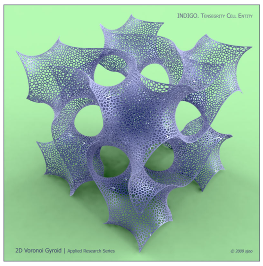
12 Manifolds in $\mathbb{R}^n$

Length of a curve and area of a surface in $\mathbb{R}^3$ are special cases of n-dimensional volume of an n-dimensional manifold in $\mathbb{R}^N$, given infinitesimally by the volume form.

Image: (CC) Jonathan Johanson, `http://cliptic.wordpress.com`



12a Planar curves

Recall the notions of relative neighborhood and relative open set.

Let $M \subset \mathbb{R}^2$.

12a1 Definition. A *chart* of M is a pair (G, ψ) of an open set $G \neq \emptyset$ in $\mathbb{R}$ and a mapping $\psi : G \to M$ such that

(a) $\psi(G)$ is (relatively) open in M;
(b) ψ is a homeomorphism from G to $\psi(G)$;
(c) $\psi \in C^1(G \to \mathbb{R}^2)$;
(d) $D\psi$ does not vanish (on G).

If a point of M belongs to $\psi(G)$, we say that (G, ψ) is a chart of M around this point.

12a2 Definition. A *co-chart*[1] of M is a pair (U, φ) of an open set U in $\mathbb{R}^2$ and a function $\varphi : U \to \mathbb{R}$ such that

(a) $M \cap U = \{x \in U : \varphi(x) = 0\} \neq \emptyset$;
(b) $\varphi \in C^1(U)$;
(c) $D\varphi$ does not vanish on $M \cap U$.

If a point of M belongs to U, we say that (U, φ) is a co-chart of M around this point.

In particular, if M is the graph of a function f of class C^1 near x_0, we may take $\psi(t) = \big(t, f(t)\big)$ and $\varphi(x, y) = y - f(x)$. The case $x = g(y)$ may be treated similarly. We'll see soon that the general case reduces to these two special cases (locally, but not globally).

[1] Not a standard terminology.

12a3 Remark. (a) If (G, ψ) is a chart of M and $G_0 \subset G$ an open subset (nonempty), then $(G_0, \psi|_{G_0})$ is a chart of M;[1]

(b) if (U, φ) is a co-chart of M and $U_0 \subset U$ is an open subset (that intersects M), then $(U_0, \varphi|_{U_0})$ is a co-chart of M.

12a4 Exercise. Let $h : \mathbb{R}^2 \to \mathbb{R}^2$ be a diffeomorphism. If (G, ψ) is a chart of M, then $(G, h \circ \psi)$ is a chart of $h(M)$. If (U, φ) is a co-chart of M, then $\big(h(U), \varphi \circ h^{-1}\big)$ is a co-chart of $h(M)$.

Prove it.

12a5 Proposition. The following three conditions on a set $M \subset \mathbb{R}^2$ and a point $(x_0, y_0) \in M$ are equivalent:

(a) there exists a chart of M around (x_0, y_0);

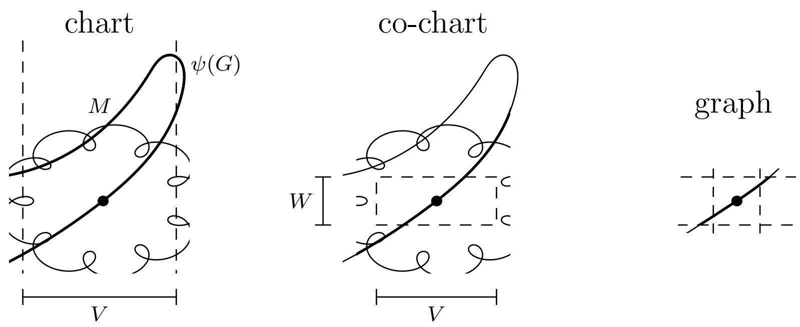
(b) there exists a co-chart of M around (x_0, y_0);

(c) there exists a local diffeomorphism $h : \mathbb{R}^2 \to \mathbb{R}^2$ near (x_0, y_0) such that

$$(x, y) \in M \quad \Longleftrightarrow \quad h(x, y) \in \mathbb{R} \times \{0\}$$

for all (x, y) near (x_0, y_0).

Proof. By 12a4, (c)$\Longrightarrow$(a) (and (c)$\Longrightarrow$(b)), since the line $\mathbb{R} \times \{0\}$ evidently has a chart (and a co-chart) near every point.



From a chart to a co-chart (and graph).

(a)$\Longrightarrow$(b): given G and ψ, $\psi(t) = \big(\psi_1(t), \psi_2(t)\big)$, $\psi(t_0) = (x_0, y_0)$, we assume that $\psi_1'(t_0) \neq 0$ (otherwise we swap the coordinates x, y) and apply to ψ_1 the inverse function theorem 4c2. Reducing G as needed we ensure that ψ_1 is a diffeomorphism from G to an open neighborhood V of x_0. Taking into account that $\psi(G)$ is a neighborhood of (x_0, y_0) in M, we reduce V and G (again) and choose a neighborhood W of y_0 such that

$$M \cap (V \times W) = \psi(G) \cap (V \times W)\,.$$

[1] $\psi(G_0)$ is open in $\psi(G)$, and $\psi(G)$ is open in M, therefore $\psi(G_0)$ is open in M (think, why).

We take $U = V \times W$, define $\varphi : U \to R$ by

$$\varphi(x,y) = y - \psi_2\bigl(\psi_1^{-1}(x)\bigr)\,,$$

and check that (U,φ) is a co-chart.

(b)$\Longrightarrow$(c): given U and φ, we assume that $(D_2\varphi)_{(x_0,y_0)} \ne 0$ (otherwise we swap the coordinates x,y). The mapping $h : (x,y) \mapsto (x, \varphi(x,y))$ fits, as was seen in the proof of Theorem 5c1. $\square$

12a6 Definition. A nonempty set $M \subset \mathbb{R}^2$ is a one-dimensional *manifold* (or 1-manifold) if for every $(x_0,y_0) \in M$ there exists a chart of M around (x_0,y_0).

"Co-chart" instead of "chart" gives an equivalent definition due to 12a5.

12a7 Definition. Let $M \subset \mathbb{R}^2$ be a 1-manifold; a function $f : M \to \mathbb{R}$ is *continuously differentiable* if for every chart (G,ψ) of M the function $f \circ \psi$ is continuously differentiable on G.

12a8 Exercise. The set $C^1(M)$ of all continuously differentiable functions on M is an algebra; that is, a vector space, and $f,g \in C^1(M) \implies fg \in C^1(M)$. Also, if $\varphi \in C^1(\mathbb{R})$ and $f \in C^1(M)$ then $\varphi \circ f \in C^1(M)$.

Prove it.

12a9 Exercise. Let $M \subset \mathbb{R}^2$ be a 1-manifold, $f : M \to \mathbb{R}$, and for every $x \in M$ there exists a chart (G,ψ) of M around x such that $f \circ \psi \in C^1(G)$. Then $f \in C^1(M)$.

Prove it.

12a10 Exercise. Which of the following subsets of $\mathbb{R}^2$ are 1-manifolds? Prove your answers, both affirmative and negative.

