9a1 Theorem. Let $U, V \subset \mathbb{R}^n$ be Jordan measurable open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$ a bounded function such that the function $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$ is also bounded. Then

(a) f is integrable on V if and only if $(f \circ \varphi)|\det D\varphi|$ is integrable on U; and

(b) if they are integrable, then $\displaystyle\int_V f = \int_U (f \circ \varphi)|\det D\varphi|$.

9b9 Proposition (the second Pappus's centroid theorem)**.** Let $\Omega \subset (0,\infty) \times \mathbb{R} \subset \mathbb{R}^2$ be a Jordan measurable set and $\tilde\Omega = \{(x,y,z) : \big(\sqrt{x^2+y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde\Omega$ is Jordan measurable and $v_3(\tilde\Omega) = v_2(\Omega) \cdot 2\pi x_{C_\Omega}$ ($C_\Omega = (x_{C_\Omega}, z_{C_\Omega})$ is the centroid of Ω).

9c1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be a linear isometry, and $f : \mathbb{R}^n \to \mathbb{R}$ a bounded function with bounded support. Then (a) ${}_*\!\int_{\mathbb{R}^n} f \circ T = {}_*\!\int_{\mathbb{R}^n} f$, ${}^*\!\int_{\mathbb{R}^n} f \circ T = {}^*\!\int_{\mathbb{R}^n} f$;
(b) $f \circ T$ is integrable if and only if f is integrable, and in this case

$$\int_{\mathbb{R}^n} f \circ T = \int_{\mathbb{R}^n} f \,.$$

9c2 Corollary. (a) $v_*\big(T(E)\big) = v_*(E)$ and $v^*\big(T(E)\big) = v^*(E)$ for all bounded $E \subset \mathbb{R}^n$;

(b) $T(E)$ is Jordan measurable if and only if E is, and then $v\big(T(E)\big) = v(E)$.

> Riemann integral and Jordan measure are well-defined on every n-dimensional Euclidean affine space, and preserved by affine isometries between these spaces.

9c3 Lemma. For every norm $\|\cdot\|$ on $\mathbb{R}^n$, the set $\{x : \|x\| = 1\}$ is of volume zero, and the sets $\{x : \|x\| < 1\}$, $\{x : \|x\| \le 1\}$ are Jordan measurable.

9d1 Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be an invertible linear operator. Then the image $T(E)$ of an arbitrary $E \subset \mathbb{R}^n$ is Jordan measurable if and only if E is Jordan measurable, and in this case

$$v\big(T(E)\big) = |\det T| v(E)\,.$$

Also, for every bounded function $f : \mathbb{R}^n \to \mathbb{R}$ with bounded support, $|\det T|\, {}_*\!\int f \circ T = {}_*\!\int f$ and $|\det T| {}^*\!\int f \circ T = {}^*\!\int f$. Thus, $f \circ T$ is integrable if and only if f is integrable, and in this case

$$|\det T| \int f \circ T = \int f\,.$$

> On an n-dimensional vector or affine space the volume is ill-defined, but Jordan measurability is well-defined, and the ratio $\frac{v(E_1)}{v(E_2)}$ of volumes is well-defined. That is, the volume is well-defined up to a coefficient.

(9f1) $$F_*(B) = v_*(\varphi^{-1}(B^\circ))\,, \quad F^*(B) = v^*(\varphi^{-1}(\overline{B}))$$

(9f2) $$J_*(x) = \inf_{(B_i)_i} \lim_i \frac{v_*(\varphi^{-1}(B_i^\circ))}{v(B_i)}\,, \quad J^*(x) = \sup_{(B_i)_i} \lim_i \frac{v^*(\varphi^{-1}(\overline{B}_i))}{v(B_i)}$$

9f3 Proposition. If J_*, J^* are locally integrable and equivalent then $F_*(B) = F^*(B) = \int_B J_* = \int_B J^*$ for every box B.

In this case (9f4) $v\big(\varphi^{-1}(B)\big) = \int_B J$ where J is *any* function equivalent to J_*, J^*.

