10 Improper integral

Riemann integral and Jordan measure are generalized to unbounded functions and sets.

10a What is the problem

The n-dimensional unit ball in the l_p metric,

$$E = \{(x_1, \dots, x_n) : |x_1|^p + \dots + |x_n|^p \le 1\},$$

is a Jordan measurable set, and its volume is a Riemann integral,

$$v(E) = \int_{\mathbb{R}^n} \mathbb{1}_E ,$$

of a bounded function with bounded support. In Sect. 10g we'll calculate it:

$$v(E) = \frac{2^n \Gamma^n\big(\frac1p\big)}{p^n \Gamma\big(\frac{n}{p}+1\big)}$$

where Γ is a function defined by

$$\Gamma(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x \quad \text{for } t > 0 ;$$

here the integrand has no bounded support; and for $t = \frac1p < 1$ it is also unbounded (near 0). Thus we need a more general, so-called improper integral, even for calculating the volume of a bounded body!

In relatively simple cases the improper integral may be treated via *ad hoc* limiting procedure adapted to the given function; for example,

$$\int_0^\infty x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x = \lim_{k\to\infty} \int_{1/k}^k x^{t-1} \mathrm{e}^{-x} \, \mathrm{d}x .$$

In more complicated cases it is better to have a theory able to integrate rather general functions on rather general n-dimensional sets. Different functions may tend to infinity on different subsets (points, lines, surfaces), and still, we expect $\int(af+bg) = a\int f + b\int g$ (linearity) to hold, as well as change of variables.[1]

10b Positive integrands

We consider an open set $G \subset \mathbb{R}^n$ and functions $f : G \to [0,\infty)$ continuous almost everywhere. We do not assume that G is bounded. We also do not assume that G is Jordan measurable, even if it is bounded.[2] "Continuous almost everywhere" means that the set $A \subset G$ of all discontinuity points of f satisfies $m^*(A) = 0$, recall Sect. 8f; but now A need not be bounded. For our purposes it is enough to know that $m^*(A) = 0$ if and only if $m^*(A_1) = 0$ for every bounded $A_1 \subset A$ (we may take this as the definition). We can use the function $f \cdot 1\!\!1_G$ equal f on G and 0 on $\mathbb{R}^n \setminus G$, but must be careful: $1\!\!1_G$ and $f \cdot 1\!\!1_G$ need not be continuous almost everywhere.

We define

$$\text{(10b1)} \quad \int_G f = \sup\left\{ \int_{\mathbb{R}^n} g \,\middle|\, g : \mathbb{R}^n \to \mathbb{R} \text{ integrable}, \right. \\ \left. 0 \le g \le f \text{ on } G,\ g = 0 \text{ on } \mathbb{R}^n \setminus G \right\} \in [0,\infty]\,.$$

The condition on g may be reformulated as $0 \le g \le f \cdot 1\!\!1_G$.

10b2 Exercise. (a) Without changing this supremum we may restrict ourselves to continuous g with bounded support; or, alternatively, to step functions g;

(b) if f is bounded and G is bounded, then $\int_G f = {}_*\!\int_{\mathbb{R}^n} f \cdot 1\!\!1_G$, and in particular, $\int_G 1 = v_*(G)$;[3]

(c) if f is bounded and G is Jordan measurable, then the integral defined by (10b1) is equal to the integral defined by (6g16).

Prove it.

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[1] Additional literature (for especially interested):
M. Pascu (2006) "On the definition of multidimensional generalized Riemann integral", Bul. Univ. Petrol **LVIII**:2, 9–16.
(*Research level*) D. Maharam (1988) "Jordan fields and improper integrals", J. Math. Anal. Appl. **133**, 163–194.

[2] A bounded open set need not be Jordan measurable, even if it is diffeomorphic to a disk, as was noted in Sect. 8e (p. 138).

[3] According to 8e, $v_*(G) = {}^\circ v(G) = m(G)$.

10b3 Exercise. Consider the case $G = \mathbb{R}^n$, and let $\|\cdot\|$ be a norm on $\mathbb{R}^n$.

(a) Prove that

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\|x\|<k} \min\big(f(x), k\big)\,\mathrm{d}x\,.$$

(b) For a locally bounded[1] f prove that

$$\int_{\mathbb{R}^n} f = \lim_{k\to\infty} \int_{\|x\|<k} f(x)\,\mathrm{d}x\,.$$

(c) Can it happen that f is locally bounded, not bounded, and $\int_{\mathbb{R}^n} f < \infty$?

10b4 Example (Poisson)**.** Consider

$$I = \int_{\mathbb{R}^2} \mathrm{e}^{-|x|^2}\,\mathrm{d}x\,.$$

On one hand, by 10b3 for the Euclidean norm,

$$I = \lim_{k\to\infty} \iint\limits_{x^2+y^2<k^2} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \lim_{k\to\infty} \int_0^k r\,\mathrm{d}r\,\mathrm{e}^{-r^2} \int_0^{2\pi} \mathrm{d}\theta = \lim_{k\to\infty} \pi \int_0^{k^2} \mathrm{e}^{-u}\,\mathrm{d}u = \pi\,.$$

On the other hand, by 10b3 for $\|(x,y)\| = \max(|x|,|y|)$,

$$I = \lim_{k\to\infty} \iint\limits_{|x|<k,|y|<k} \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y = \lim_{k\to\infty} \Big(\int_{-k}^{k} \mathrm{e}^{-x^2}\,\mathrm{d}x\Big)\Big(\int_{-k}^{k} \mathrm{e}^{-y^2}\,\mathrm{d}y\Big) = \Big(\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x\Big)^2,$$

and we obtain the celebrated Poisson formula:

$$\boxed{\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,.}$$

10b5 Exercise. Consider

$$I = \iint\limits_{x>0,y>0} x^a y^b \mathrm{e}^{-(x^2+y^2)}\,\mathrm{d}x\mathrm{d}y \in [0,\infty]$$

[1] That is, bounded on every bounded subset of $\mathbb{R}^n$.

for given $a, b \in \mathbb{R}$. Prove that, on one hand,

$$I = \biggl(\int_0^\infty r^{a+b+1} \mathrm{e}^{-r^2} \, \mathrm{d}r \biggr) \biggl(\int_0^{\pi/2} \cos^a \theta \sin^b \theta \, \mathrm{d}\theta \biggr) \,,$$

and on the other hand,

$$I = \biggl(\int_0^\infty x^a \mathrm{e}^{-x^2} \, \mathrm{d}x \biggr) \biggl(\int_0^\infty x^b \mathrm{e}^{-x^2} \, \mathrm{d}x \biggr) \,.$$

10b6 Exercise. Consider $f : \mathbb{R}^2 \to [0, \infty)$ of the form $f(x) = g(|x|)$ for a given $g : [0, \infty) \to [0, \infty)$.

(a) If g is integrable, then f is integrable and $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, \mathrm{d}r$.

(b) If g is continuous on $(0, \infty)$, then $\int_{\mathbb{R}^2} f = 2\pi \int_0^\infty g(r) \, r \, \mathrm{d}r \in [0, \infty]$.

Prove it.[1]

10b7 Exercise. Consider $f : \mathbb{R}^n \to [0, \infty)$ of the form $f(x) = g\bigl(\|x\|\bigr)$ for a given $g : [0, \infty) \to [0, \infty)$ and a given norm $\| \cdot \|$ on $\mathbb{R}^n$.