* $M_1 = \mathbb{R} \times \{0\}$;
* $M_2 = [0,1] \times \{0\}$;
* $M_3 = (0,1) \times \{0\}$;
* $M_4 = \{(0,0)\}$;
* $M_5 = \mathbb{R} \times \{0,1\}$;
* $M_6 = \mathbb{R} \times \mathbb{Z}$;
* $M_7 = \mathbb{R} \times \{1, \frac12, \frac13, \dots\}$;
* $M_8 = M_7 \cup M_1$;
* $M_9 = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, \varphi = 1/r\}$;
* $M_{10} = M_9 \cup M_4$;

* $M_{11} = \{(r\cos\varphi, r\sin\varphi) : 0 < r < 1, \varphi = 1/(1-r)\}$;
* $M_{12} = \{(x,y) : x^2 + y^2 = 1\}$;
* $M_{13} = M_{11} \cup M_{12}$;
* $M_p = \{(x,y) : x^p + y^p = 1\}$; examine all $p \in (-\infty, 0) \cup (0, \infty)$.

12b Higher dimensions; orientation; tangent space

Let $M \subset \mathbb{R}^N$, $n \in \{1, \dots, N-1\}$, and $x_0 \in M$.

12b1 Definition. A *chart* (n-chart) of M is a pair (G, ψ) of an open set $G \ne \emptyset$ in $\mathbb{R}^n$ and a mapping $\psi : G \to M$ such that
(a) $\psi(G)$ is (relatively) open in M;
(b) ψ is a homeomorphism from G to $\psi(G)$;
(c) $\psi \in C^1(G \to \mathbb{R}^N)$;
(d) for every $u \in G$ the linear operator $(D\psi)_u$ from $\mathbb{R}^n$ to $\mathbb{R}^N$ is one-to-one.
If a point of M belongs to $\psi(G)$, we say that (G, ψ) is a chart of M around this point.

12b2 Definition. A *co-chart*[1] (n-cochart) of M is a pair (U, φ) of an open set U in $\mathbb{R}^N$ and a mapping $\varphi : U \to \mathbb{R}^{N-n}$ such that
(a) $M \cap U = \{x \in U : \varphi(x) = 0\} \ne \emptyset$;
(b) $\varphi \in C^1(U \to \mathbb{R}^{N-n})$;
(c) for every $x \in M \cap U$ the linear operator $(D\varphi)_x$ from R^N to $\mathbb{R}^{N-n}$ is onto.
If a point of M belongs to U, we say that (U, φ) is a co-chart of M around this point.

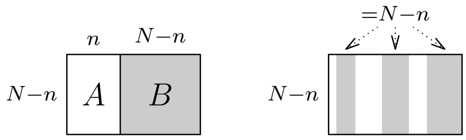
Clearly, n-charts and n-cocharts are well-defined for a subset of an N-dimensional affine space S (and then $(D\psi)_x : \mathbb{R}^n \to \vec{S}$ and $(D\varphi)_x : \vec{S} \to \mathbb{R}^{N-n}$).

In particular, if M is the graph of a mapping $f : \mathbb{R}^n \to \mathbb{R}^{N-n}$ of class C^1 near x_0, that is, $M = \{(u, f(u)) : u \in \mathbb{R}^n\}$, then we may take $\psi(u) = (u, f(u))$ and $\varphi(u, v) = v - f(u)$ for $u \in \mathbb{R}^n$, $v \in \mathbb{R}^{N-n}$.

This is one out of $\binom{N}{n}$ similar cases. Recall Sect. 5d: if a linear operator maps $\mathbb{R}^N$ onto $\mathbb{R}^{N-n}$, it does not mean that it is $(\,A \,|\, B\,)$ with invertible B. Some $(N-n) \times (N-n)$ minor is not zero, but not just the rightmost minor. That is, some $N-n$ out of the N variables are functions of the other n

[1] Not a standard terminology.

variables; but not just the last $N-n$ variables and the first n variables.



12b3 Exercise. Generalize 12a4.

12b4 Proposition. The following three conditions on a set $M \subset \mathbb{R}^N$ and a point $x_0 \in M$ are equivalent:

(a) there exists an n-chart of M around x_0;

(b) there exists an n-cochart of M around x_0;

(c) there exists a local diffeomorphism $h : \mathbb{R}^N \to \mathbb{R}^N$ near x_0 such that

$$(u,v) \in M \quad \Longleftrightarrow \quad h(u,v) \in \mathbb{R}^n \times \{0_{N-n}\}$$

for all $(u,v) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ near x_0.

I skip the proof; it is a straightforward generalization of 12a5.

As before, the general case reduces (locally) to the $\binom{N}{n}$ special cases; some $N-n$ variables are functions of the other n variables. In terms of Sect. 5d, M has a n-chart (or n-cochart) around x_0 if and only if M has n degrees of freedom at x_0.

12b5 Exercise. Let (G_1, ψ_1), (G_2, ψ_2) be two n-charts of M around x_0. Prove existence of a mapping $\varphi : G_1 \to G_2$ of class C^1 near $u_1 = \psi_1^{-1}(x_0)$ such that $\psi_1(u) = \psi_2(\varphi(u))$ for all u near u_1, and $\det(D\varphi)_{u_1} \ne 0$.[1]

12b6 Exercise. A relation $\det(D\varphi)_{u_1} > 0$ (for (G_1, ψ_1), (G_2, ψ_2), u_1 and φ as above) is an equivalence relation between n-charts of M around x_0.

Prove it.

Clearly, there exist exactly two equivalence classes (provided that M has an n-chart around x_0, of course). These equivalence classes are called the two *orientations* of M at x_0.

12b7 Exercise. If M has an n-chart at x_0 then M cannot have an m-chart at x_0 for $m \ne n$. Prove it.[2] However, M can have an m-chart for $m \ne n$ at another point; give an example.

[1] Hint: M has n degrees of freedom at x_0. Values of φ outside a neighborhood of u_1 are irrelevant.

[2] Hint: recall 2b13(b).

12b8 Definition. A nonempty set $M \subset \mathbb{R}^N$ is an n-dimensional *manifold* (or n-manifold) if for every $x_0 \in M$ there exists an n-chart of M around x_0.[1] [2]

"Co-chart" instead of "chart" gives an equivalent definition.

The same applies to a subset M of an N-dimensional affine space.

A relatively open nonempty subset of an n-manifold is an n-manifold. In particular, for every chart (G, ψ) of M the set $\psi(G)$ is an n-manifold (a *single-chart* piece of M), and for every co-chart (U, φ) of M the set $M \cap U$ is an n-manifold.

In addition, sometimes one defines an N-manifold in $\mathbb{R}^N$ as just a nonempty open subset of $\mathbb{R}^N$, and a 0-manifold as just a nonempty discrete[3] subset of $\mathbb{R}^N$.

12b9 Exercise. Let M_1 be an n_1-manifold in $\mathbb{R}^{N_1}$, and M_2 an n_2-manifold in $\mathbb{R}^{N_2}$; then $M_1 \times M_2$ is an $(n_1 + n_2)$-manifold in $\mathbb{R}^{N_1+N_2}$.

Prove it.[4]

12b10 Definition. Let $M \subset \mathbb{R}^N$ be an n-manifold; a function $f : M \to \mathbb{R}$ is *continuously differentiable* if for every chart (G, ψ) of M the function $f \circ \psi$ is continuously differentiable on G.

12b11 Exercise. Generalize 12a8, 12a9 accordingly.

12b12 Exercise. Define the notion of a function continuous *almost everywhere* on a manifold. Formulate and prove counterparts of 12a8, 12a9 for this notion.