9g1 Proposition. If $\varphi : \mathbb{R}^m \to \mathbb{R}^n$ is such that J_*, J^* are locally integrable and equivalent then for every integrable $f : \mathbb{R}^n \to \mathbb{R}$ the function $f \circ \varphi : \mathbb{R}^m \to \mathbb{R}$ is integrable and $\int_{\mathbb{R}^m} f \circ \varphi = \int_{\mathbb{R}^n} fJ$.

9h1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets and $\varphi : V \to U$ a diffeomorphism, then

$$J_*(x) = J^*(x) = |\det(D\psi)_x|$$

for all $x \in U$; here $\psi = \varphi^{-1} : U \to V$.

(10b1) $$\int_G f = \sup\left\{ \int_{\mathbb{R}^n} g \;\middle|\; \begin{array}{l} g : \mathbb{R}^n \to \mathbb{R} \text{ integrable,} \\ 0 \le g \le f \text{ on } G,\ g = 0 \text{ on } \mathbb{R}^n \setminus G \end{array} \right\} \in [0,\infty]\,.$$

(10b4) (Poisson) $$\int_{-\infty}^{+\infty} \mathrm{e}^{-x^2}\,\mathrm{d}x = \sqrt{\pi}\,.$$

10b9 Proposition (exhaustion)**.** For open sets $G, G_1, G_2, \dots \subset \mathbb{R}^n$,

$$G_k \uparrow G \quad\Longrightarrow\quad \int_{G_k} f \uparrow \int_G f \in [0,\infty]$$

for all $f : G \to [0,\infty)$ continuous almost everywhere.

10b10 Proposition. $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2 \in [0,\infty]$ for all $f_1, f_2 \ge 0$ on G, continuous almost everywhere.

(10d1) $$\Gamma(t) = \int_0^\infty x^{t-1}\mathrm{e}^{-x}\,\mathrm{d}x \quad \text{for } t > 0\,;$$ (10d2) $$\Gamma(t+1) = t\Gamma(t)\,;$$

(10d3) $$\Gamma(n+1) = n! \quad \text{for } n = 0,1,2,\dots$$ (10d5) $$\Gamma\Big(\frac12\Big) = \sqrt{\pi}\,.$$

(10d7) The volume of the n-dimensional unit ball: $\displaystyle V_n = \frac{\pi^{n/2}}{\frac{n}{2}\Gamma(\frac{n}{2})}\,.$

(10d8) $$\int_0^{\pi/2} \cos^{\alpha-1}\theta \sin^{\beta-1}\theta\,\mathrm{d}\theta = \frac12 \frac{\Gamma(\frac{\alpha}{2})\Gamma(\frac{\beta}{2})}{\Gamma(\frac{\alpha+\beta}{2})} \quad \text{for } \alpha,\beta \in (0,\infty)\,.$$

(10d9) $$\int_0^{\pi/2} \sin^{\alpha-1}\theta\,\mathrm{d}\theta = \int_0^{\pi/2} \cos^{\alpha-1}\theta\,\mathrm{d}\theta = \frac{\sqrt{\pi}}{2} \cdot \frac{\Gamma(\frac{\alpha}{2})}{\Gamma(\frac{\alpha+1}{2})}\,.$$

(10d10) $$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \mathrm{B}(\alpha,\beta) \quad \text{for } \alpha,\beta \in (0,\infty)\,,$$

(10d11) $$\mathrm{B}(\alpha,\beta) = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)} \quad \text{for } \alpha,\beta \in (0,\infty)\,.$$

$$\Gamma^{(k)}(t) = \int_0^\infty x^{t-1}\mathrm{e}^{-x}(\ln x)^k\,\mathrm{d}x \quad \text{for } k = 1,2,\dots$$

(10e4) $\displaystyle\int_G f = \int_G f^+ - \int_G f^-$ whenever $f : G \to \mathbb{R}$ is continuous almost everywhere and such that $\int_G |f| < \infty$ (*improperly integrable*).

