9b6 Lemma. Locally Jordan measurable sets are an algebra of sets (in $\mathbb{R}^n$). That is, $\emptyset$, $\mathbb{R}^n \setminus A$, $A \cap B$ (and therefore also $\mathbb{R}^n$, $A \cup B$ and $A \setminus B$) are locally Jordan measurable whenever A, B are.

9b8 Lemma. The restriction of v_* to the algebra of locally Jordan sets is additive.

9b11 Lemma. A set $A \subset \mathbb{R}^n$ is locally Jordan measurable if and only if its boundary is locally volume zero.

9c1 Theorem. *(Monotone convergence theorem for volumes)* Let $X \subset \mathbb{R}^n$, sets $A_i \subset X$ be locally Jordan in X, and $A_i \uparrow X$, then

$$v_*(A_i) \uparrow v_*(X) \qquad \text{as } i \to \infty .$$

9c3 Lemma. If $X_i \subset \mathbb{R}^n$, $X_i \downarrow \emptyset$ and $v_*(X_1) < \infty$, then $v_*(X_i) \downarrow 0$ as $i \to \infty$.

9d12 Theorem. If $f, g : \mathbb{R}^n \to \mathbb{R}$ are integrable then $f + g$ is integrable and

$$\int_{\mathbb{R}^n} (f+g) = \int_{\mathbb{R}^n} f + \int_{\mathbb{R}^n} g .$$

9e1 Example (Poisson)**.**

$$\int_{-\infty}^{\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,; \qquad \int_{\mathbb{R}^n} \mathrm{e}^{-\langle Ax,x\rangle}\,\mathrm{d}x = \frac{\pi^{n/2}}{\sqrt{\det A}} .$$

9e14 Claim (Cauchy-Schwarz)**.** Suppose $f, g \in \tilde{L}^2(U)$. Then $fg \in \tilde{L}_1(U)$ and $\big|\int_U fg\big| \le \|f\|_2 \|g\|_2$.

9e15 Claim (Hölder)**.** More generally, $fg \in \tilde{L}_1(U)$ and $\big|\int_U fg\big| \le \|f\|_p \|g\|_q$ whenever $f \in \tilde{L}^p(U)$, $g \in \tilde{L}^q(U)$, $\frac{1}{p} + \frac{1}{q} = 1$.

9e16 Claim (Minkowski)**.** If $f, g \in \tilde{L}^p(U)$ then $f+g \in \tilde{L}^p(U)$ and $\|f+g\|_p \le \|f\|_p + \|g\|_p$.

9f5 Theorem. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then f is Jordan measurable on V if and only if $f \circ \varphi$ is Jordan measurable on U, and in this case

$$\int_V f = \int_U (f \circ \varphi) |\det D\varphi| .$$

$$U(x) = \int_{B_R} \frac{\mathrm{d}\xi}{|x - \xi|} = \begin{cases} \frac{4\pi R^3}{3|x|} & \text{for } |x| \ge R, \\ \frac{2\pi}{3}(3R^2 - |x|^2) & \text{for } |x| \le R. \end{cases}$$

9h7 Theorem. *(Monotone convergence theorem for integrals)* Let $X \subset \mathbb{R}^n$ be a set, $f_i : X \to [0,\infty)$ functions Jordan measurable on X, $f_i \uparrow f$, $f : X \to [0, \infty)$. Then $\int_X f_i \uparrow {}_*\!\int_X f$.

9i2 Theorem. (Iterated improper integral for positive functions)

Let functions $f_i : \mathbb{R}^{n+m} \to [0,\infty)$ be Jordan measurable, $f_i \uparrow f$, $f : \mathbb{R}^{n+m} \to [0,\infty)$. Then

$${}_*\!\int_{\mathbb{R}^{n+m}} f = {}_*\!\int_{\mathbb{R}^n} \Big(x \mapsto {}_*\!\int_{\mathbb{R}^m} f_x \Big) .$$

9i7 Corollary. If $f : \mathbb{R}^{n+m} \to [0,\infty)$ is Jordan measurable then

$${}_*\!\int_{\mathbb{R}^n} \mathrm{d}x \; {}_*\!\int_{\mathbb{R}^m} \mathrm{d}y\, f(x,y) = \int_{\mathbb{R}^{n+m}} f = {}_*\!\int_{\mathbb{R}^m} \mathrm{d}y \; {}_*\!\int_{\mathbb{R}^n} \mathrm{d}x\, f(x,y) .$$

9i8 Corollary. For every open set $G \subset \mathbb{R}^{n+m}$,

$$v_*(G) = {}_*\!\int_{\mathbb{R}^n} v_*(G_x)\,\mathrm{d}x$$

where $G_x = \{y : (x,y) \in G\} \subset \mathbb{R}^m$.