12b13 Example. [5] Consider the set M of all 3×3 matrices A of the form

$$A = \begin{pmatrix} a^2 & ab & ac \\ ba & b^2 & bc \\ ca & cb & c^2 \end{pmatrix} \quad \text{for } a, b, c \in \mathbb{R}\,, \ a^2 + b^2 + c^2 = 1\,.$$

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[1] These are manifolds of class C^1; manifolds of class C^m are defined similarly. For M of class C^1 we can define $C^1(M)$ but not $C^2(M)$. You may reconsider the last item of 12a10: when is M_p of class C^m?

[2] "In the literature this is usually called a submanifold of Euclidean space. It is possible to define manifolds more abstractly, without reference to a surrounding vector space. However, it turns out that practically all abstract manifolds can be embedded into a vector space of sufficiently high dimension. Hence the abstract notion of a manifold is not substantially more general than the notion of a submanifold of a vector space." Sjamaar, page 69.

[3] That is, each point (and therefore each subset) is relatively open.

[4] You may choose one of the three equivalent conditions (a), (b), (c) of 12b4. Or, just for fun, you may give three proofs! On the other hand, you may prove it for $0 \le n_1 \le N_1$, $0 \le n_2 \le N_2$, not just $0 < n_1 < N_1$, $0 < n_2 < N_2$.

[5] The projective plane in disguise.

These are orthogonal projections to one-dimensional subspaces of $\mathbb{R}^3$. We treat M as a subset of the six-dimensional space of all symmetric 3×3 matrices.

The set M is invariant under transformations $A \mapsto UAU^{-1}$ where U runs over all orthogonal matrices (linear isometries); these are linear transformations of the six-dimensional space of matrices. If A corresponds to $x = (a, b, c)$ then UAU^{-1} corresponds to Ux. For arbitrary $A, B \in M$ there exists U such that $UAU^{-1} = B$ ("transitive action").

Thus, M looks the same around all its points ("homogeneous space"). In order to prove that M is a 2-manifold (in $\mathbb{R}^6$) it is sufficient to find a chart (or co-chart) around a single point of M, say,

$$A_1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \in M\,.$$

12b14 Exercise. Find a 2-chart of M around A_1.[1]

12b15 Exercise. Locally, near A_1, four coordinates should be smooth functions of the other two coordinates. Which two? Calculate explicitly these four functions of two variables.[2]

Recall the two orientations of M at x_0 introduced after 12b6.

12b16 Definition. (a) An *orientation* of an n-manifold $M \subset \mathbb{R}^N$ is a family $(\mathcal{O}_x)_{x\in M}$ of orientations $\mathcal{O}_x$ of M at points x such that for every $x_0 \in M$ and every $(G, \psi) \in \mathcal{O}_{x_0}$ the relation $(G, \psi) \in \mathcal{O}_x$ holds for all x near x_0.

(b) M is *orientable* if it has (at least one) orientation.

The same applies to $M \subset S$ where S is an N-dimensional affine space.

We will see that a sphere is orientable but the Möbius strip (see 12c20) is not, as well as M of 12b13. However, a single-chart piece of a manifold is orientable.

An *oriented* manifold is, by definition, a pair $(M, \mathcal{O})$ of a manifold and its orientation. By a chart of an oriented manifold $(M, \mathcal{O})$ we mean a chart (G, ψ) of M such that $(G, \psi) \in \mathcal{O}_x$ for all $x \in \psi(G)$.

If two orientations of M agree at x, then they agree near x (think, why). Thus, they agree on a relatively open subset of M. Similarly, they disagree on a relatively open subset of M. These two sets are relatively clopen. If M is connected, then it has at most two orientations. If M is connected and

[1] Hint: $(b, c) \mapsto (\sqrt{1 - b^2 - c^2}, b, c) = x \mapsto A = \psi(b, c)$.

[2] Hint: solve a quadratic equation.

orientable (in particular, connected and single-chart), then it has exactly two orientations. For instance, $\mathbb{R}^n$ has exactly two orientations; and the same holds for arbitrary n-dimensional affine (sub)space.

When $M = V$ is an n-dimensional vector subspace of $\mathbb{R}^N$ (or of arbitrary N-dimensional vector space), *linear charts* are convenient: $G = \mathbb{R}^n$ and $\psi : \mathbb{R}^n \to V$ is a linear bijection. Two such linear charts $(\mathbb{R}^n, \psi_1)$, $(\mathbb{R}^n, \psi_2)$ are related via a matrix $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ such that $\psi_1 = \psi_2 \circ \varphi$, that is, $\varphi = \psi_2^{-1} \circ \psi_1$. If $\det \varphi > 0$, then these two charts give the same orientation of V; if $\det \varphi < 0$, they give the two different orientations. Note that the linear operators $\psi : \mathbb{R}^n \to V$ correspond bijectively to bases $\big(\psi(e_1), \dots, \psi(e_n)\big)$ of V (here $(e_1, \dots, e_n)$ is the usual basis of $\mathbb{R}^n$), and two such bases are related via the matrix $\varphi = (\varphi_{i,j})_{i,j}$:

$$\psi_1(e_k) = \psi_2\big(\varphi(e_k)\big) = \psi_2(\varphi_{1,k}e_1 + \dots + \varphi_{n,k}e_n) = \varphi_{1,k}\psi_2(e_1) + \dots + \varphi_{n,k}\psi_2(e_n) \, .$$

Thus, an orientation of $M = V$ may be thought of as an equivalence class of bases. The same applies to an n-dimensional affine subspace $M = S$ of $\mathbb{R}^N$ (or of arbitrary N-dimensional affine space); the two orientations of S correspond evidently to the two orientations of the difference space $\vec{S}$.

If in addition V (or S) is endowed with a Euclidean metric, then it is convenient to use linear isometries $\psi : \mathbb{R}^n \to V$ and the corresponding orthonormal bases of V (or $\vec{S}$).

12b17 Example. (a) $M = \mathbb{R}$; there are two orthonormal bases, (1) and (-1); they give the two orientations of $\mathbb{R}$.

(b) $M = \mathbb{R}^2$; an orthonormal basis is either

$$\big((\cos\theta, \sin\theta), (\cos(\theta + \tfrac{\pi}{2}), \sin(\theta + \tfrac{\pi}{2}))\big) = \big((\cos\theta, \sin\theta), (-\sin\theta, \cos\theta)\big)$$

or

$$\big((\cos\theta, \sin\theta), (\cos(\theta - \tfrac{\pi}{2}), \sin(\theta - \tfrac{\pi}{2}))\big) = \big((\cos\theta, \sin\theta), (\sin\theta, -\cos\theta)\big) \, ;$$

these two cases give the two orientations of $\mathbb{R}^2$.

(c) $M = \mathbb{R}^3$; an orthonormal basis is either $(a, b, a \times b)$ or $(a, b, -a \times b)$ for $|a| = |b| = 1$, $\langle a, b \rangle = 0$; these two cases give the two orientations of $\mathbb{R}^3$. [1]

12b18 Definition. Let M be an n-manifold in $\mathbb{R}^N$.

(a) A vector $h \in \mathbb{R}^N$ is *tangent* to M at $x_0 \in M$ if $\operatorname{dist}(x_0 + \varepsilon h, M) = o(\varepsilon)$ (as $\varepsilon \to 0$);

(b) the *tangent space* $T_{x_0}M$ (to M at x_0) is the set of all tangent vectors (to M at x_0).

[1] About relevance of orientations of our three-dimensional space to physics, chemistry and biology see Wikipedia:Chirality (and follow the links there).

The same applies to $M \subset S$ where S is an N-dimensional affine space; and then $T_{x_0}M \subset \vec{S}$.[1] Though, the distance needs a metric; but $o(\cdot)$ does not depend on the choice of a norm on $\vec{S}$.