(a) If g is integrable then f is integrable, and $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, \mathrm{d}r$ where V is the volume of $\{x : \|x\| < 1\}$.

(b) If g is continuous on $(0, \infty)$, then $\int_{\mathbb{R}^n} f = nV \int_0^\infty g(r) \, r^{n-1} \, \mathrm{d}r \in [0, \infty]$.

c) Let g be continuous on $(0, \infty)$ and satisfy

$$g(r) \sim r^a \quad \text{for } r \to 0+ \,, \qquad g(r) \sim r^b \quad \text{for } r \to +\infty \,.$$

Then $\int f < \infty$ if and only if $b < -n < a$.

Prove it.[2]

10b8 Example. $\int_{\mathbb{R}^n} \mathrm{e}^{-\|x\|^2} \, \mathrm{d}x = nV \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r$; in particular, $\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \, \mathrm{d}x = nV_n \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r$ where V_n is the volume of the (usual) n-dimensional unit ball. On the other hand, $\int_{\mathbb{R}^n} \mathrm{e}^{-|x|^2} \, \mathrm{d}x = \bigl(\int_{\mathbb{R}} \mathrm{e}^{-x^2} \, \mathrm{d}x \bigr)^n = \pi^{n/2}$. Therefore

$$V_n = \frac{\pi^{n/2}}{n \int_0^\infty r^{n-1} \mathrm{e}^{-r^2} \, \mathrm{d}r} \,.$$

Not unexpectedly, $V_2 = \frac{\pi}{2 \int_0^\infty r \mathrm{e}^{-r^2} \, \mathrm{d}r} = \pi$.

Clearly, $\int_G cf = c \int_G f$ for $c \in (0, \infty)$.

10b9 Proposition. $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2 \in [0, \infty]$ for all $f_1, f_2 \ge 0$ on G, continuous almost everywhere.

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[1] Hint: (a) either polar coordinates, or 9g4; (b) use (a).

[2] Hint: (a), (b) similar to 9g4, using also 9c3 and 6g12; (c) use (b).

Proof. First we prove that $\int_G (f_1 + f_2) \ge \int_G f_1 + \int_G f_2$.[1] Given integrable g_1, g_2 such that $0 \le g_1 \le f_1 \cdot 1\!\!1_G$ and $0 \le g_2 \le f_2 \cdot 1\!\!1_G$, we have $\int g_1 + \int g_2 = \int (g_1 + g_2) \le \int_G (f_1 + f_2)$, since $g_1 + g_2$ is integrable and $0 \le g_1 + g_2 \le (f_1 + f_2) \cdot 1\!\!1_G$. The supremum in g_1, g_2 gives the claim.

It remains to prove that $\int_G (f_1 + f_2) \le \int_G f_1 + \int_G f_2$, that is, $\int g \le \int_G f_1 + \int_G f_2$ for every integrable g such that $0 \le g \le (f_1 + f_2) \cdot 1\!\!1_G$. We introduce $g_1 = \min(f_1, g)$, $g_2 = \min(f_2, g)$ (pointwise minimum on G; and 0 on $\mathbb{R}^n \setminus G$) and prove that they are continuous almost everywhere (on $\mathbb{R}^n$, not just on G). For almost every $x \in G$, both f_1 and g are continuous at x and therefore g_1 is continuous at x. For almost every $x \in \partial G$, g is continuous at x, which ensures continuity of g_1 at x (irrespective of continuity of f_1), since $g(x) = 0$ ($x \notin G$). Thus, g_1 is continuous almost everywhere; the same holds for g_2.

By Theorem 8f1, the functions g_1, g_2 are integrable. We have $g_1 + g_2 \ge \min(f_1 + f_2, g) = g$, since generally, $\min(a, c) + \min(b, c) \ge \min(a + b, c)$ for all $a, b, c \in [0, \infty)$ (think, why). Thus, $\int g \le \int (g_1 + g_2) = \int g_1 + \int g_2 \le \int_G f_1 + \int_G f_2$, since $0 \le g_1 \le f_1 \cdot 1\!\!1_G$, $0 \le g_2 \le f_2 \cdot 1\!\!1_G$. □

10b10 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

$$G_k \uparrow G \quad \Longrightarrow \quad \int_{G_k} f \uparrow \int_G f \in [0, \infty]$$

for all $f : G \to [0, \infty)$ continuous almost everywhere.

Proof. First of all, $\int_{G_k} f \le \int_{G_{k+1}} f$ (since $0 \le g \le f \cdot 1\!\!1_{G_k}$ implies $0 \le g \le f \cdot 1\!\!1_{G_{k+1}}$), and similarly, $\int_{G_k} f \le \int_G f$, thus $\int_{G_k} f \uparrow$ and $\lim_k \int_{G_k} f \le \int_G f$. We have to prove that $\int_G f \le \lim_k \int_{G_k} f$.

Let a step function $g : \mathbb{R}^n \to \mathbb{R}$ satisfy $0 \le g \le f \cdot 1\!\!1_G$; we have to prove that $\int g \le \lim_k \int_{G_k} f$, but we'll prove that moreover, $\int g \le \lim_k \int_{G_k} g$. By linearity, WLOG, $g = 1\!\!1_{C^\circ}$ for a box C, $C^\circ \subset G$. By 10b2, $\int_{G_k} g = {}_*\int_{\mathbb{R}^n} 1\!\!1_{C^\circ \cap G_k} = v_*(C^\circ \cap G_k)$ and $\int_G g = v(C)$; by 8e9, $v_*(C^\circ \cap G_k) \uparrow v_*(C^\circ \cap G) = v(C^\circ)$. □

10b11 Exercise. Let $G_1 \subset G_2 \subset \mathbb{R}^n$ be two open sets, and $f : G_2 \to [0, \infty)$ continuous almost everywhere. If $f = 0$ on $G_2 \setminus G_1$, then $\int_{G_2} f = \int_{G_1} f$.

Prove it.[2]

10b12 Exercise. The following four conditions on a function $f : G \to [0, \infty)$ continuous almost everywhere are equivalent:

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[1] Compare it with (6d10).

[2] Hint: just (10b1).

(a) $\int_G f = 0$;
(b) $f(x) = 0$ for every continuity point x of f;
(c) $f(x) = 0$ for almost all $x \in G$;
(d) the set $\{x \in G : f(x) = 0\}$ is dense in G.
Prove it.[1]

10c Newton potential

By the celebrated Newton's law of universal gravitation, the gravitational force exerted by a particle of mass m at point ξ on a particle of mass m_0 at point x is $-\mathcal{G} m_0 m g_\xi(x)$, and $-\mathcal{G} m g_\xi(\cdot)$ is the gravitational field generated by m,

$$(10c1) \qquad g_\xi(x) = g_0(x-\xi) = \frac{x-\xi}{|x-\xi|^3} = -\nabla U_0(x-\xi)\,;$$

here the function $U_0 : x \mapsto \frac{1}{|x|}$is proportional to the gravitational potential (energy), and $\mathcal{G}$ is the gravitational constant.[2] The reason to replace the force by the potential is simple: it is easier to work with scalar functions than with the vector ones.[3]

What happens if we have a system of point masses μ_1, ..., μ_k at points ξ_1, ..., ξ_k? The forces are to be added, and the corresponding potential is

$$U(x) = \sum_{j=1}^{k} \frac{\mu_j}{|x-\xi_j|}\,.$$

A continuously distributed mass is described in physics by its density ρ. Mathematically it means that the density is a point function, the mass is an additive box function, and these two functions are related according to Sect. 6a (and 8c): the mass within a box B is $\int_B \rho$. Generally, ρ is not quite integrable but improperly integrable; and still, the mass within a box B is assumed to be $\int_B \rho$ (improper integral) for evident physical reasons; and the total mass is $\int_{\mathbb{R}^3} \rho$.