9i9 Corollary. For every compact set $K \subset \mathbb{R}^{n+m}$,

$$v^*(K) = {}^*\!\!\int_{\mathbb{R}^n} v^*(K_x)\,\mathrm{d}x$$

where $K_x = \{y : (x,y) \in K\} \subset \mathbb{R}^m$.

$$\Gamma(s) = \int_0^\infty t^{s-1}\mathrm{e}^{-t}\,\mathrm{d}t \quad \text{for } s > 0\,; \quad \Gamma(s+1) = s\Gamma(s)\,; \quad \Gamma(n) = (n-1)!\,; \quad \Gamma(\tfrac12) = \sqrt{\pi} .$$

$$\mathrm{B}(\alpha,\beta) = \int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \qquad \text{for } \alpha, \beta > 0 .$$

$$\int_0^{\pi/2} \sin^{\alpha-1}\theta \cos^{\beta-1}\theta\,\mathrm{d}\theta = \frac12 \mathrm{B}\Big(\frac{\alpha}{2}, \frac{\beta}{2}\Big) = \frac12 \cdot \frac{\Gamma\big(\frac{\alpha}{2}\big)\Gamma\big(\frac{\beta}{2}\big)}{\Gamma\big(\frac{\alpha+\beta}{2}\big)} .$$

$$\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}} \Gamma(x)\, \Gamma\Big(x + \frac12\Big) .$$

9j12 Proposition. For $p_1, \dots p_n > 0$,

$$\underset{\substack{x_1,\dots x_n \ge 0,\\ x_1+\dots+x_n \le 1}}{\int \cdots \int} x_1^{p_1-1} \dots x_n^{p_n-1}\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)} .$$

$$\underset{\substack{x_1,\dots,x_n \ge 0\\ x_1^p+\dots+x_n^p \le 1}}{\int \cdots \int} \mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma^n\big(\frac1p\big)}{p^n \Gamma\big(\frac{n}{p}+1\big)} .$$

The volume of the unit ball in the metric l_p, and l_2:

$$v_n\big(B_p(1)\big) = \frac{2^n \Gamma^n\big(\frac1p\big)}{p^n \Gamma\big(\frac{n}{p}+1\big)}\,; \quad v_n\big(B_2(1)\big) = \frac{2\pi^{n/2}}{n\Gamma\big(\frac{n}{2}\big)} .$$

9j13 Exercise.

$$\underset{\substack{x_1+\dots+x_n\le 1\\ x_1,\dots,x_n\ge 0}}{\int\cdots\int} \varphi(x_1+\dots+x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{(n-1)!}\int_0^1 \varphi(s)s^{n-1}\,\mathrm{d}s\,.$$

9k2 Theorem. The following three conditions on a bounded function $f : B \to \mathbb{R}$ on a box $B \subset \mathbb{R}^n$ are equivalent:
(a) f is integrable;
(b) ${}^*\!\!\int_B \operatorname{Osc}_f = 0$;
(c) for every $\varepsilon > 0$ the set $\{x \in B : \operatorname{Osc}_f(x) \ge \varepsilon\}$ is of volume zero.

Integral of a k-form over a singular k-box: $\displaystyle\int_\Gamma \omega = \int_B \omega\big(\Gamma(u),(D_1\Gamma)_u,\dots,(D_k\Gamma)_u\big)\,\mathrm{d}u\,.$

11c3 Proposition. (*Stokes' theorem for $k=1$*)
Let C be a 1-chain in $\mathbb{R}^n$, and ω a 0-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C}\omega\,.$$

11c5 Lemma. For every $f \in C^0(\mathbb{R}^n)$ there exist $f_i \in C^1(\mathbb{R}^n)$ such that $f_i \to f$ uniformly on bounded sets.