The next exercise shows (in particular) that the tangent space is indeed a vector subspace of $\mathbb{R}^N$, of dimension n, and may be defined without mentioning a distance.

12b19 Exercise. Let (G, ψ) be a chart around $x_0 = \psi(u_0)$ and (U, φ) a co-chart around x_0. Prove that the following three conditions on a vector $h \in \mathbb{R}^N$ are equivalent:

(a) h is a tangent vector (at x_0);

(b) h belongs to the image of the linear operator $(D\psi)_{u_0} : \mathbb{R}^n \to \mathbb{R}^N$;

(c) h belongs to the kernel of the linear operator $(D\varphi)_{x_0} : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

12b20 Example. Let $M \subset \mathbb{R}^2$ be the graph of a function $f \in C^1(\mathbb{R})$. Then $T_{(x,f(x))}M = \{\big(\lambda, \lambda f'(x)\big) : \lambda \in \mathbb{R}\}$.

12b21 Exercise. Generalize 12b20 to curves and surfaces in $\mathbb{R}^3$ (that are graphs).

If $M = S$ is an affine subspace then $T_x S = \vec{S}$ for every $x \in S$; and if $M = V$ is a vector subspace then $T_x V = V$ for every $x \in V$.

If (G, ψ) is a chart of M around $x_0 = \psi(u_0)$ then $(D\psi)_{u_0} : \mathbb{R}^n \to T_{x_0}M$ is a linear chart of $T_{x_0}M$ ("the tangent chart"). For two charts (G_1, ψ_1), (G_2, ψ_2) of M around x_0, $\psi_1 = \psi_2 \circ \varphi$, the chain rule gives $(D\psi_1)_{u_1} = (D\psi_2)_{u_2} \circ (D\varphi)_{u_1}$, where $\psi_1(u_1) = x_0 = \psi_2(u_2)$. Clearly, the charts (G_1, ψ_1) and (G_2, ψ_2) of M give the same orientation of M at x_0 if and only if the tangent charts $\big(\mathbb{R}^n, (D\psi_1)_{u_1}\big)$ and $\big(\mathbb{R}^n, (D\psi_2)_{u_2}\big)$ give the same orientation of $T_{x_0}M$. This way the two orientations of M at x_0 correspond to the two orientations of $T_{x_0}M$.

Thus, an orientation of M may be thought of as a family of orientations of the tangent spaces $T_x M$, $x \in M$.[2]

12b22 Exercise (CYLINDER)**.** Let M_1 be an n-manifold in $\mathbb{R}^N$, and $h \in \mathbb{R}^N$ satisfy

$$\forall x \in M_1 \;\; h \notin T_x M_1 \,.$$

Consider the set

$$M = \{x + \lambda h : x \in M_1, \lambda \in \mathbb{R}\} \,.$$

[1]Geometrically, it looks more natural to define $T_{x_0}M$ as the affine subspace of all $x_0 + h$. But the version $T_{x_0}M \subset \vec{S}$ is algebraically natural and widely used.

[2]But not an arbitrary family; indeed, the family $(\mathcal{O}_x)_{x \in M}$ in Def. 12b16 is not arbitrary.

Assume that the mapping $(x, \lambda) \mapsto x+\lambda h$ is a homeomorphism $M_1 \times \mathbb{R} \to M$. Then

(a) M is an $(n+1)$-manifold in $\mathbb{R}^N$;

(b) if (G, ψ_1) is a chart of M_1, then $(G \times \mathbb{R}, \psi)$ for $\psi : (u, \lambda) \mapsto \psi_1(u) + \lambda h$ is a chart of M.

Prove it. And show by counterexamples that no one of the two conditions ($h \notin T_x M_1$, and homeomorphism) can be dropped.

12b23 Exercise (CONE)**.** Let M_1 be an n-manifold in $\mathbb{R}^N$ such that

$$\forall x \in M_1 \;\; x \notin T_x M_1 \, .$$

Consider the set

$$M = \{\lambda x : x \in M_1, \lambda \in (0, \infty)\} \, .$$

Assume that the mapping $(x, \lambda) \mapsto \lambda x$ is a homeomorphism $M_1 \times (0, \infty) \to M$. Then

(a) M is an $(n+1)$-manifold in $\mathbb{R}^N$;

(b) if (G, ψ_1) is a chart of M_1, then $(G \times (0, \infty), \psi)$ for $\psi : (u, \lambda) \mapsto \lambda \psi_1(u)$ is a chart of M.

Prove it.

12b24 Exercise (SURFACE OF REVOLUTION OR BODY OF REVOLUTION)**.** Let M_1 be an n-manifold in $\mathbb{R}^3$ (here $n = 1$ or $n = 2$) such that

$$\forall (x, y, z) \in M_1 \;\; (0, -z, y) \notin T_{(x,y,z)} M_1 \, .$$

Consider the set

$$M = \{(x, cy - sz, sy + cz) : (x, y, z) \in M_1, (c, s) \in S\}$$

where $S = \{(c, s) \in \mathbb{R}^2 : c^2 + s^2 = 1\}$ (the circle). Assume that the mapping $\big((x, y, z), (c, s)\big) \mapsto (x, cy - sz, sy + cz)$ is a homeomorphism $M_1 \times S \to M$. Then

(a) M is an $(n+1)$-manifold in $\mathbb{R}^3$;

(b) if (G_1, ψ_1) is a chart of M_1 and (G_2, ψ_2) is a chart of S, then $(G_1 \times G_2, \psi)$ is a chart of M; here $\psi(u_1, u_2) = (x, cy - sz, sy + cz)$ whenever $\psi_1(u_1) = (x, y, z)$ and $\psi_2(u_2) = (c, s)$.

Prove it.

12c Forms on manifolds

12c1 Definition. A *differential form* of order k (or k-form)[1] on an n-manifold $M \subset \mathbb{R}^N$ is a continuous function ω on the set $\{(x, h_1, \dots, h_k) : x \in M,\, h_1, \dots, h_k \in T_x M\}$ such that for every $x \in M$ the function $\omega(x, \cdot, \dots, \cdot)$ is an antisymmetric multililear k-form on $T_x M$.

Given a k-form ω on M, the integral $\int_\Gamma \omega$ is well-defined for every singular k-box Γ in M (that is, k-box $\Gamma : B \to \mathbb{R}^N$ such that $\Gamma(B) \subset M$); recall (11e12) and note that $(D_i \Gamma)_u \in T_{\Gamma(u)} M$.

The case $k = n$ is important.

Let us compare two notions, singular n-box in M and n-chart of M. These are $\Gamma : B \to M$ and $\psi : G \to M$; both B and G are subsets of $\mathbb{R}^n$; both Γ and ψ are continuously differentiable; but B is a closed box, while G is an open set; and ψ is a homeomorphism (and more), while Γ may degenerate (even be constant). Anyway, let us define $\int_{(G,\psi)} \omega$ similarly to $\int_\Gamma \omega$:

$$\text{(12c2)} \qquad \int_{(G,\psi)} \omega = \int_G \omega\bigl(\psi(u), (D_1\psi)_u, \dots, (D_n\psi)_u\bigr)\, \mathrm{d}u\,.$$

The integrand is continuous, but may be unbounded; also G may be unbounded; thus, the integral is interpreted as improper, and may converge or diverge.

Here is parametrization invariance, similar to (11b6).