[1] Hint: (b)$\Longrightarrow$(c)$\Longrightarrow$(d): easy; (d)$\Longrightarrow$(a): use 10b2(a); (a)$\Longrightarrow$(b): otherwise $f(\cdot) \ge \varepsilon$ on some neighborhood of x.

[2] $\mathcal{G} \approx 6.674 \cdot 10^{-11}\,\mathrm{N(m/kg)^2}$; that is, if $m = \mu = 1\,\mathrm{kg}$ and $|x-\xi| = 1\,\mathrm{m}$ then the force is $\approx 6.674 \cdot 10^{-11}$ newtons.

[3] Knowing the force F one can write down the differential equations of motion of the particle (Newton's second law) $m_0 \ddot{x} = F$, or $\ddot{x} = \mathcal{G}\nabla U$ (note that m_0 does not matter). Then one hopes to integrate these equations, thus finding out where is the particle at time t.

Similarly, the potential is assumed to be $-\mathcal{G}U_\rho$ where $U_\rho(x) = \int_{\mathbb{R}^3} \frac{\rho(\xi)}{|x-\xi|}\,\mathrm{d}\xi$; this integral is improper (in general) and must be finite.[1]

Let us compute the potential of the homogeneous mass distribution, of density 1, within the ball of radius R centered at the origin:

$$U_R(x) = \int_{|\xi|<R} \frac{\mathrm{d}\xi}{|x-\xi|}\,.$$

Due to rotation invariance (Theorem 9c1), U_R is a radial function, that is, depends only on $|x|$. Thus, it suffices to compute $U_R(x)$ at the point $x = (0,0,a)$, $a \in [0,\infty)$. The integral is proper for $a \in (R,\infty)$ and improper for $a \in [0,R]$.

First, consider the proper integral, for $a > R$. Using the spherical coordinates $\xi = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$ (recall 9b3) we have

$$U_R(x) = \int_0^R \mathrm{d}r\, 2\pi \int_0^\pi \frac{r^2\sin\theta\,\mathrm{d}\theta}{\sqrt{(a-r\cos\theta)^2 + r^2\sin^2\theta}} = \\ = \int_0^R \mathrm{d}r\, \underbrace{2\pi \int_0^\pi \frac{r^2\sin\theta\,\mathrm{d}\theta}{\sqrt{a^2 - 2ar\cos\theta + r^2}}}_{V_a(r)}\,.$$

Intuitively, the under-braced expression $V_a(r)$ is the potential of the homogeneous sphere of radius r; but rigorously, integration over spheres and other surfaces will be treated much later. We compute $V_a(r)$ using the variable

$$t = \sqrt{a^2 - 2ar\cos\theta + r^2}\,.$$

Then $a - r < t < a + r$, and $t\,\mathrm{d}t = ar\sin\theta\,\mathrm{d}\theta$. We get

$$V_a(r) = 2\pi r^2 \int_{a-r}^{a+r} \frac{t\,\mathrm{d}t}{art} = \frac{2\pi r}{a}\cdot 2r = 4\pi\frac{r^2}{a}\,.$$

[1]Mathematical rigorosity is of little interest to physicists, and still, the distinction between proper and improper integrals may be physically sound. Imagine a material ball of mass M and radius R, consisting of a large number of uniformy distributed "particles" that are balls of mass m and radius r. Outside the (large) ball, near its surface, the gravitational field is $\mathcal{G}M/R^2$ in a good approximation. Inside the ball, near the surface of a "particle", the gravitational field of this single "particle" is $\mathcal{G}m/r^2$. Let $M = 1\,\mathrm{kg}$, $R = 0.1\,\mathrm{m}$, $m = 10^{-25}\,\mathrm{kg}$, $r = 10^{-14}\,\mathrm{m}$; then $M/R^2 = 100\,\mathrm{kg/m^2}$ while $m/r^2 = 1000\,\mathrm{kg/m^2}$. Here, a single "particle" generates a field 10 times stronger than the improper integral that will be calculated! Do not think that such parameters are physically unrealistic; these m, r are the parameters of a typical atomic nucleus.

Now we easily find $U_R(x)$ by integration:

$$U_R(x) = \int_0^R V_a(r)\,\mathrm{d}r = 4\pi \int_0^R \frac{r^2}{a}\,\mathrm{d}r = \frac{4\pi R^3}{3a} = \frac{4\pi R^3}{3|x|} \quad \text{for } |x| > R\,.$$

We turn to the case $a < R$, and treat the improper integral by exhaustion:

$$U_R(x) = \lim_{\varepsilon\to 0+} \biggl(\int_{|\xi|<a-\varepsilon} \frac{\mathrm{d}\xi}{|x-\xi|} + \int_{a+\varepsilon<|\xi|<R} \frac{\mathrm{d}\xi}{|x-\xi|} \biggr) =$$
$$= \lim_{\varepsilon\to 0+} \biggl(\int_0^{a-\varepsilon} V_a(r)\,\mathrm{d}r + \int_{a+\varepsilon}^R V_a(r)\,\mathrm{d}r \biggr) = \int_0^R V_a(r)\,\mathrm{d}r \in [0,\infty]\,,$$

the latter integral being improper, since V_a need not be bounded near a. For $r < a$ we have $V_a(r) = 4\pi\frac{r^2}{a}$ as before. For $r > a$ we still use $t = \sqrt{a^2 - 2ar\cos\theta + r^2}$, and t is still strictly increasing in $\theta \in (0,\pi)$, but now $\sqrt{a^2 - 2ar + r^2} = r - a$, thus $r - a < t < r + a$, and we get

$$V_a(r) = 2\pi r^2 \int_{r-a}^{r+a} \frac{t\,\mathrm{d}t}{art} = \frac{2\pi r}{a} \cdot 2a = 4\pi r\,.$$

A surprise: V_a appears to be bounded near a, and extends by continuity to $(0,R)$, thus the one-dimensional integral may be treated as proper. We have

$$U_R(x) = \int_0^R V_a(r)\,\mathrm{d}r = \int_0^a 4\pi\frac{r^2}{a}\,\mathrm{d}r + \int_a^R 4\pi r\,\mathrm{d}r =$$
$$= 4\pi\biggl(\frac{a^2}{3} + \frac{R^2}{2} - \frac{a^2}{2}\biggr) = \frac{2\pi}{3}(3R^2 - a^2) = \frac{2\pi}{3}(3R^2 - |x|^2) \quad \text{for } 0 \le |x| < R\,.$$

The case $a = R$ is easy: $U_R(x) = \int_0^R V_a(r)\,\mathrm{d}r = \int_0^R 4\pi\frac{r^2}{a}\,\mathrm{d}r = 4\pi\frac{R^3}{3a} = \frac{4\pi R^2}{3}$ for $|x| = R$. The function U_R appears to be continuous. Finally,

$$U_R(x) = \begin{cases} \frac{4\pi R^3}{3|x|} & \text{for } |x| \ge R, \\ \frac{2\pi}{3}(3R^2 - |x|^2) & \text{for } |x| \le R. \end{cases}$$

Observe that $4\pi R^3/3$ is exactly the total mass of the ball. That is, together with Newton, we arrived at the conclusion that *the gravitational potential, and hence the gravitational force exerted by the homogeneous ball on a particle is the same as if the whole mass of the ball were concentrated at its center, as long as the point is outside the ball.* Of course, you heard about this already in the high-school.