11d2 Definition. The *exterior derivative* of a 1-form ω of class C^1 is a 2-form $d\omega$ defined by

$$(d\omega)(\cdot,h,k) = D_h\omega(\cdot,k) - D_k\omega(\cdot,h)\,.$$

11d3 Theorem. (*Stokes' theorem for $k=2$*)
Let C be a 2-chain in $\mathbb{R}^n$, and ω a 1-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C}\omega\,.$$

$$(\omega_1\wedge\omega_2)(x,h,k) = \omega_1(x,h)\omega_2(x,k) - \omega_1(x,k)\omega_2(x,h)\,;\quad (dx_i\wedge dx_j)(x,h,k) = \begin{vmatrix} h_i & k_i\\ h_j & k_j\end{vmatrix}\,.$$

$$d(df)=0\,;\quad d(f\omega) = df\wedge\omega + f\,d\omega\,;\quad d(f\,dg) = df\wedge dg\,.$$

11e7 Definition. (*Equivalent to 11d2*) The *exterior derivative* of a 1-form ω of class C^1 is a 2-form $d\omega$ defined by

$$d\omega = \sum_{i=1}^n df_i\wedge dx_i \quad\text{for } \omega = \sum_{i=1}^n f_i\,dx_i\,.$$

11e8 Exercise.

$$\int_\Gamma \omega = \int_B \sum_{i<j} f_{i,j}(x)\frac{\partial(x_i,x_j)}{\partial(u_1,u_2)}\,\mathrm{d}u_1\mathrm{d}u_2 \quad \text{for } \omega = \sum_{i<j} f_{i,j}\,dx_i\wedge dx_j\,;$$

here $x=(x_1,\dots,x_n)=\Gamma(u_1,u_2)$ and $\frac{\partial(x_i,x_j)}{\partial(u_1,u_2)} = \begin{vmatrix} \frac{\partial x_i}{\partial u_1} & \frac{\partial x_i}{\partial u_2}\\ \frac{\partial x_j}{\partial u_1} & \frac{\partial x_j}{\partial u_2}\end{vmatrix}$.

11f1 Definition. The *pullback* of ω along φ is a k-form $\varphi^*\omega$ defined by

$$(\varphi^*\omega)(x,h_1,\dots,h_k) = \omega\big(\varphi(x),(D\varphi)_x(h_1),\dots,(D\varphi)_x(h_k)\big)\,.$$

(11f2) $$\int_\Gamma\omega = \int_B \Gamma^*\omega\,.$$

(11f3) $$\int_{\varphi\circ C}\omega = \int_C \varphi^*\omega\,.$$

11f4 Lemma. For every 0-form $f \in C^1(\mathbb{R}^n)$ and $\varphi \in C^1(\mathbb{R}^\ell\to\mathbb{R}^n)$,

$$\varphi^*(df) = d(\varphi^*f)\,.$$

11f5 Lemma. For all 1-forms ω_1,ω_2 on $\mathbb{R}^n$ and $\varphi \in C^1(\mathbb{R}^\ell\to\mathbb{R}^n)$,

$$\varphi^*(\omega_1\wedge\omega_2) = (\varphi^*\omega_1)\wedge(\varphi^*\omega_2)\,.$$

11f6 Lemma. For every 1-form ω of class C^1 on $\mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^\ell\to\mathbb{R}^n)$,

$$\varphi^*(d\omega) = d(\varphi^*\omega)\,.$$

11g3 Lemma. For every $\Gamma \in C^1(B\to\mathbb{R}^n)$ there exist $\Gamma_i \in C^2(B\to\mathbb{R}^n)$ such that $\Gamma_i\to\Gamma$ in C^1.

11h1 Corollary.

$$C_1\sim C_2 \quad\text{implies}\quad \partial C_1\sim\partial C_2$$

for arbitrary 2-chains C_1,C_2 in $\mathbb{R}^n$.

11h2 Proposition. Assume that $\gamma,\gamma_1,\gamma_2,\dots \in C^1\big([t_0,t_1]\to\mathbb{R}^n\big)$, γ_k are bounded in C^1 (that is, $\sup_k\max_t|\gamma_k'(t)|<\infty$), and $\gamma_k\to\gamma$ in C^0 (that is, $\max_t|\gamma_k(t)-\gamma(t)|\to 0$ as $k\to\infty$). Then

$$\int_{\gamma_k}\omega \to \int_\gamma\omega \quad\text{as } k\to\infty$$

for every 1-form ω (of class C^0) on $\mathbb{R}^n$.

11h3 Remark. The condition that γ_k are bounded in C^1 cannot be dropped.