12c3 Proposition. Let (G_1, ψ_1), (G_2, ψ_2) be two charts of an oriented manifold $(M, \mathcal{O})$. If $\psi_1(G_1) = \psi_2(G_2)$ then

$$\int_{(G_1,\psi_1)} \omega = \int_{(G_2,\psi_2)} \omega$$

for every n-form ω on M; that is, either these two integrals converge and are equal, or both integrals diverge.

Some observations before the proof.

The space of all antisymmetric multililear n-forms L on $\mathbb{R}^n$ (or on arbitrary n-dimensional vector space) is one-dimensional (recall the paragraph before 11e13), thus, an n-form ω on an n-manifold M is basically a (scalar) function on M. More exactly, such ω corresponds to the scalar function $x \mapsto \omega\bigl(x, e_1(x), \dots, e_n(x)\bigr)$ where $x \in M$ and $(e_1(x), \dots, e_n(x))$ is a basis of $T_x M$. Be warned: such a basis, continuous in x, need not exist (on the whole

[1]These are forms of class C^0.

M) even if M is orientable.[1] But clearly, it exists on a single-chart piece of M.

On $\mathbb{R}^n$, the determinant is an antisymmetric multililear n-form; and therefore (by the one-dimensionality), every such form L is

$$L(a_1,\dots,a_n) = c\det(a_1,\dots,a_n) \quad \text{for } a_1,\dots,a_n \in \mathbb{R}^n \,.$$

A linear operator $\mathbb{R}^n \to \mathbb{R}^n$ corresponds to a matrix,

$$\mathbb{R}^n \ni x \mapsto Ax \in \mathbb{R}^n \,,$$

and leads to such an antisymmetric multililear n-form L on $\mathbb{R}^n$:

$$L(a_1,\dots,a_n) = \det(Aa_1,\dots,Aa_n) \quad \text{for } a_1,\dots,a_n \in \mathbb{R}^n \,.$$

For the usual basis $(e_1,\dots.e_n)$ of $\mathbb{R}^n$ we have $L(e_1,\dots,e_n) = \det A$, since $Ae_1,\dots,Ae_n$ are the columns of the matrix A. By the one-dimensionality, $L(a_1,\dots,a_n) = (\det A)\det(a_1,\dots,a_n)$, that is,

$$\det(Aa_1,\dots,Aa_n) = (\det A)\det(a_1,\dots,a_n)\,. \tag{12c4}$$

By the one-dimensionality (again),

$$L(Aa_1,\dots,Aa_n) = (\det A)L(a_1,\dots,a_n) \tag{12c5}$$

for every antisymmetric multililear n-form L on $\mathbb{R}^n$.

Applying Theorem 9d1 to the unit cube $[0,1]^n \subset \mathbb{R}^n$ we get the volume of the *parallelotope*

$$\mathcal{P}(a_1,\dots,a_n) = A([0,1]^n) = \{\lambda_1 a_1 + \dots + \lambda_n a_n : \lambda_1,\dots,\lambda_n \in [0,1]\} \subset \mathbb{R}^n$$

generated by vectors $a_1 = Ae_1,\dots,a_n = Ae_n$ (the columns of the matrix A) emanating from the vertex 0 (the corner point):

$$v\big(\mathcal{P}(a_1,\dots,a_n)\big) = |\det(a_1,\dots,a_n)|\,.$$

Taking into account that the sign of $\det(a_1,\dots,a_n)$ is related to an orientation of $\mathbb{R}^n$ (as explained before 12b17), one says that $\det(a_1,\dots,a_n)$ is the *oriented volume* (or signed volume) of the parallelotope generated by $a_1,\dots,a_n$.

On an n-dimensional vector space V "the" volume (Jordan measure) is defined up to a (positive) coefficient (recall the end of Sect. 9d). On the

[1] In particular, it does not exist for the sphere $M = \mathcal{S}^2 \subset \mathbb{R}^3$; see Wikipedia:Hairy ball theorem.

other hand, "the" antisymmetric multililear n-form L on V is defined up to a coefficient (not just positive). They correspond naturally by the formula

$$v\big(\mathcal{P}(a_1,\dots,a_n)\big) = |L(a_1,\dots,a_n)|\,;$$

each Jordan measure v (a special set function) corresponds to two forms ($\pm L$) that in turn correspond to the two orientations of V.

On an n-dimensional Euclidean vector space E we have a single Jordan measure and two "normalized" antisymmetric multililear n-forms $\pm L$ (corresponding to the two orientations of E); $L(e_1,\dots,e_n) = \pm 1$ for every orthonormal basis of E. In particular, this holds in every n-dimensional vector subspace of $\mathbb{R}^N$.

Proof *of Prop. 12c3.*
The mapping $\varphi = \psi_2^{-1}\circ\psi_1 : G_1 \to G_2$ is a homeomorphism (the composition of two homeomorphisms $G_1 \to \psi_1(G_1) = \psi_2(G_2) \to G_2$), and moreover, a diffeomorphism (since 12b5 applies near every point). By Theorem 10f1 it is sufficient to prove that

$$\begin{aligned}\omega\big(\psi_1(u_1), (D_1\psi_1)_{u_1}, \dots, (D_n\psi_1)_{u_1}\big) = \\ = \omega\big(\psi_2(u_2), (D_1\psi_2)_{u_2}, \dots, (D_n\psi_2)_{u_2}\big)|\det(D\varphi)_{u_1}|\end{aligned}$$

whenever $u_2 = \varphi(u_1)$. Also, $\det(D\varphi)_{u_1} > 0$, since both charts conform to the given orientation $\mathcal{O}$.

Let $x \in M$, $u_1 \in G_1$, $u_2 \in G_2$ satisfy $\psi_1(u_1) = x = \psi_2(u_2)$, then $\varphi(u_1) = u_2$. We introduce an antisymmetric multililear n-form

$$L(a_1,\dots,a_n) = \omega\big(x, (D\psi_2)_{u_2}a_1, \dots, (D\psi_2)_{u_2}a_n\big) \quad \text{for } a_1,\dots,a_n \in \mathbb{R}^n\,.$$

By the chain rule, the relation $\psi_1 = \psi_2\circ\varphi$ implies $(D\psi_1)_{u_1} = (D\psi_2)_{u_2}\circ(D\varphi)_{u_1}$; therefore,

$$\begin{aligned}\omega\big(x, (D_1\psi_1)_{u_1}, \dots, (D_n\psi_1)_{u_1}\big) &= \omega\big(x, (D\psi_1)_{u_1}e_1, \dots, (D\psi_1)_{u_1}e_n\big) = \\ &= \omega\big(x, (D\psi_2)_{u_2}(D\varphi)_{u_1}e_1, \dots, (D\psi_2)_{u_2}(D\varphi)_{u_1}e_n\big) = \\ &= L\big((D\varphi)_{u_1}e_1, \dots, (D\varphi)_{u_1}e_n\big) = \big(\det(D\varphi)_{u_1}\big)L(e_1,\dots,e_n) = \\ &= \big(\det(D\varphi)_{u_1}\big)\omega\big(x, (D_1\psi_2)_{u_2}, \dots, (D_n\psi_2)_{u_2}\big)\end{aligned}$$

by (12c5). $\square$

Thus, we may write $\int_{\psi(G)}\omega$ instead of $\int_{(G,\psi)}\omega$. Also, we may write $\int_U\omega$ whenever a relatively open set $U \subset M$ is such that $U = \psi(G)$ for some n-chart (G,ψ) of (M,O). However, the orientation of M is essential. The opposite orientation leads to the opposite value of the integral.