Another important conclusion is that the potential of the homogeneous sphere does not depend on the point inside the sphere![1] Hence, *the gravitational force is zero inside the sphere.* The same is true for the homogeneous shell $\{\xi : a < |\xi| < b\}$: there is no gravitational force inside the shell.

10c2 Exercise. Check that all the conclusions are true when the mass distribution ρ is radial: $\rho(\xi) = \rho(\xi')$ whenever $|\xi| = |\xi'|$.

10c3 Exercise. Find the potential of the homogeneous solid ellipsoid $(x^2 + y^2)/b^2 + z^2/c^2 < 1$ at its center.

10c4 Exercise. Find the potential of the homogeneous solid cone of height h and radius of the base r at its vertex.

10c5 Problem. Show that at sufficiently large distances the potential of a solid is approximated by the potential of a point with the same total mass located at the center of mass of the solid with an error less than a constant divided by the square of the distance. The potential itself decays as the distance, so the approximation is good: its *relative* error is small.[2]

10d Special functions gamma and beta

Integrating a function of two variables in one variable we get a function of the other variable. An interesting example was seen in 7e2: the function $F(t) = \int_0^{\pi/2} \ln(t^2 - \sin^2 x)\,\mathrm{d}x$ appeared to be the elementary function $F(t) = \pi \ln \frac{t+\sqrt{t^2-1}}{2}$. But generally it is not elementary. Here is a much more important example. The Euler *gamma function* Γ is defined by[3]

$$\Gamma(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x}\,\mathrm{d}x \quad \text{for } t \in (0,\infty)\,. \tag{10d1}$$

This integral is not proper for two reasons. First, the integrand is bounded near 0 for $t \in [1,\infty)$ but unbounded for $t \in (0,1)$. Second, the integrand has no bounded support. In every case, using 10b10,

$$\Gamma(t) = \lim_{k\to\infty} \int_{1/k}^{k} x^{t-1} \mathrm{e}^{-x}\,\mathrm{d}x < \infty\,,$$

since the integrand (for a given t) is continuous on $(0,\infty)$, is $O(x^{t-1})$ as $x \to 0$, and (say) $O(\mathrm{e}^{-x/2})$ as $x \to \infty$. Thus, $\Gamma : (0,\infty) \to (0,\infty)$.

[1] Since $V_a(r)$ does not depend on a for $a < r$.

[2] This estimate is rather straightforward. A more accurate argument shows that the error is of order constant divided by the *cube* of the distance.

[3] This is rather $\Gamma|_{(0,\infty)}$.

Clearly, $\Gamma(1) = 1$. Integration by parts gives

$$\int_{1/k}^{k} x^t \mathrm{e}^{-x}\,\mathrm{d}x = -x^t \mathrm{e}^{-x}\Big|_{x=1/k}^{k} + t\int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x\,;$$

(10d2) $$\Gamma(t+1) = t\Gamma(t) \quad \text{for } t \in (0,\infty)\,.$$

In particular,

(10d3) $$\Gamma(n+1) = n! \quad \text{for } n = 0,1,2,\dots$$

We note that

(10d4) $$\int_0^\infty x^a \mathrm{e}^{-x^2}\,\mathrm{d}x = \frac12 \Gamma\Big(\frac{a+1}{2}\Big) \quad \text{for } a \in (-1,\infty)\,,$$

since $\int_0^\infty x^a \mathrm{e}^{-x^2}\,\mathrm{d}x = \int_0^\infty u^{a/2}\mathrm{e}^{-u}\,\frac{\mathrm{d}u}{2\sqrt{u}}$. For $a = 0$ the Poisson formula (recall 10b4) gives

(10d5) $$\Gamma\Big(\frac12\Big) = \sqrt{\pi}\,.$$

Thus,

(10d6) $$\Gamma\Big(\frac{2n+1}{2}\Big) = \frac12 \cdot \frac32 \cdot \dots \cdot \frac{2n-1}{2}\sqrt{\pi}\,.$$

The volume V_n of the n-dimensional unit ball (recall 10b8) is thus calculated:

(10d7) $$V_n = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})}\,.$$

Not unexpectedly, $V_3 = \frac{\pi^{3/2}}{\frac32\Gamma(\frac32)} = \frac{\pi^{3/2}}{\frac32\cdot\frac12\sqrt{\pi}} = \frac43\pi$.

By 10b5, $\frac12\Gamma\big(\frac{a+b+2}{2}\big)\int_0^{\pi/2}\cos^a\theta\sin^b\theta\,\mathrm{d}\theta = \frac12\Gamma\big(\frac{a+1}{2}\big)\cdot\frac12\Gamma\big(\frac{b+1}{2}\big)$ for $a,b \in (-1,\infty)$; that is,

(10d8) $$\int_0^{\pi/2}\cos^{\alpha-1}\theta\sin^{\beta-1}\theta\,\mathrm{d}\theta = \frac12\frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})} \quad \text{for } \alpha,\beta \in (0,\infty)\,.$$

In particular,

(10d9) $$\int_0^{\pi/2}\sin^{\alpha-1}\theta\,\mathrm{d}\theta = \int_0^{\pi/2}\cos^{\alpha-1}\theta\,\mathrm{d}\theta = \frac{\sqrt{\pi}}{2}\cdot\frac{\Gamma\big(\frac{\alpha}{2}\big)}{\Gamma\big(\frac{\alpha+1}{2}\big)}\,.$$

The trigonometric functions can be eliminated: $\int_0^{\pi/2} \cos^{\alpha-1}\theta \sin^{\beta-1}\theta \,\mathrm{d}\theta = \frac12 \int_0^{\pi/2} \cos^{\alpha-2}\theta \sin^{\beta-2}\theta \cdot 2\sin\theta\cos\theta \,\mathrm{d}\theta = \frac12 \int_0^1 (1-u)^{\frac{\alpha-2}{2}} u^{\frac{\beta-2}{2}} \,\mathrm{d}u$; thus,

(10d10) $$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \mathrm{B}(\alpha,\beta) \quad \text{for } \alpha,\beta \in (0,\infty)\,,$$

where

(10d11) $$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha,\beta \in (0,\infty)$$

is another special function, the *beta function.*

10d12 Exercise. Check that $\mathrm{B}(x,x) = 2^{1-2x}B(x,\frac12)$.
Hint: $\int_0^{\pi/2} \big(\frac{2\sin\theta\cos\theta}{2}\big)^{2x-1}\,\mathrm{d}\theta$.

10d13 Exercise. Check the *duplication formula*:
$$\boxed{\Gamma(2x) = \frac{2^{2x-1}}{\sqrt{\pi}}\Gamma(x)\,\Gamma\Big(x+\frac12\Big)}.$$
Hint: use 10d12.

10d14 Exercise. Calculate $\int_0^1 x^4\sqrt{1-x^2}\,\mathrm{d}x$.
Answer: $\frac{\pi}{32}$.

10d15 Exercise. Calculate $\int_0^\infty x^m \mathrm{e}^{-x^n}\,\mathrm{d}x$.
Answer: $\frac1n \Gamma\big(\frac{m+1}{n}\big)$.

