_0+\varepsilon) \to \mathbb{R}^2$ such that $\gamma(z_0) = (x_0,y_0)$, $f(\gamma(t)) = t$ and $g(\gamma(t)) = 0$ for all $t \in (z_0-\varepsilon, z_0+\varepsilon)$.

3f4 Proposition. Assume that $f, g_1, g_2 : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable near a given point (x_0,y_0,z_0); vectors $\nabla f(x_0,y_0,z_0)$, $\nabla g_1(x_0,y_0,z_0)$ and $\nabla g_2(x_0,y_0,z_0)$ are linearly independent; and $g_1(x_0,y_0,z_0) = g_2(x_0,y_0,z_0) = 0$. Denote $w_0 = f(x_0,y_0,z_0)$. Then there exist $\varepsilon > 0$ and a path $\gamma : (w_0-\varepsilon, w_0+\varepsilon) \to \mathbb{R}^3$ such that $\gamma(w_0) = (x_0,y_0,z_0)$, $f(\gamma(t)) = t$ and $g_1(\gamma(t)) = g_2(\gamma(t)) = 0$ for all $t \in (w_0-\varepsilon, w_0+\varepsilon)$.

3g1 Proposition. Assume that $f,g : \mathbb{R}^3 \to \mathbb{R}$ are continuously differentiable near a given point (x_0,y_0,z_0); vectors $\nabla f(x_0,y_0,z_0)$ and $\nabla g(x_0,y_0,z_0)$ are linearly independent; and $g(x_0,y_0,z_0) = 0$. Denote $w_0 = f(x_0,y_0,z_0)$. Then there exist $\varepsilon > 0$ and a path $\gamma : (w_0-\varepsilon, w_0+\varepsilon) \to \mathbb{R}^3$ such that $\gamma(w_0) = (x_0,y_0,z_0)$, $f(\gamma(t)) = t$ and $g(\gamma(t)) = 0$ for all $t \in (w_0-\varepsilon, w_0+\varepsilon)$.

3g2 Corollary. If f, g, x_0, y_0, z_0 are as in 3g1 then (x_0,y_0,z_0) cannot be a local constrained extremum of f on Z_g.

3g3 Exercise. Generalize 3g1 and 3g2 to $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$, $1 \le m \le n-1$.

3h1 Theorem. Assume that $x_0 \in \mathbb{R}^n$, functions $f, g_1, \dots, g_m : \mathbb{R}^n \to \mathbb{R}$ are continuously differentiable near x_0, $g_1(x_0) = \dots = g_m(x_0) = 0$, and vectors $\nabla g_1(x_0), \dots, \nabla g_m(x_0)$ are linearly independent. If x_0 is a local constrained extremum of f subject to $g_1(\cdot) = \dots = g_m(\cdot) = 0$ then there exist $\lambda_1, \dots, \lambda_m \in \mathbb{R}$ such that

$$\nabla f(x_0) = \lambda_1 \nabla g_1(x_0) + \dots + \lambda_m \nabla g_m(x_0) \,.$$

$$\frac{\partial}{\partial c_k}\Big|_{c=0} f(x(c)) = \lambda_k(0) \,.$$

It means that $\lambda_k = \lambda_k(0)$ is the sensitivity of the critical value to the level c_k of the constraint $g_k(x) = c_k$.

4c1 Theorem. Assume that a mapping $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable near x_0, and the operator $(Df)_{x_0}$ is invertible. Then there exists an open neighborhood U of x_0 and an open neighborhood V of $y_0 = f(x_0)$ such that $f|_U$ is a homeomorphism $U \to V$, continuously differentiable on U, and the inverse mapping $(f|_U)^{-1} : V \to U$ is continuously differentiable on V.

$$(Dg)_y = \big((Df)_x\big)^{-1} \quad \text{for } g = (f|_U)^{-1} \,, \; y = f(x) \,.$$

4c5 Theorem. Assume that $U, V \subset \mathbb{R}^n$ are open, $f : U \to V$ is a homeomorphism, continuously differentiable, and the operator $(Df)_x$ is invertible for all $x \in U$. Then the inverse mapping $f^{-1} : V \to U$ is continuously differentiable.

4c9 Exercise. (a) Let $f : U \to V$ be as in Theorem 4c5 and in addition $f \in C^2(U)$. Then $f^{-1} \in C^2(V)$.