12c6 Definition. An n-form μ on an oriented n-manifold $(M,\mathcal{O})$ in $\mathbb{R}^N$ is the *volume form,* if for every $x \in M$ the antisymmetric multililear n-form $\mu(x,\cdot,\dots,\cdot)$ on T_xM is normalized and corresponds to the orientation $\mathcal{O}_x$.

"Normalized" means that it corresponds to the Jordan measure on the Euclidean subspace T_xM of the Euclidean space $\mathbb{R}^N$. "Corresponds to the orientation $\mathcal{O}_x$" means that for some (therefore, every) chart $(G,\psi) \in \mathcal{O}_x$,

$$\mu\big(\psi(u),(D_1\psi)_u,\dots,(D_n\psi)_u\big) > 0 \quad \text{where } u = \psi^{-1}(x)\,.$$

The same applies to a manifold in an N-dimensional Euclidean affine space (but fails in the absence of a Euclidean metric).

Clearly, such μ is unique. Is it clear that μ exists? Surely, $\mu(x,\cdot,\dots,\cdot)$ is well-defined for each x; but is it continuous in all the variables (including x)? An affirmative answer will be given (after Example 12c17).

Having the volume form μ on $(M,\mathcal{O})$ we define the n-dimensional volume

$$v(U) = \int_{(G,\psi)} \mu \in (0,\infty] \tag{12c7}$$

whenever $U = \psi(G)$ for an n-chart (G,ψ) of $(M,\mathcal{O})$.

Also, for a function $f : M \to \mathbb{R}$ continuous almost everywhere we define[1]

$$\int_U f = \int_G f\big(\psi(u)\big)\mu\big(\psi(u),(D_1\psi)_u,\dots,(D_n\psi)_u\big)\,\mathrm{d}u\,; \tag{12c8}$$

the integral is interpreted as improper, and may converge or diverge.

12c9 Exercise. Formulate and prove parametrization invariance for $\int_U f$ (similar to 12c3).[2]

12c10 Example. Let $M \subset \mathbb{R}^2$ be the graph of a function $f \in C^1(\mathbb{R})$. The whole M is covered by the chart $\mathbb{R} = G_+ \ni x \mapsto \psi_+(x) = (x,f(x)) \in M$; denote by $\mathcal{O}_+$ the corresponding orientation of M, and by $\mathcal{O}_-$ the other orientation. The two volume forms on M are $\mu_\pm\big((x,f(x)),(\lambda,\lambda f'(x))\big) = \pm\lambda\sqrt{1+f'^2(x)}$ (clearly, continuous functions of x and λ); thus,

$$v\big(\psi_+(G)\big) = \int_G \mu_+\big((x,f(x)),(1,f'(x))\big)\,\mathrm{d}x = \int_G \sqrt{1+f'^2(x)}\,\mathrm{d}x$$

[1] Surely, such $\int_U f$ is never interpreted as $\int_{\mathbb{R}^N} f\cdot 1\!\!1_U$ (unless $n=N$); indeed, U cannot be a Jordan set of non-zero volume (since $U^\circ = \emptyset$). On the other hand, for $n=N$, this $\int_U f$ is the same as the improper integral of Sect. 10 (just use the trivial chart). Many authors include n into the notation; say, $V_n(U)$ rather than $v(U)$, and $\int_U f\,\mathrm{d}V_n$ rather than $\int_U f$. When $N=3$, one often uses $\mathrm{d}\ell$ (or $\mathrm{d}s$) for $n=1$; $\mathrm{d}A$ (or $\mathrm{d}S$, or $\mathrm{d}\sigma$) for $n=2$; and $\mathrm{d}v$ for $n=3$.

[2] For a continuous f we may just apply 12c3 to the n-form $f\mu$.

is the 1-dimensional volume (just the length) of a part of the curve M. Note that

$$\text{(12c11)} \qquad v\big(\{(x, f(x)) : a < x < b\}\big) = \int_a^b \sqrt{1 + f'^2(x)}\, \mathrm{d}x$$

is an additive function of a box $(a, b) \subset \mathbb{R}$, and $x \mapsto \sqrt{1 + f'^2(x)}$ is the derivative of this box function. Informally,

$$(\mathrm{d}\ell)^2 = (\mathrm{d}x)^2 + (\mathrm{d}y)^2\,, \quad \text{where } y = f(x)\,.$$

Another chart $\mathbb{R} = G_- \ni x \mapsto \psi_-(x) = (-x, f(-x)) \in M$ corresponds to $\mathcal{O}_-$; we have $v\big(\psi_-(G)\big) = \int_G \mu_-\big((-x, f(-x)), (-1, -f'(-x))\big)\, \mathrm{d}x = \int_G \sqrt{1 + f'^2(-x)}\, \mathrm{d}x$; taking $G = (-b, -a)$ we get (12c11) again. The same length via the other orientation.

Can we generalize 12c10 to a surface M in $\mathbb{R}^3$ (the graph of a function $f \in C^1(\mathbb{R}^2)$)? We know the tangent space (recall 12b21) $T_{(x,y,f(x,y))}M$, it is spanned by two vectors, $(1, 0, (D_1 f)_{(x,y)})$ and $(0, 1, (D_2 f)_{(x,y)})$, but they are not orthogonal. How to know that a form is normalized? We could apply the orthogonalization process, but it leads to unpleasant formulas already for $n = 2$ (and even worse for higher n). Fortunately a better way exists.

For arbitrary n vectors $a_1, \dots, a_n \in \mathbb{R}^n$,

$$\begin{aligned}\big(\det(a_1, \dots, a_n)\big)^2 = \big(\det(A)\big)^2 = \det(A^{\mathrm{t}} A) = \\ = \det\big(\langle a_i, a_j \rangle\big)_{i,j} = \begin{vmatrix} \langle a_1, a_1 \rangle & \dots & \langle a_1, a_n \rangle \\ \langle a_2, a_1 \rangle & \dots & \langle a_2, a_n \rangle \\ \dots & \dots & \dots \\ \langle a_n, a_1 \rangle & \dots & \langle a_n, a_n \rangle \end{vmatrix}\,;\end{aligned}$$

here $A = (\, a_1 \,|\, \dots \,|\, a_n \,)$ is the matrix whose columns are the vectors $a_1, \dots, a_n$; accordingly, $A^{\mathrm{t}} A$ is the matrix of scalar products (think, why), the so-called Gram matrix, and its determinant is called the Gram determinant, or Gramian of $a_1, \dots, a_n$. We see that the volume of a parallelotope is the root of the Gramian,

$$\text{(12c12)} \qquad v\big(\mathcal{P}(a_1, \dots, a_n)\big) = \sqrt{\det\big(\langle a_i, a_j \rangle\big)_{i,j}}$$

in $\mathbb{R}^n$, and therefore, in every n-dimensional Euclidean vector space. In particular, in every n-dimensional subspace of $\mathbb{R}^N$.