10d16 Exercise. Calculate $\int_0^1 x^m (\ln x)^n\,\mathrm{d}x$.
Answer: $\frac{(-1)^n n!}{(m+1)^{n+1}}$.

10d17 Exercise. Calculate $\int_0^{\pi/2} \frac{\mathrm{d}x}{\sqrt{\cos x}}$.
Answer: $\frac{\Gamma^2(1/4)}{2\sqrt{2\pi}}$.

10d18 Exercise. Check that $\Gamma(p)\Gamma(1-p) = \int_0^\infty \frac{x^{p-1}}{1+x}\,\mathrm{d}x$.
Hint: change x to t via $(1+x)(1-t)=1$.

We mention without proof another useful formula

$$\int_0^\infty \frac{x^{p-1}}{1+x}\,\mathrm{d}x = \frac{\pi}{\sin\pi p} \qquad \text{for } 0<p<1\,.$$

There is a simple proof that that uses the residues theorem from the complex analysis course. This formula yields that $\Gamma(t)\Gamma(1-t) = \frac{\pi}{\sin\pi t}$.

Is the function Γ continuous?

For every compact interval $[t_0, t_1] \subset (0, \infty)$ the given function of two variables $(t, x) \mapsto x^{t-1}\mathrm{e}^{-x}$ is Lipschitz continuous on $[t_0, t_1] \times \big[\frac{1}{k}, k\big]$, therefore the integral is Lipschitz continuous on $[t_0, t_1]$ (recall 7b). Also,

$$\int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \to \Gamma(t) \quad \text{uniformly on } [t_0, t_1]\,,$$

since $\int_0^{1/k} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \le \int_0^{1/k} x^{t_0-1}\,\mathrm{d}x \to 0$ as $k \to \infty$ and $\int_k^\infty x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \le \int_k^\infty x^{t_1-1}\mathrm{e}^{-x}\,\mathrm{d}x \to 0$ as $k \to \infty$. It follows that Γ is continuous on arbitrary $[t_0, t_1]$, therefore, on the whole $(0, \infty)$.

In particular, $t\Gamma(t) = \Gamma(t+1) \to \Gamma(1) = 1$ as $t \to 0+$; that is,

$$\Gamma(t) = \frac{1}{t} + o\Big(\frac{1}{t}\Big) \quad \text{as } t \to 0+\ .$$

Is the function Γ differentiable?

By Theorem 7e1 the function $t \mapsto \int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x$ is continuously differentiable, and its derivative is $t \mapsto \int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x$; this relation results from application of Prop. 7b4 (iterated integral) to the function $(t, x) \mapsto \frac{\partial}{\partial t} x^{t-1}\mathrm{e}^{-x} = x^{t-1}\mathrm{e}^{-x}\ln x$ on $[t_0, t_1] \times \big[\frac{1}{k}, k\big]$. Regretfully, iterated improper integral is not an easy matter.[1] Instead, we use exhaustion, as follows. As before,

$$\int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x \to \int_0^\infty x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x \quad \text{uniformly on } [t_0, t_1]$$

(check it), therefore

$$\int_{t_0}^{t_1} \mathrm{d}t \int_{1/k}^{k} x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x \to \int_{t_0}^{t_1} \mathrm{d}t \int_0^\infty x^{t-1}\mathrm{e}^{-x}\ln x\,\mathrm{d}x\,.$$

On the other hand,

$$\int_{1/k}^{k} \mathrm{d}x \int_{t_0}^{t_1} \mathrm{d}t\, x^{t-1}\mathrm{e}^{-x}\ln x = \int_{1/k}^{k} \Big(x^{t-1}\mathrm{e}^{-x}\Big|_{t=t_0}^{t_1} \Big)\,\mathrm{d}x =$$
$$= \int_{1/k}^{k} x^{t_1-1}\mathrm{e}^{-x}\,\mathrm{d}x - \int_{1/k}^{k} x^{t_0-1}\mathrm{e}^{-x}\,\mathrm{d}x \to \Gamma(t_1) - \Gamma(t_0)\,.$$

[1] If $f : [0,1] \times [0,1] \to [0, \infty)$ is improperly integrable, then $f_x : y \to f(x, y)$ is improperly integrable on $[0, 1]$ for almost every x; however, the function $\varphi : x \to \int f_x$ need not be improperly integrable. Rather, φ is equivalent to a function semicontinuous from below (possibly, unbounded on every interval), and $_*\!\int \varphi = \int f$.

Thus, $\Gamma(t_1) - \Gamma(t_0) = \int_{t_0}^{t_1} \mathrm{d}t \int_0^\infty x^{t-1} \mathrm{e}^{-x} \ln x \, \mathrm{d}x$, which implies

$$\Gamma'(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x} \ln x \, \mathrm{d}x \, .$$

Similarly, Γ' is differentiable; continuing this way we get

$$\Gamma^{(k)}(t) = \int_0^\infty x^{t-1} \mathrm{e}^{-x} (\ln x)^k \, \mathrm{d}x \quad \text{for } k = 1, 2, \dots$$

10e Normed space of equivalence classes

In order to integrate signed functions we reuse the simple trick of (8e1). We define

$$\int_G (g - h) = \int_G g - \int_G h$$

whenever $g, h : G \to [0, \infty)$ are continuous almost everywhere and $\int_G g < \infty$, $\int_G h < \infty$; this definition is correct, that is,

$$\int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2 \quad \text{whenever } g_1 - h_1 = g_2 - h_2 \, ;$$

proof:

(10e1)

$$g_1 - h_1 = g_2 - h_2 \implies g_1 + h_2 = g_2 + h_1 \implies \int_G (g_1 + h_2) = \int_G (g_2 + h_1) \implies$$
$$\implies \int_G g_1 + \int_G h_2 = \int_G g_2 + \int_G h_1 \implies \int_G g_1 - \int_G h_1 = \int_G g_2 - \int_G h_2 \, .$$

10e2 Lemma. The following two conditions on a function $f : G \to \mathbb{R}$ continuous almost everywhere are equivalent:

(a) there exist $g, h : G \to [0, \infty)$, continuous almost everywhere, such that $\int_G g < \infty$, $\int_G h < \infty$ and $f = g - h$;

(b) $\int_G |f| < \infty$.

Proof. (a)$\Longrightarrow$(b): $\int_G |g - h| \le \int_G \big(|g| + |h| \big) = \int_G |g| + \int_G |h| < \infty$.

(b)$\Longrightarrow$(a): we introduce the positive part f^+ and the negative part f^- of f,

(10e3)
$$f^+(x) = \max\big(0, f(x)\big) \, , \quad f^-(x) = \max\big(0, -f(x)\big) \, ;$$
$$f^- = (-f)^+ \, ; \quad f = f^+ - f^- \, ; \quad |f| = f^+ + f^- \, ;$$

they are continuous almost everywhere (think, why); $\int_G f^+ \le \int_G |f| < \infty$, $\int_G f^- \le \int_G |f| < \infty$; and $f^+ - f^- = f$. □

We summarize:

$$\int_G f = \int_G f^+ - \int_G f^- \tag{10e4}$$

whenever $f : G \to \mathbb{R}$ is continuous almost everywhere and such that $\int_G |f| < \infty$. Such functions will be called *improperly integrable*[1] (on G).

10e5 Exercise. Prove linearity: $\int_G cf = c \int_G f$ for $c \in \mathbb{R}$, and $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2$.