(b) The same for $C^k(\dots)$ where $k = 3, 4, \dots$

4d1 Proposition. Assume that $x_0 \in \mathbb{R}^n$, $f : \mathbb{R}^n \to \mathbb{R}^n$ is differentiable near x_0, Df is continuous at x_0, and the operator $T = (Df)_{x_0}$ is invertible. Then for every y near $y_0 = f(x_0)$ the iterative process

$$x_{n+1} = x_n + T^{-1}\big(y - f(x_n)\big) \quad \text{for } n = 0, 1, 2, \dots$$

is well-defined and converges to a solution x of the equation $f(x) = y$. In addition, $|x - x_0| = O(|y - y_0|)$.

5c1 Theorem. Assume that $r, c \in \{1, 2, 3, \dots\}$, $n = r+c$, $x_0 \in \mathbb{R}^r$, $y_0 \in \mathbb{R}^c$, $g : \mathbb{R}^n \to \mathbb{R}^c$ is continuously differentiable near (x_0, y_0), $g(x_0, y_0) = 0$, and the operator $B = \frac{\partial g}{\partial y}\Big|_{(x_0,y_0)}$ is invertible. Then there exist open neighborhoods U of x_0 and V of y_0 such that

(a) for every $x \in U$ there exists one and only one $y \in V$ satisfying $g(x, y) = 0$;

(b) a function $\varphi : U \to V$ defined by $g\big(x, \varphi(x)\big) = 0$ is continuously differentiable, and $(D\varphi)_{x_0} = -B^{-1}A$ where $A = \frac{\partial g}{\partial x}\Big|_{(x_0,y_0)}$.

(6d8) $$\overline{\int_B} (f + g) \le \overline{\int_B} f + \overline{\int_B} g ;$$

(6d9) $$\underline{\int_B} (f + g) \ge \underline{\int_B} f + \underline{\int_B} g ;$$

(6d10) if f, g are integrable then $f + g$ is, and $\displaystyle\int_B (f + g) = \int_B f + \int_B g$.

6d15 Proposition. Let $f, f_n : B \to \mathbb{R}$ be bounded functions such that

$$\overline{\int_B} |f_n - f| \to 0 \quad \text{as } n \to \infty .$$

Then

$$\underline{\int_B} f_n \to \underline{\int_B} f \quad \text{and} \quad \overline{\int_B} f_n \to \overline{\int_B} f \quad \text{as } n \to \infty .$$

If each f_n is integrable then f is integrable and $\int_B f_n \to \int_B f$.

(6f1) $$v_*(E) = \underline{\int_{\mathbb{R}^n}} \mathbb{1}_E , \quad v^*(E) = \overline{\int_{\mathbb{R}^n}} \mathbb{1}_E , \quad v(E) = \int_{\mathbb{R}^n} \mathbb{1}_E .$$

(6f3) $$v^*(E_1 \cup E_2) \le v^*(E_1) + v^*(E_2) ,$$

(6f4) $$v_*(E_1 \uplus E_2) \ge v_*(E_1) + v_*(E_2) ;$$

(6f5) if E_1, E_2 are Jordan measurable then $E_1 \uplus E_2$ is, and
$$v(E_1 \uplus E_2) = v(E_1) + v(E_2) .$$

6g1 Lemma. If bounded functions $f, g : \mathbb{R}^n \to \mathbb{R}$ with bounded support differ only on a set of volume zero then $\underline{\int} f = \underline{\int} g$ and $\overline{\int} f = \overline{\int} g$.

(6g7) $$\overline{\int_B} f = \inf_{h \ge f} \int_B h , \qquad \underline{\int_B} f = \sup_{h \le f} \int_B h$$

where h runs over all step functions, and the inequalities $h \ge f$, $h \le f$ are required on the domain of h.

6h1 Proposition. Let $f : B \to [0, \infty)$ be an integrable function on a box $B \subset \mathbb{R}^n$, and

$$E = \{(x, t) : x \in B,\, 0 \le t \le f(x)\} \subset \mathbb{R}^{n+1} .$$

Then E is Jordan measurable (in $\mathbb{R}^{n+1}$), and $v(E) = \displaystyle\int_B f$.

(6i1) $$|f(x) - f(y)| \le L|x - y| \quad \text{for all } x, y .$$

6i2 Proposition. For every bounded function f on a box B,

$$\underline{\int_B} f = \sup_{g \le f} \int_B g , \qquad \overline{\int_B} f = \inf_{g \ge f} \int_B g ,$$

where g runs over all Lipschitz functions.