11h4 Remark. Prop. 11h2 generalizes readily to paths γ_k,γ that are only *piecewise* continuously differentiable.

$$\omega = \sum_{i<j} f_{i,j}\, dx_i \wedge dx_j ; \quad \omega(x,h,k) = \det\big(H(x),h,k\big), \quad H(x) = \big(f_{2,3}(x), f_{3,1}(x), f_{1,2}(x)\big).$$

$$\int_\Gamma \omega = \int_B \det\big(H(\Gamma(u)), (D_1\Gamma)_u, (D_2\Gamma)_u\big)\, du . \qquad \text{flux through}$$

$$\omega = \sum_i f_i\, dx_i ; \quad \omega(x,h) = \langle E(x), h\rangle , \quad E(x) = \big(f_1(x), f_2(x), f_3(x)\big) .$$

$$\int_\gamma \omega = \int_{t_0}^{t_1} \omega\big(\gamma(t), \gamma'(t)\big)\, dt = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\, dt . \qquad \begin{array}{l}\text{integral along a path;}\\ \text{circulation around a loop}\end{array}$$

If $\omega = \omega_1 \wedge \omega_2$ then $H = E_1 \times E_2$. If $\omega = dg$ then $E = \nabla g$. $\operatorname{curl}(\nabla f) = 0$.

12a2 Exercise. Let ω_1 be a 1-form on $\mathbb{R}^3$, $\omega_2 = d\omega_1$, E dual to ω_1, and H dual to ω_2. Then $H = \operatorname{curl} E$,
that is, $H_1 = D_2E_3 - D_3E_2$, $H_2 = D_3E_1 - D_1E_3$, $H_3 = D_1E_2 - D_2E_1$.

(12a8)
$$\int_{\partial\Gamma} E = \int_\Gamma \operatorname{curl} E .$$

$\omega = f_1\, dx_1 + f_2\, dx_2$, $\omega(x,h) = \det(H(x),h) = \langle E(x), h\rangle$,
$H = (f_2, -f_1)$, $E = (f_1, f_2)$; the rotation by $\pi/2$ turns $H(x)$ into $E(x)$.

(12c1-2)
$$\int_{\text{along } \gamma} E = \int_{t_0}^{t_1} \langle E(\gamma(t)), \gamma'(t)\rangle\, dt , \qquad \int_{\text{through } \gamma} H = \int_{t_0}^{t_1} \det\big(H(\gamma(t)), \gamma'(t)\big)\, dt .$$

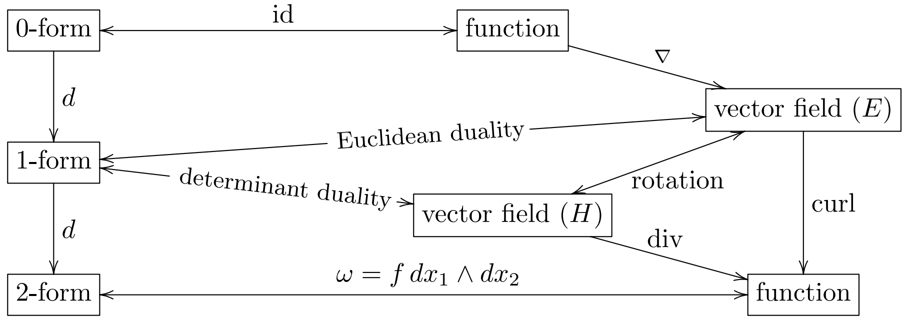
If $\omega_2 = d\omega_1$ then: $\omega_1 = E_1\, dx_1 + E_2\, dx_2$, $\omega_2 = (\operatorname{curl} E)\, dx_1 \wedge dx_2$, $\operatorname{curl} E = D_1E_2 - D_2E_1$.

(12c5)
$$\int_{\text{along } \partial\Gamma} E = \int_\Gamma \operatorname{curl} E .$$

$$\operatorname{curl} E = D_1E_2 - D_2E_1 = D_1H_1 + D_2H_2 = \operatorname{div} H \quad (\text{since } H_1 = E_2 \text{ and } H_2 = -E_1) .$$

(12c8)
$$\int_{\text{through } \partial\Gamma} H = \int_\Gamma \operatorname{div} H$$

(12c9)



12d1 Exercise. $\bar{z}w = \langle z, w\rangle + \mathrm{i}\det(z,w)$;