10e5 Exercise. Linearity: $\int_G cf = c\int_G f$ for $c \in \mathbb{R}$, and $\int_G (f_1 + f_2) = \int_G f_1 + \int_G f_2$.

10e7 Corollary. Let $G_1 \subset G_2 \subset \mathbb{R}^n$ be two open sets, and $f : G_2 \to \mathbb{R}$ improperly integrable. If $f = 0$ almost everywhere on $G_2 \setminus G_1$, then $\int_{G_2} f = \int_{G_1} f$.

10e8 Proposition (Exhaustion)**.** Let open sets $G_1 \subset G_2 \subset \dots \subset G \subset \mathbb{R}^n$ be such that $\cup_k G_k$ contains almost all points of G. Then

$$\int_{G_k} f \to \int_G f \quad \text{as } k \to \infty$$

for all f improperly integrable on G.

10e9 Proposition. Let $G \subset \mathbb{R}^n$ be an open set, and f an improperly integrable function on G. Then there exist Jordan measurable open sets $G_1 \subset G_2 \subset \dots$ such that $G_k \subset G$, $\cup_k G_k$ contains almost all points of G, and f is defined and bounded on every G_k.

We consider the vector space of all square integrable equivalence classes, with the inner product $\langle [f],[g] \rangle = \int fg$ and the corresponding norm $\|[f]\|_2 = \|f\|_2 = \big(\int f^2 \big)^{1/2}$.

The triangle inequality: $\|f+g\|_2 \le \|f\|_2 + \|g\|_2$.

The Cauchy-Schwarz inequality: $-\|f\|_2 \|g\|_2 \le \langle f,g \rangle \le \|f\|_2 \|g\|_2$.

10f1 Theorem. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$. Then

(a) f is improperly integrable on V if and only if $(f \circ \varphi)|\det D\varphi|$ is improperly integrable on U; and

(b) in this case

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \,.$$

(10g1)
$$\underset{\substack{x_1,\dots x_n > 0,\\ x_1+\dots+x_n<1}}{\int \dots \int} x_1^{p_1-1} \dots x_n^{p_n-1} \,\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{\Gamma(p_1)\dots\Gamma(p_n)}{\Gamma(p_1+\dots+p_n+1)}$$

for all $p_1, \dots p_n > 0$.

The volume of the unit ball in the metric l_p: $\quad v\big(B_p(1)\big) = \dfrac{2^n \Gamma^n\big(\frac1p\big)}{p^n \Gamma\big(\frac np + 1\big)}$.

(10g3)
$$\underset{\substack{x_1+\dots+x_n<1\\ x_1,\dots,x_n>0}}{\int \dots \int} \varphi(x_1+\dots+x_n)\,\mathrm{d}x_1 \dots \mathrm{d}x_n = \frac{1}{(n-1)!} \int_0^1 \varphi(s) s^{n-1} \,\mathrm{d}s \,.$$

11e10 Definition. A *differential form* of order k and of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x,\cdot,\dots,\cdot)$ is an antisymmetric multililear k-form on $\mathbb{R}^n$.

(11e12)
$$\int_\Gamma \omega = \int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\,\mathrm{d}u \,.$$

Antisymmetric multililear k-forms on $\mathbb{R}^n$ are a vector space of dimension $\binom{n}{k}$.

12b4 Proposition. The following three conditions on a set $M \subset \mathbb{R}^N$ and a point $x_0 \in M$ are equivalent:

(a) there exists an n-chart of M around x_0;

(b) there exists an n-cochart of M around x_0;

(c) there exists a local diffeomorphism $h : \mathbb{R}^N \to \mathbb{R}^N$ near x_0 such that

$$(u,v) \in M \quad \Longleftrightarrow \quad h(u,v) \in \mathbb{R}^n \times \{0_{N-n}\}$$

for all $(u,v) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ near x_0.

12b8 Definition. A nonempty set $M \subset \mathbb{R}^N$ is an n-dimensional *manifold* (or n-manifold) if for every $x_0 \in M$ there exists an n-chart of M around x_0.