Given a one-to-one linear operator $B : \mathbb{R}^n \to \mathbb{R}^N$, we have $v_n(B(E)) = c v_n(E)$ for all Jordan sets $E \subset \mathbb{R}^n$, with some $c > 0$ that depends on B (but does not depend on E). Taking a parallelotope $E = \mathcal{P}(a_1, \dots, a_n)$ we have $B(E) = \mathcal{P}(Ba_1, \dots, Ba_n)$. Thus, the ratio

$$\frac{\sqrt{\det\big(\langle Ba_i, Ba_j\rangle\big)_{i,j}}}{|\det(a_1, \dots, a_n)|} \quad \text{does not depend on a basis } (a_1, \dots, a_n) \text{ of } \mathbb{R}^n\,.$$

In particular,

$$\text{(12c13)} \qquad \sqrt{\det\big(\langle Be_i, Be_j\rangle\big)_{i,j}} \quad \text{does not depend on an orthonormal basis } (e_1, \dots, e_n) \text{ of } \mathbb{R}^n\,.$$

Let μ be a volume form on $(M, \mathcal{O})$, and (G, ψ) a chart of $(M, \mathcal{O})$. By (12c12), $v_n\big(\mathcal{P}((D_1\psi)_u, \dots, (D_n\psi)_u)\big) = J_\psi(u)$, where

$$J_\psi(u) = \sqrt{\det\big(\langle (D_i\psi)_u, (D_j\psi)_u\rangle\big)_{i,j}}$$

is the (generalized) Jacobian of ψ. Clearly, $J_\psi : G \to (0, \infty)$ is continuous. Normalization of μ becomes

$$\text{(12c14)} \qquad \mu\big(\psi(u), (D_1\psi)_u, \dots, (D_n\psi)_u\big) = J_\psi(u)\,.$$

By (12c7) and (12c2),

$$\text{(12c15)} \qquad v(U) = \int_{\psi(G)} \mu = \int_G J_\psi(u)\,\mathrm{d}u\,.$$

By (12c8),

$$\text{(12c16)} \qquad \int_U f = \int_G f(\psi(u)) J_\psi(u)\,\mathrm{d}u\,.$$

Here $U = \psi(G)$ for an n-chart (G, ψ) of $(M, \mathcal{O})$.

Now we are in position to generalize 12c10.

12c17 Example. Let $M \subset \mathbb{R}^3$ be the graph of a function $f \in C^1(\mathbb{R}^2)$; that is, $M = \{(x, y, f(x, y)) : x, y \in \mathbb{R}\}$. The whole M is covered by the chart $\mathbb{R}^2 = G \ni (x, y) \mapsto \psi(x, y) = (x, y, f(x, y)) \in M$; denote by $\mathcal{O}$ the corresponding orientation of M. We have

$$(D_1\psi)_{(x,y)} = \big(1, 0, (D_1 f)_{(x,y)}\big)\,; \quad (D_2\psi)_{(x,y)} = \big(0, 1, (D_2 f)_{(x,y)}\big)\,;$$

$$\begin{aligned} J_\psi^2(x, y) &= \begin{vmatrix} 1 + (D_1 f)^2 & D_1 f \cdot D_2 f \\ D_1 f \cdot D_2 f & 1 + (D_2 f)^2 \end{vmatrix} = \\ &= 1 + (D_1 f)^2 + (D_2 f)^2 + (D_1 f)^2 (D_2 f)^2 - (D_1 f)^2 (D_2 f)^2 = \\ &\qquad = 1 + (D_1 f)^2 + (D_2 f)^2 = 1 + |\nabla f(x, y)|^2\,. \end{aligned}$$

The volume form μ must satisfy

$$\mu\big((x,y,f(x,y)),(1,0,(D_1f)_{(x,y)}),(0,1,(D_2f)_{(x,y)})\big)=\sqrt{1+|\nabla f(x,y)|^2}\,.$$

Given $h,k\in T_{(x,y,f(x,y))}M$, we have $h=(h_1,h_2,h_3)=h_1(1,0,(D_1f)_{(x,y)})+h_2(0,1,(D_2f)_{(x,y)})$ (think, why), and the same for k; thus,

$$\begin{aligned}\mu\big((x,y,f(x,y)),h,k\big)&=(h_1k_2-k_1h_2)\mu\big(\cdot,(1,0,D_1f),(0,1,D_2f)\big)=\\&=\sqrt{1+|\nabla f(x,y)|^2}\begin{vmatrix}h_1&h_2\\k_1&k_2\end{vmatrix}\,;\end{aligned}$$

clearly, a continuous function of x,y,h and k. Existence of the volume form is thus verified (for the considered case), and

$$v\big(\psi(G)\big)=\int_G\sqrt{1+|\nabla f(x,y)|^2}\,\mathrm{d}x\mathrm{d}y$$

is the 2-dimensional volume (just the area) of a part of the surface M. Once again,

$$v\big(\psi(B)\big)=\int_B J_\psi \tag{12c18}$$

is an additive function of a box $B\subset\mathbb{R}^2$, and J_ψ is its derivative. Informally,

$$(\mathrm{d}A)^2=(\mathrm{d}x\mathrm{d}y)^2+(\mathrm{d}x\mathrm{d}z)^2+(\mathrm{d}y\mathrm{d}z)^2\,,\quad\text{where }z=f(x,y)\,.$$

The other orientation leads to the same area.

Existence of the volume form in general is proved similarly. Locally, M is the graph $\{(x,f(x))\}$ of a mapping $f:\mathbb{R}^n\to\mathbb{R}^{N-n}$. Using a chart (G,ψ), $\psi(x)=\big(x,f(x)\big)$, we see that $\mu(\psi(x),h_1,\dots,h_n)$ is the (continuous) $J_\psi(x)$ multiplied by a polynomial (in fact, just the determinant) of the projections on $h_1,\dots,h_n$ from $T_{(x,f(x))}\subset\mathbb{R}^N$ onto $\mathbb{R}^n$.

The case $n=N-1$ (a hypersurface) is important. In this case $f:\mathbb{R}^n\to\mathbb{R}$ has the gradient ∇f, and we wonder, whether the formula $J_\psi^2=1+|\nabla f|^2$ still holds, or was it a good luck in low dimensions.

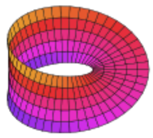
12c19 Lemma. $J_\psi=\sqrt{1+|\nabla f|^2}$.

Proof. We have $D_k\psi=e_k+(D_kf)e_N$ for $k=1,\dots,n$. According to (12c13), we are free to choose an orthonormal basis in $\mathbb{R}^n$. We choose it such that $\nabla f=|\nabla f|e_1$. Then $(D_1\psi)_{(x,f(x))}=e_1+|\nabla f(x)|e_N$, $(D_2\psi)_{(x,f(x))}=e_2$, ..., $(D_n\psi)_{(x,f(x))}=e_n$; these vectors being orthogonal, we get the determinant of a diagonal matrix: $J_\psi^2=\big|e_1+|\nabla f(x)|e_N\big|^2\cdot|e_2|^2\dots|e_n|^2=1+|\nabla f(x)|^2$. $\square$

12c20 Exercise. Consider a Möbius strip[1] (without the edge),

$$M = \{\Gamma(s,\theta) : s \in (-1,1),\, \theta \in [0,2\pi]\}\,,$$

$$\Gamma(s,\theta) = \begin{pmatrix} (R+rs\cos\frac{\theta}{2})\cos\theta \\ (R+rs\cos\frac{\theta}{2})\sin\theta \\ rs\sin\frac{\theta}{2} \end{pmatrix},$$



for given $R > r > 0$. Prove that it is a non-orientable 2-manifold in $\mathbb{R}^3$.[2]

Two facts without proofs: every 1-manifold in $\mathbb{R}^N$ is orientable; every *compact* 2-manifold in $\mathbb{R}^3$ is orientable.

12c21 Exercise. Continuing 12b13 prove that the compact 2-manifold $M \subset \mathbb{R}^6$ is non-orientable.[3]

12c22 Exercise. Let $f \in C^1(\mathbb{R})$, M_a be the graph of $f(\cdot)+a$ for $a \in \mathbb{R}$, and $g \in C(\mathbb{R}^2)$ compactly supported. Prove that
(a) $\int_{\mathbb{R}} \mathrm{d}a \int_{M_a} g^2 \ge \int_{\mathbb{R}^2} g^2$;
(b) the equality holds if and only if $\forall x, y \;\; f'(x)g(x,y) = 0$.