$f(x+\mathrm{i}y) = u(x,y) + \mathrm{i}v(x,y)$, $\operatorname{Re}(f\,dz) = u\,dx - v\,dy$, $\operatorname{Im}(f\,dz) = v\,dx + u\,dy$;

f analytic $\Longrightarrow$ $u_x = v_y$, $u_y = -v_x$ (Cauchy-Riemann); $d\operatorname{Re}(f\,dz) = 0$, $d\operatorname{Im}(f\,dz) = 0$.

$$\operatorname{div} \nabla f = \Delta f , \quad \Delta = D_1D_1 + D_2D_2 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} ; \quad f \text{ harmonic: } \Delta f = 0 .$$

$$f \text{ analytic} \implies \operatorname{Re} f, \operatorname{Im} f \text{ harmonic};$$

(12d4)
$$u \text{ harmonic} \implies u(z) = \frac{1}{2\pi}\int_0^{2\pi} u(z + r e^{\mathrm{i}\theta})\, d\theta .$$

Green formulas:

(12d5)
$$\int_{\text{through } \partial\Gamma} \nabla u = \int_\Gamma \Delta u \quad \text{for all } u \in C^2(\mathbb{R}^2) .$$

(12d7)
$$\int_{\text{through } \partial\Gamma} u\nabla v = \int_\Gamma \big(u\Delta v + \langle \nabla u, \nabla v\rangle\big) \quad \text{for all } u \in C^1(\mathbb{R}^2) \text{ and } v \in C^2(\mathbb{R}^2) ,$$

(12d8)
$$\int_{\text{through } \partial\Gamma} (u\nabla v - v\nabla u) = \int_\Gamma (u\Delta v - v\Delta u) \quad \text{for all } u, v \in C^2(\mathbb{R}^2) .$$

13a2 Lemma. A 1-form ω on $\mathbb{R}^n$ satisfies $\int_\gamma \omega = 0$ for all loops γ if and only if $\omega = df$ for some $f \in C^1$.

The same holds over an open subset of $\mathbb{R}^n$.

13b6 Lemma. A 1-form ω of class C^1 on G is closed if and only if $\int_{\partial\Gamma} \omega = 0$ for all singular 2-boxes Γ in G.

13b9 Proposition. If loops γ_1, γ_2 in an open set $G \subset \mathbb{R}^n$ are homotopic in G then $\int_{\gamma_1} \omega = \int_{\gamma_2} \omega$ for all closed 1-forms ω on G.

13b14 Corollary. If γ is null homotopic in G then $\int_\gamma \omega = 0$ for all closed 1-forms ω on G.

13b16 Proposition. Every closed 1-form ω on a simply connected G is exact.

13b18 Exercise. If α is a closed 1-form and β is an exact 1-form then the 2-form $\alpha \wedge \beta$ is exact.

13b19 Proposition. If ω is an exact 2-form on a simply connected open set $G \subset \mathbb{R}^n$, then for every loop γ in G, $\int_\gamma \alpha$ does not depend on the choice of α such that $d\alpha = \omega$.

13c4 Exercise. For a radial function $g : \mathbb{R}^n \ni x \mapsto f(|x|) \in \mathbb{R}$, $f \in C^2[0,\infty)$, $f'(0) = 0$,

$$\operatorname{div} \nabla g(x) = f''(|x|) + \frac{n-1}{|x|} f'(|x|) .$$

$$B_\gamma(x) = \frac{1}{4\pi} \int_{t_0}^{t_1} \frac{\gamma'(t) \times (x - \gamma(t))}{|x - \gamma(t)|^3}\, dt ; \quad \operatorname{curl} B_\gamma(x) = E_0(x - \gamma(t_1)) - E_0(x - \gamma(t_0))$$

for all $x \in \mathbb{R}^3 \setminus \gamma([t_0,t_1])$; here $E_0(x) = \frac{1}{4\pi}\frac{x}{|x|^3}$.

$$\operatorname{Lk}(\gamma_1,\gamma_2) = \frac{1}{4\pi} \iint \frac{\det(\gamma_1'(s), \gamma_2'(t), \gamma_1(s) - \gamma_2(t))}{|\gamma_1(s) - \gamma_2(t)|^3} ds dt .$$

$$\operatorname{div} E = \frac{1}{\varepsilon_0}\rho , \ \operatorname{div} B = 0 , \ \operatorname{curl} E = -\frac{\partial B}{\partial t} , \ \operatorname{curl} B = \mu_0 j + \frac{1}{c^2}\frac{\partial E}{\partial t} ; \qquad \text{exact 2-form:}$$

$$\omega = (E_1\, dx_1 + E_2\, dx_2 + E_3\, dx_3) \wedge dt + B_1\, dx_2 \wedge dx_3 + B_2\, dx_3 \wedge dx_1 + B_3\, dx_1 \wedge dx_2 .$$