Similarly to Sect. 6e, a function $f : G \to \mathbb{R}$ continuous almost everywhere will be called *negligible* if $\int_G |f| = 0$. Functions f, g continuous almost everywhere and such that $f - g$ is negligible will be called equivalent. The equivalence class of f will be denoted $[f]$.

Improperly integrable functions $f : G \to \mathbb{R}$ are a vector space. On this space, the functional $f \mapsto \int_G |f|$ is a seminorm. The corresponding equivalence classes are a normed space (therefore also a metric space). Similarly to 6e3, the integral is a continuous linear functional on this space.

If G is Jordan measurable then the space of improperly integrable functions on G is embedded into the space of improperly integrable functions on $\mathbb{R}^n$ by $f \mapsto f \cdot 1\!\!1_G$.

10e6 Lemma. Let $G_1 \subset G_2 \subset \mathbb{R}^n$ be two open sets, and $f : G_2 \to \mathbb{R}$ continuous almost everywhere. If $f = 0$ almost everywhere on $G_2 \setminus G_1$, then $f \cdot 1\!\!1_{G_1}$ is continuous almost everywhere on G_2 and equivalent to f.

Proof. The set $A = \{x \in G_2 \setminus G_1 : f(x) \ne 0\}$ is of Lebesgue measure 0. If f is continuous at $x \in G_2$ while $f \cdot 1\!\!1_{G_1}$ is not, then clearly $x \in G_2 \setminus G_1$; and moreover, $x \in A$ (since $\lim_{t \to x} f(t) = 0$ implies $\lim_{t \to x} \big(f(t) \cdot 1\!\!1_{G_1}(t) \big) = 0$). Thus, $f \cdot 1\!\!1_{G_1}$ is continuous almost everywhere on G_2. Finally, $f \cdot 1\!\!1_{G_1} = f$ on $G_2 \setminus A$. $\square$

In particular, if G_1 contains almost all points of G_2 (that is, $G_2 \setminus G_1$ is of Lebesgue measure 0),[2] then the condition "$f = 0$ almost everywhere on $G_2 \setminus G_1$" holds vacuously; in this case the values of f on $G_2 \setminus G_1$ do not influence the equivalence class of f.

10e7 Corollary. Let $G_1 \subset G_2 \subset \mathbb{R}^n$ be two open sets, and $f : G_2 \to \mathbb{R}$ improperly integrable. If $f = 0$ almost everywhere on $G_2 \setminus G_1$, then $\int_{G_2} f = \int_{G_1} f$.

[1] In one dimension they are usually called absolutely (improperly) integrable.

[2] Warning: this condition implies $v_*(G_1) = v_*(G_2)$ and is implied by $v_*(G_1) = v_*(G_2) < \infty$, but is not implied by $v_*(G_1) = v_*(G_2) = \infty$.

Proof. First, $\int_{G_2} f = \int_{G_2} f \cdot 1\!\!1_{G_1}$ since $[f] = [f \cdot 1\!\!1_{G_1}]$ by 10e6. Second, $\int_{G_2} f^+ \cdot 1\!\!1_{G_1} = \int_{G_1} f^+$ by 10b11; the same holds for f^-, and therefore for $f^+ - f^- = f$. $\square$

Once again, if G_1 contains almost all points of G_2, then we get $\int_{G_2} f = \int_{G_1} f$ for all f improperly integrable on G_2.

We may admit a function f partially defined on G, provided that for almost every $x \in G$, f is defined near x.[1] [2] In other words: $f : G \setminus A \to \mathbb{R}$, and the (relative) closure of A in G is of Lebesgue measure 0. In this case almost all points of G belong to $(G \setminus A)^\circ$. Such partially defined functions may be used as well as functions defined on the whole G, whenever only equivalence classes matter.

Thus, we need not hesitate saying that, for instance, $\int_{-1}^{1} \frac{\mathrm{d}x}{|x|^\alpha} = \frac{2}{1-\alpha}$ for $\alpha < 1$, even though the integrand is undefined at 0.

10e8 Proposition (Exhaustion)**.** Let open sets $G_1 \subset G_2 \subset \dots \subset G \subset \mathbb{R}^n$ be such that $\cup_k G_k$ contains almost all points of G. Then

$$\int_{G_k} f \to \int_G f \quad \text{as } k \to \infty$$

for all f improperly integrable on G.

Proof. First, the open set $\tilde{G} = \cup_k G_k$ contains almost all points of G, therefore $\int_G f = \int_{\tilde{G}} f$. Second, $G_k \uparrow \tilde{G}$; 10b10 gives $\int_{G_k} f = \int_{G_k} f^+ - \int_{G_k} f^- \to \int_{\tilde{G}} f^+ - \int_{\tilde{G}} f^- = \int_{\tilde{G}} f$. $\square$

In particular, if G_k are also Jordan measurable and such that f is defined and bounded on each G_k, then $\int_{G_k} f$ is the proper (Riemann) integral, and we obtain the improper integral $\int_G f$ as the limit of proper integrals.

10e9 Proposition. Let $G \subset \mathbb{R}^n$ be an open set, and f an improperly integrable function on G.[3] Then there exist Jordan measurable open sets $G_1 \subset G_2 \subset \dots$ such that $G_k \subset G$, $\cup_k G_k$ contains almost all points of G, and f is defined and bounded on every G_k.

[1] Not just "at x"!

[2] In fact, for every set $A \subset G$ of Lebesgue measure 0 (even if dense in G), every function $f : G \setminus A \to \mathbb{R}$ continuous almost everywhere can be extended to a function $G \to \mathbb{R}$ continuous almost everywhere (and all such extentions evidently are mutually eqiuvalent). Hint: $\liminf_{t \to x, t \in G \setminus A} f(t) \le \tilde{f}(x) \le \limsup_{t \to x, t \in G \setminus A} f(t)$ for every $x \in A$ such that f is bounded near x. Such $\tilde{f}$ is continuous at every continuity point of f.

[3] We admit partially defined f, as explained above.

Proof. We'll prove that for every box $B \subset \mathbb{R}^n$ and every $\varepsilon > 0$ there exists a Jordan measurable open set $G_{B,\varepsilon} \subset B^\circ \cap G$ such that f is defined and bounded on $G_{B,\varepsilon}$ and $v(G_{B,\varepsilon}) \ge v_*(G \cap B) - \varepsilon$. This is sufficient, since we may take $B_k \uparrow \mathbb{R}^n$ and $\varepsilon_k \to 0$, and then the sets $G_k = G_{B_1,\varepsilon_1} \cup \dots \cup G_{B_k,\varepsilon_k}$ fit.

We take a set $A \subset G$ of Lebesgue measure 0 such that for each $x \in G \setminus A$, f is defined near x and continuous at x.[1] Given B and ε, we take an open $U \subset G$ such that $A \cap \overline{B} \subset U$ and $v_*(U \cap B^\circ) \le \varepsilon/2$. We also take a compact set $K \subset B^\circ \cap G$ such that $v^*(K) \ge v_*(B^\circ \cap G) - \varepsilon/2$. Then the compact set $K \setminus U$ satisfies $v^*(K \setminus U) \ge v^*(K) - v_*(U \cap B^\circ) \ge v_*(B^\circ \cap G) - \varepsilon$, and every point of $K \setminus U$ has a Jordan measurable open neighborhood (just a ball, or a box) on which f is defined and bounded. We choose a finite subcovering; the union of the chosen neighborhoods (intersected with $B^\circ \cap G$) is $G_{B,\varepsilon}$. $\square$

The normed space of equivalence classes, introduced above, does not admit an inner product.[2] Now we turn to improperly *square integrable* functions; these are functions $f : G \to \mathbb{R}$ continuous almost everywhere and such that $\int f^2 < \infty$. If $[f] = [g]$ then $\int f^2 = \int g^2$ (check it via 10b12), thus, square integrability applies to equivalence classes. We denote the set of all square integrable equivalence classes by $\tilde{L}^2(G)$,[3] and often write $f \in \tilde{L}^2(G)$ instead of $[f] \in \tilde{L}^2(G)$. This set is a vector space (since $(f+g)^2 \le (f+g)^2 + (f-g)^2 = 2f^2 + 2g^2$).