12c23 Exercise. Find J_ψ given $\psi(\varphi,\theta) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta)$. Compare your answer with (9b3).

12c24 Exercise. Find J_ψ given $\psi(x) = \big(x, \sqrt{1-|x|^2}\big) \in \mathbb{R}^{n+1}$ for $x \in \mathbb{R}^n$, $|x| < 1$.
Answer: $1/\sqrt{1-|x|^2}$.

12c25 Exercise. Consider spherical caps $M_a = \{x : |x| = 1, x_N > a\}$ in $\mathbb{R}^N$ (for $0 < a < 1$).
(a) $v(M_a) = \displaystyle\int_{|u|^2 < 1-a^2} \frac{\mathrm{d}u}{\sqrt{1-|u|^2}}$ (n-dimensional integral, $n = N-1$);
(b) $v(M_a) = nV_n \displaystyle\int_0^{\sqrt{1-a^2}} \frac{r^{n-1}\,\mathrm{d}r}{\sqrt{1-r^2}}$, where $V_n = \frac{2\pi^{n/2}}{n\Gamma(n/2)}$ is the volume of the n-dimensional unit ball;
(c) $\frac{v(M_a)}{NV_N/2} \to 0$ as $N \to \infty$ (but not uniformly in a, of course);
(d) *(Archimedes)* $v(M_a) = 2\pi(1-a)$ for $N = 3$.
Prove it.[4]

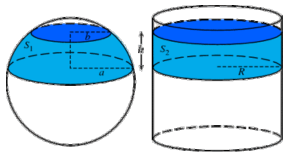
[1]Images from Wikipedia.
[2]Hint: think about the function $\theta \mapsto \mu\big(\Gamma(0,\theta), D_1\Gamma(0,\theta), D_2\Gamma(0,\theta)\big)$.
[3]Hint: similar to 12c20. (In fact, a part of M is diffeomorphic to the Möbius strip.)
[4]Hint: (a) use 12c24; (b) recall 10b7.

Here is a probabilistic interpretation. Let a point $(x_1, \dots, x_N)$ on the unit sphere in $\mathbb{R}^N$ be chosen at random, uniformly;

(c) if N is large, then x_N is usually small;

(d) if $N = 3$, then x_N is distributed uniformly.

Geometric interpretation of Item (d) is Archimedes' Hat-Box Theorem.[1]



12c26 Exercise. Consider a half-space $G = \mathbb{R}^{N-1} \times (0, \infty) \subset \mathbb{R}^N$, semi-spheres $M_r = \{x \in G : |x| = r\}$ for $r > 0$, and a compactly supported $f \in C(G)$. Prove that

(a) $\displaystyle \int_{M_r} f = \int_{|u|<r} \frac{r}{\sqrt{r^2 - |u|^2}} f\big(u, \sqrt{r^2 - |u|^2}\big) \, \mathrm{d}u;$

(b) $\displaystyle \int_0^\infty \mathrm{d}r \int_{M_r} f = \int_G f.$

12c27 Exercise (PRODUCT)**.** Let M_1 be an n_1-manifold in $\mathbb{R}^{N_1}$ and M_2 an n_2-manifold in $\mathbb{R}^{N_2}$. By 12b9, the set $M = M_1 \times M_2$ is an n-manifold in $\mathbb{R}^N$ where $n = n_1 + n_2$ and $N = N_1 + N_2$. Let (G_1, ψ_1) be a chart of M_1 and (G_2, ψ_2) a chart of M_2; consider the product-chart (G, ψ) of M, that is, $G = G_1 \times G_2$ and $\psi(u_1, u_2) = \big(\psi_1(u_1), \psi_2(u_2)\big)$. Prove that

(a) $J_\psi(u_1, u_2) = J_{\psi_1}(u_1) J_{\psi_2}(u_2)$;

(b) $v(U_1 \times U_2) = v(U_1) v(U_2) \in (0, \infty]$, where $U_1 = \psi_1(G_1)$, $U_2 = \psi_2(G_2)$.

12c28 Exercise (SCALING)**.** Let M be an n-manifold in $\mathbb{R}^N$, and $s \in (0, \infty)$. By 12b3, the set $sM = \{sx : x \in M\}$ is an n-manifold. Let (G, ψ) be a chart of M; consider the scaled chart $(G, s\psi)$ of sM. Prove that

(a) $J_{s\psi}(u) = s^n J_\psi(u)$;

(b) $v(sU) = s^n v(U) \in (0, \infty]$, where $U = \psi(G)$.

12c29 Exercise (MOTION)**.** Let M be an n-manifold in $\mathbb{R}^N$, and $T : \mathbb{R}^N \to \mathbb{R}^N$ an isometric affine mapping. By 12b3, the set $T(M)$ is an n-manifold. Let (G, ψ) be a chart of M; consider the corresponding chart $(G, T \circ \psi)$ of $T(M)$. Prove that

(a) $J_{T \circ \psi}(u) = J_\psi(u)$;

(b) $v(T(U)) = v(U) \in (0, \infty]$, where $U = \psi(G)$.

[1]Weisstein, Eric W. "Archimedes' Hat-Box Theorem." From MathWorld – A Wolfram Web Resource.

12c30 Exercise (CYLINDER). Let $M_1, h, M, (G_1, \psi_1), (G \times \mathbb{R}, \psi)$ be as in 12b22(b). Then

$$J_\psi(u, \lambda) = J_{\psi_1}(u) \operatorname{dist}(h, T_{\psi_1(u)} M_1) \, .$$

In particular, if $\langle h, \cdot \rangle$ is constant on M_1, then $h \perp T_x M_1$, thus,

$$J_\psi(u, \lambda) = |h| J_{\psi_1}(u) \, .$$

Prove it.[1]

12c31 Exercise (CONE). Let $M_1, M, (G, \psi_1), (G \times (0, \infty), \psi)$ be as in 12b23(b). Then

$$J_\psi(u, \lambda) = \lambda^n J_{\psi_1}(u) \operatorname{dist}(x, T_x M_1) \quad \text{where } x = \psi_1(u) \, .$$

In particular, if $\forall x \in M_1 \;\; |x| = c$, then $x \perp T_x M_1$, thus,

$$J_\psi(u, \lambda) = c \lambda^n J_{\psi_1}(u) \, .$$

Prove it.

12c32 Exercise (SURFACE OF REVOLUTION OR BODY OF REVOLUTION). Let $M_1, n, M, S, (G_1, \psi_1), (G_2, \psi_2), (G_1 \times G_2, \psi)$ be as in 12b24(b). Then

$$J_\psi(u_1, u_2) = J_{\psi_1}(u_1) \operatorname{dist}((0, -z, y), T_{(x,y,z)} M_1) \quad \text{where } (x, y, z) = \psi_1(u_1) \, .$$

In particular, if $M_1 \subset \mathbb{R}^2 \times \{0\}$, then also $T_{(x,y,z)} M_1 \subset \mathbb{R}^2 \times \{0\}$; $(0, -z, y) = (0, 0, y) \perp \mathbb{R}^2 \times \{0\}$; thus,

$$J_\psi(u_1, u_2) = |y| J_{\psi_1}(u_1) \quad \text{where } (x, y, 0) = \psi_1(u_1) \, .$$

Prove it.

[1] Hint: first, try $N = 2$, $n = 1$.

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