Solutions to selected exercises

1b14 Exercise. In an affine space, an affine combination does not depend on the choice of the origin.

Prove it.[1]

Solution. First, the relevant part of the solution of 1b13 (according to the two hints).

We upgrade the affine space to a vector space (which is possible by Def. 1b12): $x \lfloor\lambda\rfloor y = x + \lambda(y - x)$ for all x, y and λ. In particular, $a \lfloor 2 \rfloor b = a + 2(b - a) = 2b - a$ and $a \lfloor \frac12 \rfloor b = a + \frac12(b - a) = \frac12(a + b)$ for all a, b. Thus, $\big(O \lfloor 2 \rfloor x\big) \lfloor \frac12 \rfloor \big(O \lfloor 2 \rfloor y\big) = \frac12\big((2x - O) + (2y - O)\big) = x + y - O$. Similarly, $O \lfloor\lambda\rfloor x = O + \lambda(x - O)$.

Now, the solution of 1b14.

We can upgrade the affine space to a vector space, $x \lfloor\lambda\rfloor y = x + \lambda(y - x)$, with some origin 0. By 1b13, given O, we can also upgrade the same affine space to another vector space, $x \lfloor\lambda\rfloor y = x +_O \lambda \cdot_O (y - x)$, with operations $+_O, \cdot_O$ and the origin O. We also know (see the solution to 1b13) that $\lambda \cdot_O x - O = \lambda(x - O)$ and $x +_O y - O = (x - O) + (y - O)$. By induction, $x_1 +_O \dots +_O x_k - O = (x_1 - O) + \dots + (x_k - O)$. Thus,

$$
\begin{aligned}
&\lambda_1 \cdot_O x_1 +_O \dots +_O \lambda_k \cdot_O x_k - O = \\
&\qquad = \big((\lambda_1 \cdot_O x_1) - O\big) + \dots + \big((\lambda_k \cdot_O x_k) - O\big) = \\
&= \lambda_1(x_1 - O) + \dots + \lambda_k(x_k - O) = \lambda_1 x_1 + \dots + \lambda_k x_k - (\lambda_1 + \dots + \lambda_k)O \,,
\end{aligned}
$$

therefore

$$\lambda_1 \cdot_O x_1 +_O \dots +_O \lambda_k \cdot_O x_k = \lambda_1 x_1 + \dots + \lambda_k x_k - (\lambda_1 + \dots + \lambda_k - 1)O \,.$$

For an affine combination $\lambda_1 + \dots + \lambda_k - 1 = 0$, and we get

$$\lambda_1 \cdot_O x_1 +_O \dots +_O \lambda_k \cdot_O x_k = \lambda_1 x_1 + \dots + \lambda_k x_k \,.$$

□

[1] Hint: use the hint to 1b13; calculating a linear combination relative to O we get $O + \lambda_1(x_1 - O) + \dots + \lambda_k(x_k - O)$.