If $f, g \in \tilde{L}^2(G)$ then their pointwise product fg is improperly integrable (since $f^2 - 2|fg| + g^2 \ge 0$), and we define the inner product

$$\langle [f], [g] \rangle = \int fg \tag{10e10}$$

and the corresponding norm

$$\|[f]\|_2 = \sqrt{\langle [f], [f] \rangle}\,, \quad \text{that is,} \quad \|f\|_2 = \sqrt{\int f^2} \tag{10e11}$$

satisfying $[f] \ne [0] \implies \|[f]\|_2 > 0$ (check it via 10b12). We often write $\langle f, g \rangle$ and $\|f\|_2$ instead of $\langle [f], [g] \rangle$ and $\|[f]\|_2$. Every 2-dimensional subspace of $\tilde{L}^2(G)$ is a Euclidean plane, which ensures the triangle inequality

$$\|f+g\|_2 \le \|f\|_2 + \|g\|_2 \tag{10e12}$$

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[1] Not "continuous near x"!

[2] Its 2-dimensional subspace of step functions is not the Euclidean plane; you may check it similarly to the paragraph before 1e3.

[3] The widely used notation L^2 is reserved for the corresponding notion in the framework of Lebesgue integration.

and the Cauchy-Schwarz inequality

$$-\|f\|_2\|g\|_2 \le \langle f,g\rangle \le \|f\|_2\|g\|_2\,. \tag{10e13}$$

More generally, for arbitrary $p \in [1,\infty)$ we introduce the norm

$$\|f\|_p = \Big(\int |f|^p \Big)^{1/p}$$

on the vector space $\tilde L^p(G)$ of $[f]$ such that $\int |f|^p < \infty$ (two special cases $p=1$ and $p=2$ being already treated). The triangle inequality

$$\|f+g\|_p \le \|f\|_p + \|g\|_p$$

follows from convexity of the ball $\{f : \int |f|^p \le 1\}$ (recall 1e14); convexity of the ball follows from convexity of the functional $f \mapsto \int |f|^p$ (recall 1e13); and convexity of this functional follows from convexity of the function $t \mapsto |t|^p$ (similarly to 1e15). The triangle inequality ensures that $\tilde L^p(G)$ is a vector space. The Hölder inequality

$$\Big| \int fg \Big| \le \|f\|_p\|g\|_q \quad \text{for } f \in \tilde L^p(G),\, g \in \tilde L^q(G),\, \frac1p + \frac1q = 1\,,$$

is obtained similarly to 6d15(b) (but harder); first, $ab \le \frac{a^p}{p} + \frac{b^q}{q}$ for $a,b \in [0,\infty)$; second,

$$\Big| \int fg \Big| \le \min_{c>0} \Big(\frac1p \|cf\|_p^p + \frac1q \Big\| \frac1c g \Big\|_q^q \Big) = \|f\|_p\|g\|_q\,.$$

10f Change of variables

10f1 Theorem. Let $U,V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then

(a) f is improperly integrable on V if and only if $(f\circ\varphi)|\det D\varphi|$ is improperly integrable on U; and

(b) in this case

$$\int_V f = \int_U (f\circ\varphi)|\det D\varphi|\,.$$

Proof. We reuse the arguments from the proof of Theorem 9a1. There, U and V are assumed to be Jordan measurable, but this assumption is used only in the last paragraph of the proof. Before that we constructed Jordan measurable open sets $V_k \uparrow V$ (denoted there by K_i°)[1] such that $\overline{V}_k \subset V$,

[1] But notations U,V are swapped there; compare 9a1 and 9a2.

the sets $\varphi^{-1}(V_k) = U_k \uparrow U$ are Jordan measurable, $\overline{U}_k \subset U$, and we showed that the claim of the theorem (for Riemann integral) holds for every f whose support is contained in some V_k, therefore, for every f with a compact support inside V.

Now, given $f : V \to \mathbb{R}$, we note that

f is improperly integrable on V if and only if
it is improperly integrable on each V_k, and
$\lim_k \int_{V_k} |f| < \infty$,
and in this case $\int_V f = \lim_k \int_{V_k} f$.

Similarly,

$(f \circ \varphi)|\det D\varphi|$ is improperly integrable on U if and only if
it is improperly integrable on each U_k, and
$\lim_k \int_{U_k} (f \circ \varphi)|\det D\varphi| < \infty$,
and in this case $\int_U (f \circ \varphi)|\det D\varphi| = \lim_k \int_{U_k} (f \circ \varphi)|\det D\varphi|$.

Thus, in order to prove the theorem for arbitrary f it is sufficient to prove it for f whose support is contained in some V_k.

Theorem 9a1, applied to the diffeomorphism $\varphi|_{U_k} : U_k \to V_k$, gives the needed claim for proper integration, that is, for bounded f (boundedness of $(f \circ \varphi)|\det D\varphi|$ follows, since the determinant is bounded on U_k). It remains to generalize this claim to unbounded $f : V_k \to \mathbb{R}$. Taking into account that $f = f^+ - f^-$ we may assume that $f : V_k \to [0, \infty)$. We note that

f is improperly integrable on V_k if and only if
each $f_\ell = \min(f, \ell)$ is integrable on V_k,
and in this case $\int_{V_k} f = \lim_\ell \int_{V_k} f_\ell$

(since every integrable g such that $0 \le g \le f \cdot 1\!\!1_{V_k}$ satisfies $g \le \ell$ for some ℓ). Similarly, taking into account that $|\det D\varphi|$ is bounded away from 0 on U_k, we see that

$(f \circ \varphi)|\det D\varphi|$ is improperly integrable on U_k if and only if
each $(f_\ell \circ \varphi)|\det D\varphi|$ is improperly integrable on U_k,
and in this case $\int_{U_k} (f \circ \varphi)|\det D\varphi| = \lim_\ell \int_{U_k} (f_\ell \circ \varphi)|\det D\varphi|$.

The claim follows. □

10f2 Exercise. Prove the equality (10d10) once again, avoiding (10b5) and triginometric functions; to this end, consider

$$\Big(\int_0^\infty u^{\alpha+\beta-1} \mathrm{e}^{-u} \, \mathrm{d}u \Big) \Big(\int_0^1 x^{\alpha-1} (1-x)^{\beta-1} \, \mathrm{d}x \Big)$$

and change the variables u, x to t_1, t_2 as follows:

$$\begin{cases} t_1 &= ux \\ t_2 &= u(1-x) \end{cases} \qquad \begin{cases} u = t_1 + t_2 \\ x = \frac{t_1}{t_1+t_2} \end{cases}$$

10g Multidimensional beta integrals of Dirichlet

10g1 Proposition.

$$\underset{\substack{x_1,\dots x_n>0,\\ x_1+\dots+x_n<1}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}$$

for all $p_1,\dots p_n > 0$.