(6i3) $$f_L^+(x) = \sup_{y \in B} \big(f(y) - L|x - y|\big) \quad \text{for } x \in B$$
$$f_L^-(x) = \inf_{y \in B} \big(f(y) + L|x - y|\big) \quad \text{for } x \in B$$

(6i4) $$(\mathbb{1}_E)_L^+(x) = \max\big(0, 1 - L \operatorname{dist}(x, E)\big) = 1 - \min\big(1, L \operatorname{dist}(x, B \setminus E)\big) ,$$
$$(\mathbb{1}_E)_L^-(x) = \min\big(1, L \operatorname{dist}(x, B \setminus E)\big)$$

6i9 Lemma.

$$\int_B h_L^- \uparrow \int_B h \text{ and } \int_B h_L^+ \downarrow \int_B h \quad \text{as } L \to \infty$$

for every step function h on B.

6i10 Lemma.

$$\int_B f_L^- \uparrow \underline{\int_B} f \text{ and } \int_B f_L^+ \downarrow \overline{\int_B} f \quad \text{as } L \to \infty$$

for every bounded function f on B.

6i12 Exercise. A function f is integrable on B if and only if there exist Lipschitz functions f_n on B such that ${}^*\!\!\int_B |f_n - f| \to 0$.

6j1 Lemma. Let $\varphi : \mathbb{R} \to \mathbb{R}$ be a Lipschitz function satisfying $\varphi(0) = 0$, and $f : \mathbb{R}^n \to \mathbb{R}$ an integrable function. Then the function $\varphi \circ f : \mathbb{R}^n \to \mathbb{R}$ is integrable.

6j3 Exercise. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable then $\min(f,g)$, $\max(f,g)$ and fg are integrable.

6j4 Exercise. If E, F are Jordan measurable then $E \cap F$, $E \cup F$ and $E \setminus F$ are Jordan measurable.

(6j5)
$$\int_E f = \int_{\mathbb{R}^n} f \mathbb{1}_E \,.$$

(6j6)
$$\int_{E_1 \uplus E_2} f = \int_{E_1} f + \int_{E_2} f$$

whenever E_1, E_2 are Jordan measurable and disjoint.

6k3 Corollary. $v_*(E) + v^*(\partial E) = v^*(E)$ for all bounded $E \subset \mathbb{R}^n$.

(6k7)
$$v(E_1 \cup E_2) + v(E_1 \cap E_2) = v(E_1) + v(E_2)$$

6k8 Proposition. If f is integrable on B then
$$L(f,P) \to \int_B f \quad \text{and} \quad U(f,P) \to \int_B f \qquad \text{as } \operatorname{mesh}(P) \to 0 \,.$$

6k10 Exercise. For every integrable $f : \mathbb{R}^n \to \mathbb{R}$,
$$\varepsilon^n \sum_{k_1,\dots,k_n \in \mathbb{Z}} f(\varepsilon k_1, \dots, \varepsilon k_n) \to \int f \quad \text{as } \varepsilon \to 0 \,.$$

6k11 Exercise. (a) For every $\varepsilon > 0$ and Jordan measurable $E \subset \mathbb{R}^n$, for all $\delta > 0$ small enough there exist closed δ-pixelated sets E_-, E_+ such that $E_- \subset E \subset E_+$ and $v(E_+) - v(E_-) \le \varepsilon$.

(b) The same holds for non-closed δ-pixelated sets.

6l1 Proposition. If a map $w : \mathcal{J}(\mathbb{R}^n) \to [0, \infty)$ satisfies additivity and translation invariance then
$$\exists c \ge 0 \;\; \forall E \in \mathcal{J}(\mathbb{R}^n) \quad w(E) = cv(E) \,.$$

6m1 Proposition. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry (that is, a linear operator satisfying $\forall x \;\; |T(x)| = |x|$). Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case
$$v\big(T(E)\big) = v(E) \,.$$

6m4 Proposition. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded function with bounded support. Then
$$\sideset{_*}{}\int f \circ T = \sideset{_*}{}\int f \quad \text{and} \quad \sideset{}{^*}\int f \circ T = \sideset{}{^*}\int f \,.$$

Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case
$$\int f \circ T = \int f \,.$$

6n1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case
$$v\big(T(E)\big) = |\det T| v(E) \,.$$