14a4 Definition. The *exterior derivative* of a 2-form ω of class C^1 is a 3-form $d\omega$ defined by

$$(d\omega)(\cdot,h_1,h_2,h_3) = D_{h_1}\omega(\cdot,h_2,h_3) + D_{h_2}\omega(\cdot,h_3,h_1) + D_{h_3}\omega(\cdot,h_1,h_2)\,.$$

14a10 Definition. *(Equivalent to 14a4)* The *exterior derivative* of a 2-form ω of class C^1 is a 3-form $d\omega$ defined by

$$d\omega = \sum_{i<j} df_{i,j}\wedge dx_i \wedge dx_j \quad \text{for } \omega = \sum_{i<j} f_{i,j}\,dx_i\wedge dx_j\,.$$

14a12 Lemma. For every 2-form ω of class C^1 on $\mathbb{R}^n$ and $\varphi \in C^2(\mathbb{R}^\ell \to \mathbb{R}^n)$,

$$\varphi^*(d\omega) = d(\varphi^*\omega)\,.$$

14a13 Theorem. (*Stokes' theorem for* $k=3$)

Let C be a 3-chain in $\mathbb{R}^n$, and ω a 2-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C}\omega\,.$$

14a14 Corollary.

$$C_1 \sim C_2 \quad \text{implies} \quad \partial C_1 \sim \partial C_2$$

for arbitrary 3-chains C_1, C_2 in $\mathbb{R}^n$. (Similar to 11h1.)

14a17 Exercise. $d(\omega_1\wedge\omega_2) = (d\omega_1)\wedge\omega_2 - \omega_1\wedge d\omega_2$ for 1-forms ω_1,ω_2 on $\mathbb{R}^n$.

14a18 Exercise. *(a generalization of the formula for integration by parts)*

$$\int_C f\,d\omega = \int_{\partial C} f\omega - \int_C df\wedge\omega$$

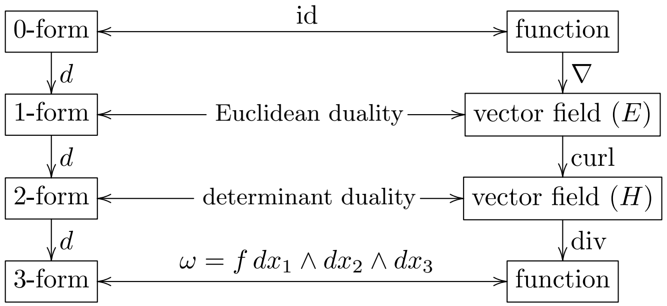
for arbitrary 2-form ω (of class C^1) on $\mathbb{R}^n$, function $f\in C^1(\mathbb{R}^n)$, and 3-chain C in $\mathbb{R}^n$.

$$\omega(x,h_1,h_2) = \det(H(x),h_1,h_2)\,,\quad H(x) = \big(f_{2,3}(x), f_{3,1}(x), f_{1,2}(x)\big)\,,$$

$$\omega = f_{1,2}\,dx_1\wedge dx_2 + f_{2,3}\,dx_2\wedge dx_3 + f_{3,1}\,dx_3\wedge dx_1\,;$$

$$d\omega = (\operatorname{div} H)\,dx_1\wedge dx_2\wedge dx_3\,,\quad \operatorname{div} H = D_1H_1 + D_2H_2 + D_3H_3\,.$$

(14b2)



$$\operatorname{div}(fH) = \langle\nabla f, H\rangle + f\operatorname{div}H\,;\quad \operatorname{div}(E_1\times E_2) = \langle\operatorname{curl}E_1, E_2\rangle - \langle E_1, \operatorname{curl}E_2\rangle\,.$$

(14b5) $$\int_{\partial\Gamma} H = \int_\Gamma \operatorname{div} H \qquad \textit{three-dimensional divergence theorem}$$

for every vector field H (of class C^1) on $\mathbb{R}^3$ and every singular 3-box Γ in $\mathbb{R}^3$.