For the proof, we denote

$$I(p_1,\dots,p_n) = \underset{\substack{x_1,\dots x_n>0,\\ x_1+\dots+x_n<1}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

This integral is improper, unless $p_1,\dots,p_n \ge 1$.

10g2 Lemma. $I(p_1,\dots,p_n) = \mathrm{B}(p_n, p_1+\dots+p_{n-1}+1)I(p_1,\dots,p_{n-1})$.

Proof. We introduce proper integrals

$$I_\varepsilon(p_1,\dots,p_n) = \underset{\substack{x_1,\dots x_n>\varepsilon,\\ x_1+\dots+x_n<1}}{\int\cdots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n$$

for $\varepsilon > 0$.[1] Clearly, $I_\varepsilon(p_1,\dots,p_n) \le I(p_1,\dots,p_n)$, and $I_\varepsilon(p_1,\dots,p_n) \to I(p_1,\dots,p_n)$ as $\varepsilon \to 0+$.

The change of variables $\xi = ax$ (that is, $\xi_1 = ax_1,\dots,\xi_n = ax_n$) gives (by Theorem 10f1)

$$\underset{\substack{\xi_1,\dots x_n>a\varepsilon,\\ x_1+\dots+x_n<a}}{\int\cdots\int} \xi_1^{p_1-1}\dots\xi_n^{p_n-1}\,\mathrm{d}\xi_1\dots\mathrm{d}\xi_n = a^{p_1+\dots+p_n} I_\varepsilon(p_1,\dots,p_n) \quad \text{for } a>0\,.$$

[1] If $n\varepsilon \ge 1$ then $I_\varepsilon(p_1,\dots,p_n) = 0$, of course.

We use iterated integral (proper!):

$$I_\varepsilon(p_1,\dots,p_n) = \int_\varepsilon^1 \mathrm{d}x_n\, x_n^{p_n-1} \underset{\substack{x_1,\dots x_{n-1}>\varepsilon,\\ x_1+\dots+x_{n-1}<1-x_n}}{\int\dots\int} x_1^{p_1-1}\dots x_{n-1}^{p_{n-1}-1}\,\mathrm{d}x_1\dots\mathrm{d}x_{n-1} = \\ = \int_\varepsilon^1 x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}} I_{\varepsilon/(1-x_n)}(p_1,\dots,p_{n-1})\,\mathrm{d}x_n\,.$$

On one hand,

$$I_\varepsilon(p_1,\dots,p_n) \le I(p_1,\dots,p_{n-1}) \int_0^1 x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}}\,\mathrm{d}x_n = \\ = I(p_1,\dots,p_{n-1})\mathrm{B}(p_n, p_1+\dots+p_{n-1}+1)$$

for all ε, therefore $I(p_1,\dots,p_n) \le \mathrm{B}(p_n,p_1+\dots+p_{n-1}+1)I(p_1,\dots,p_{n-1})$.

On the other hand, for arbitrary $\delta > 0$,

$$I_\varepsilon(p_1,\dots,p_n) \ge \int_\varepsilon^{1-\delta} x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}} I_{\varepsilon/(1-x_n)}(p_1,\dots,p_{n-1})\,\mathrm{d}x_n \ge \\ \ge \int_\varepsilon^{1-\delta} x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}} I_{\varepsilon/\delta}(p_1,\dots,p_{n-1})\,\mathrm{d}x_n$$

for all ε, therefore

$$I(p_1,\dots,p_n) \ge \int_0^{1-\delta} x_n^{p_n-1}(1-x_n)^{p_1+\dots+p_{n-1}} I(p_1,\dots,p_{n-1})\,\mathrm{d}x_n$$

for all δ, and finally, $I(p_1,\dots,p_n) \ge \mathrm{B}(p_n,p_1+\dots+p_{n-1}+1)I(p_1,\dots,p_{n-1})$.

□

Proof *of Prop. 10g1.*

Induction in the dimension n. For $n=1$ the formula is obvious:

$$\int_0^1 x_1^{p_1-1}\,\mathrm{d}x_1 = \frac{1}{p_1} = \frac{\Gamma(p_1)}{\Gamma(p_1+1)}\,.$$

From $n-1$ to n: using 10g2,

$$I(p_1,\dots,p_n) = \frac{\Gamma(p_n)\Gamma(p_1+\dots+p_{n-1}+1)}{\Gamma(p_1+\dots+p_n+1)} \cdot \frac{\Gamma(p_1)\dots\Gamma(p_{n-1})}{\Gamma(p_1+\dots+p_{n-1}+1)} = \\ = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}\,.$$

□

There is a seemingly more general formula,

$$\underset{\substack{x_1,\dots,x_n>0,\\ x_1^{\gamma_1}+\dots+x_n^{\gamma_n}<1}}{\int\dots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{\gamma_1\dots\gamma_n}\cdot\frac{\Gamma\big(\frac{p_1}{\gamma_1}\big)\dots\Gamma\big(\frac{p_n}{\gamma_n}\big)}{\Gamma\big(\frac{p_1}{\gamma_1}+\dots+\frac{p_n}{\gamma_n}+1\big)}\,,$$

easily obtained from the previous one by the change of variables $y_j = x_j^{\gamma_j}$.

A special case: $p_1 = \dots = p_n = 1$, $\gamma_1 = \dots = \gamma_n = p$;

$$\underset{\substack{x_1,\dots,x_n>0\\ x_1^p+\dots+x_n^p<1}}{\int\dots\int} \mathrm{d}x_1\dots\mathrm{d}x_n = \frac{\Gamma^n\big(\frac1p\big)}{p^n\Gamma\big(\frac np+1\big)}\,.$$

We've found the volume of the unit ball in the metric l_p:

$$\boxed{v\big(B_p(1)\big) = \frac{2^n\Gamma^n\big(\frac1p\big)}{p^n\Gamma\big(\frac np+1\big)}\,.}$$

If $p = 2$, the formula gives us (again; see (10d7)) the volume of the standard unit ball:

$$\boxed{V_n = v\big(B_2(1)\big) = \frac{2\pi^{n/2}}{n\Gamma\big(\frac n2\big)}\,.}$$

We also see that the volume of the unit ball in the l_1-metric equals $\frac{2^n}{n!}$.

Question: what does the formula give in the $p\to\infty$ limit?

10g3 Exercise. Show that

$$\underset{\substack{x_1+\dots+x_n<1\\ x_1,\dots,x_n>0}}{\int\dots\int} \varphi(x_1+\dots+x_n)\,\mathrm{d}x_1\dots\mathrm{d}x_n = \frac{1}{(n-1)!}\int_0^1 \varphi(s)s^{n-1}\,\mathrm{d}s$$

for every "good" function $\varphi : [0,1]\to\mathbb{R}$ and, more generally,

$$\underset{\substack{x_1+\dots+x_n<1\\ x_1,\dots,x_n>0}}{\int\dots\int} \varphi(x_1+\dots+x_n)x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n =$$

$$= \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n)}\int_0^1 \varphi(u)u^{p_1+\dots p_n-1}\,\mathrm{d}u\,.$$

Hint: consider

$$\int_0^1 \mathrm{d}s\,\varphi'(s)\underset{\substack{x_1+\dots+x_n<s\\ x_1,\dots,x_n>0}}{\int\dots\int} x_1^{p_1-1}\dots x_n^{p_n-1}\,\mathrm{d}x_1\dots\mathrm{d}x_n\,.$$

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