Also, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support,
$$|\det T| \sideset{_*}{}\int f \circ T = \sideset{_*}{}\int f \quad \text{and} \quad |\det T| \sideset{}{^*}\int f \circ T = \sideset{}{^*}\int f \,.$$

Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case
$$|\det T| \int f \circ T = \int f \,.$$

7b1 Proposition. Let $f : B \to \mathbb{R}$ be a Lipschitz function on a box $B = I_1 \times I_2 \subset \mathbb{R}^2$. Then

(a) for every $x \in I_1$ the function f_x is Lipschitz continuous on I_2;

(b) the function $x \mapsto \int_{I_2} f_x$ is Lipschitz continuous on I_1;

(c)
$$\int_B f = \int_{I_1} \bigg(x \mapsto \int_{I_2} f_x \bigg) \,.$$

7b3 Proposition. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and a Lipschitz function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then

(a) for every $x \in B_1$ the function f_x is Lipschitz continuous on B_2;

(b) the function $x \mapsto \int_{B_2} f_x$ is Lipschitz continuous on B_1;

(c)
$$\int_B f = \int_{B_1} \bigg(x \mapsto \int_{B_2} f_x \bigg) \,.$$

7b5 Exercise.
$$\int_{B_1 \times B_2} f(x_1, \dots, x_m) g(y_1, \dots, y_n) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \, \mathrm{d}y_1 \dots \mathrm{d}y_n =$$
$$= \bigg(\int_{B_1} f(x_1, \dots, x_m) \, \mathrm{d}x_1 \dots \mathrm{d}x_m \bigg) \bigg(\int_{B_2} g(y_1, \dots, y_n) \, \mathrm{d}y_1 \dots \mathrm{d}y_n \bigg)$$

for Lipschitz functions $f : B_1 \to \mathbb{R}$, $g : B_2 \to \mathbb{R}$.

7d1 Theorem. Let two boxes $B_1 \subset \mathbb{R}^m$, $B_2 \subset \mathbb{R}^n$ be given, and an integrable function f on a box $B = B_1 \times B_2 \subset \mathbb{R}^{m+n}$. Then the iterated integrals
$$\int_{B_1} \mathrm{d}x \sideset{_*}{}\int_{B_2} \mathrm{d}y \, f(x,y) \,, \qquad \int_{B_1} \mathrm{d}x \sideset{}{^*}\int_{B_2} \mathrm{d}y \, f(x,y) \,,$$
$$\int_{B_2} \mathrm{d}y \sideset{_*}{}\int_{B_1} \mathrm{d}x \, f(x,y) \,, \qquad \int_{B_2} \mathrm{d}y \sideset{}{^*}\int_{B_1} \mathrm{d}x \, f(x,y)$$

are well-defined and equal to
$$\iint_B f(x,y) \, \mathrm{d}x\mathrm{d}y \,.$$

7d3 Exercise. Generalize 7b5 to integrable functions

(a) assuming integrability of the function $(x,y) \mapsto f(x)g(y)$,

(b) deducing integrability of the function $(x,y) \mapsto f(x)g(y)$ from integrability of f and g (via sandwich).

7d4 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^n$ are Jordan measurable sets then the set $E = E_1 \times E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

7d5 Exercise. If $E_1 \subset \mathbb{R}^m$ and $E_2 \subset \mathbb{R}^{m+n}$ are Jordan measurable sets then the set $E = \{(x,y) \in E_2 : x \in E_1\} = (E_1 \times \mathbb{R}^n) \cap E_2 \subset \mathbb{R}^{m+n}$ is Jordan measurable.

7d6 Corollary. Let $f : \mathbb{R}^{m+n} \to \mathbb{R}$ be integrable on every box, and $E \subset \mathbb{R}^{m+n}$ a Jordan measurable set; then

$$\int_E f = \int_{\mathbb{R}^m} \Big(x \mapsto \int_{E_x} f_x \Big)$$

where $E_x = \{y : (x,y) \in E\} \subset \mathbb{R}^n$ for $x \in \mathbb{R}^m$.

7d7 Corollary. (Cavalieri) If Jordan measurable sets $E, F \subset \mathbb{R}^3$ satisfy $v_2(E_x) = v_2(F_x)$ for all x then $v_3(E) = v_3(F)$.