(14b11) $$\int_{\partial B_R} f = \int_0^\pi d\theta \int_0^{2\pi} d\varphi\cdot R^2\sin\theta\cdot f(R\sin\theta\cos\varphi, R\sin\theta\sin\varphi, R\cos\theta)\,.$$

$$\operatorname{div}\nabla f = \Delta f\,,\quad \Delta = D_1D_1 + D_2D_2 + D_3D_3 = \frac{\partial^2}{\partial x_1^2} + \frac{\partial^2}{\partial x_2^2} + \frac{\partial^2}{\partial x_3^2}\,;$$

functions $f\in C^2(\mathbb{R}^3)$ such that $\Delta f = 0$ are called *harmonic*. For harmonic u,

(14b13) $$u(0) = \frac{1}{4\pi R^2}\int_{\partial B_R} u\,;\quad u(x) = \frac{1}{4\pi R^2}\int_{\partial B_R} u(x+\cdot)\,.$$

The mean value may be taken on the ball rather than the sphere.

14b21 Proposition. Every harmonic function $\mathbb{R}^3\to[0,\infty)$ is constant.

14c7 Definition. For a $(k-1)$-form ω of class C^1,

$$(d\omega)(\cdot,h_1,\dots,h_k) = \sum_{i=1}^k (-1)^{i-1} D_{h_i}\omega(\cdot,h_1,\dots,h_{i-1},h_{i+1},\dots,h_k)\,.$$

14c8 Theorem. (*Stokes' theorem*)

Let C be a k-chain in $\mathbb{R}^n$, and ω a $(k-1)$-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C}\omega\,.$$

$$d\omega = (\operatorname{div}H)\,dx_1\wedge\dots\wedge dx_n\,,\quad \operatorname{div}H = D_1H_1+\dots+D_nH_n\,;$$

(14c9) $$\int_{\partial\Gamma} H = \int_\Gamma \operatorname{div} H\,.$$

A chart: $\psi(0) = x_0$; $\psi(G)$ is an open neighborhood of x_0 in M; ψ is a homeomorphism from G to $\psi(G)$; $\psi\in C^1(G\to\mathbb{R}^N)$; for every $x\in G$ the linear operator $(D\psi)_x$ from $\mathbb{R}^n$ to $\mathbb{R}^N$ is one-to-one.

A co-chart: $M\cap U = \{x\in U : \varphi(x) = 0\}$; $\varphi\in C^1(U\to\mathbb{R}^{N-n})$; for every $x\in U$ the linear operator $(D\varphi)_x$ from R^N to $\mathbb{R}^{N-n}$ is onto.

15b3 Lemma. Existence of a chart (n-chart of M around x_0) is equivalent to existence of a co-chart (n-cochart of M around x_0).

(15c2) $$\int_{(M,\mathcal{O})}\omega = \int_{(G,\psi)}\omega = \int_G f\,;\qquad \psi^*\omega = f\,du_1\wedge\dots\wedge du_n\,.$$

(15d5) $$\psi^*\mu = J_\psi\,du_1\wedge\dots\wedge du_n\,,\qquad J_\psi(u) = \sqrt{\det\big(\langle(D_i\psi)_u,(D_j\psi)_u\rangle\big)_{i,j}}\,.$$

(15d6) $$\int_{(M,\mathcal{O})} f = \int_{(M,\mathcal{O})} f\mu = \int_{(G,\psi)} f\mu = \int_G (f\circ\psi)J_\psi\,.$$

16b9 Theorem. $\int_{\varphi(G)} \mathrm{d}c \int_{M_c} f = \int_G f|\nabla\varphi|\,;\quad M_c = \{x\in G : \varphi(x) = c\}$.

16b15 Theorem. Let $G\subset\mathbb{R}^n$ be a bounded regular open set, $M\subset\mathbb{R}^n$ an $(n-1)$-manifold, $\partial G = M$, and orientations $\mathcal{O}$ of G and $\tilde{\mathcal{O}}$ of M conform (at every point of M). Then

$$\int_{(G,\mathcal{O})} d\omega = \int_{(M,\tilde{\mathcal{O}})}\omega$$

for every $(n-1)$-form ω of class C^1 on $\mathbb{R}^n$.

16b16 Theorem.

$$\int_G \operatorname{div}H = \int_M \langle H,\vec{n}\,\rangle\,. \qquad \textit{(the divergence theorem)}$$