7e1 Theorem. Let $B \subset \mathbb{R}^n$ be a box, and $f, g : B \times [0,1] \to \mathbb{R}$ Lipschitz functions such that $f'_x(t) = g_x(t)$ for all $x \in B$, $t \in (0,1)$. Then $F'(t) = G(t)$ for all $t \in (0,1)$, where $F(t) = \int_B f(x,t)\,\mathrm{d}x$ and $G(t) = \int_B g(x,t)\,\mathrm{d}x$.

8a2 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $E \subset U$. Then the following two conditions are equivalent.

(a) E is Jordan measurable and contained in a compact subset of U;

(b) $\varphi(E)$ is Jordan measurable and contained in a compact subset of V.

8a5 Theorem. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, $E \subset U$ a Jordan measurable set contained in a compact subset of U, and $f : \varphi(E) \to \mathbb{R}$ an integrable function. Then $f \circ \varphi : E \to \mathbb{R}$ is integrable, and

$$\int_{\varphi(E)} f = \int_E (f \circ \varphi) |\det D\varphi| \, .$$

8a6 Corollary. If, in addition, U and V are Jordan measurable and $D\varphi$ is bounded on U then integrability of $f : V \to \mathbb{R}$ implies integrability of $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$, and

$$\int_V f = \int_U (f \circ \varphi) |\det D\varphi| \, .$$

8b8 Proposition. (Pappus) Let $\Omega \subset (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$ be a Jordan measurable set and $\tilde\Omega = \{(x,y,z) : \big(\sqrt{x^2+y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde\Omega$ is Jordan measurable, and

$$v_3(\tilde\Omega) = v_2(\Omega) \cdot 2\pi x_{C_E} \, ;$$

here $C_E = (x_{C_E}, y_{C_E}, z_{C_E})$ is the centroid of E.

8d1 Proposition. If $F : B \mapsto \int_B f$ for a locally integrable function $f : \mathbb{R}^n \to \mathbb{R}$, then the three functions ${}_*F'$, f, ${}^*F'$ are (pairwise) equivalent.

8e1 Proposition. (a) If an additive box function F is differentiable on a box B then

$$v(B) \inf_{x \in B} F'(x) \le F(B) \le v(B) \sup_{x \in B} F'(x) \, .$$

(b) For every additive box function F,

$$v(B) \inf_{x \in B} {}_*F'(x) \le F(B) \le v(B) \sup_{x \in B} {}^*F'(x) \, .$$

(8e4) $$F(B) = \int_B F'$$

whenever F' exists and is integrable on B.

8e5 Exercise.

$$\underset{*}{\int_B} {}_*F' \le F(B) \le \overset{*}{\int_B} {}^*F'$$

for every box B and additive box function F such that ${}_*F'$ and ${}^*F'$ are bounded on B.

(8f1) $$F_*(B) = v_*(\varphi^{-1}(B^\circ)) \, , \quad F^*(B) = v^*(\varphi^{-1}(B)) \, ,$$

(8f2) $$J^-(x) = \liminf_{B \to x} \frac{F_*(B)}{v(B)} \, , \quad J^+(x) = \limsup_{B \to x} \frac{F^*(B)}{v(B)} \, .$$

8f3 Proposition. If J^-, J^+ are locally integrable and equivalent then

$$F_*(B) = F^*(B) = \int_B J^- = \int_B J^+$$

for every box B.

In this case

(8f4) $$v\big(\varphi^{-1}(B)\big) = \int_B J$$

where J is *any* function equivalent to J^-, J^+.

8g1 Proposition. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that J^-, J^+ are locally integrable and equivalent then for every integrable $f : \mathbb{R}^n \to \mathbb{R}$ the function $f \circ \varphi : \mathbb{R}^m \to \mathbb{R}$ is integrable and

$$\int_{\mathbb{R}^m} f \circ \varphi = \int_{\mathbb{R}^n} fJ \, .$$

8g2 Corollary. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that J^-, J^+ are locally integrable and equivalent then:

(a) for every Jordan measurable set $E \subset \mathbb{R}^n$ the set $\varphi^{-1}(E) \subset \mathbb{R}^m$ is Jordan measurable;

(b) for every integrable $f : E \to \mathbb{R}$ the function $f \circ \varphi$ is integrable on $\varphi^{-1}(E)$, and

$$\int_{\varphi^{-1}(E)} f \circ \varphi = \int_E fJ \, .$$

8h1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets and $\varphi : V \to U$ a diffeomorphism, then

$$J^-(x) = J^+(x) = |\det(D\psi)_x|$$

for all $x \in U$; here $\psi = \varphi^{-1} : U \to V$.