12b9 Exercise. Let M_1 be an n_1-manifold in $\mathbb{R}^{N_1}$, and M_2 an n_2-manifold in $\mathbb{R}^{N_2}$; then $M_1 \times M_2$ is an (n_1+n_2)-manifold in $\mathbb{R}^{N_1+N_2}$.

12b10 Definition. Let $M \subset \mathbb{R}^N$ be an n-manifold; a function $f : M \to \mathbb{R}$ is *continuously differentiable* if for every chart (G,ψ) of M the function $f \circ \psi$ is continuously differentiable on G.

12b19 Exercise. Let (G,ψ) be a chart around $x_0 = \psi(u_0)$ and (U,φ) a co-chart around x_0. The following three conditions on a vector $h \in \mathbb{R}^N$ are equivalent:

(a) h is a tangent vector (at x_0);

(b) h belongs to the image of the linear operator $(D\psi)_{u_0} : \mathbb{R}^n \to \mathbb{R}^N$;

(c) h belongs to the kernel of the linear operator $(D\varphi)_{x_0} : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

12c1 Definition. A *differential form* of order k (or k-form) on an n-manifold $M \subset \mathbb{R}^N$ is a continuous function ω on the set $\{(x,h_1,\dots,h_k) : x \in M,\, h_1,\dots,h_k \in T_xM\}$ such that for every $x \in M$ the function $\omega(x,\cdot,\dots,\cdot)$ is an antisymmetric multililear k-form on T_xM.

(12c2)
$$\int_{(G,\psi)} \omega = \int_G \omega\big(\psi(u), (D_1\psi)_u, \dots, (D_n\psi)_u\big)\,\mathrm{d}u \,.$$

12c3 Proposition. Let (G_1,ψ_1), (G_2,ψ_2) be two charts of an oriented manifold $(M,\mathcal{O})$. If $\psi_1(G_1) = \psi_2(G_2)$ then

$$\int_{(G_1,\psi_1)} \omega = \int_{(G_2,\psi_2)} \omega$$

for every n-form ω on M; that is, either these two integrals converge and are equal, or both integrals diverge.

12c6 Definition. An n-form μ on an oriented n-manifold $(M,\mathcal{O})$ in $\mathbb{R}^N$ is the *volume form,* if for every $x \in M$ the antisymmetric multililear n-form $\mu(x,\cdot,\dots,\cdot)$ on T_xM is normalized and corresponds to the orientation $\mathcal{O}_x$.

$$J_\psi(u) = \sqrt{\det\big(\langle (D_i\psi)_u, (D_j\psi)_u \rangle\big)_{i,j}} \quad \text{the (generalized) Jacobian}$$

(12c16)
$$\int_U f = \int_G f(\psi(u)) J_\psi(u)\,\mathrm{d}u \,.$$

Here $U = \psi(G)$ for an n-chart (G,ψ) of $(M,\mathcal{O})$.

12c19 Lemma. $J_\psi = \sqrt{1+|\nabla f|^2}$.

13a3 Lemma. Let $M \subset \mathbb{R}^N$ be an n-manifold and $K \subset M$ a compact set. Then there exist single-chart continuous functions $\rho_1,\dots,\rho_i : M \to [0,1]$ such that $\rho_1 + \dots + \rho_i = 1$ on K.

(13a7)
$$\int_M f = \int_{(G,\psi)} f \mu_{(G,\psi)} = \int_G (f \circ \psi) J_\psi \,.$$

(13a13)	product	$v(M_1 \times M_2) = v(M_1)v(M_2)\,.$
(13a14)	scaling	$v(sM) = s^n v(M)\,.$
(13a15)	motion	$v(T(M)) = v(M)\,;\quad \int_{T(M)} f \circ T^{-1} = \int_M f\,.$
(13a16)	cylinder	$v(M) = (b-a)\|h\|v(M_1)\,.$
(13a17)	cone	$v(M) = \frac{c}{n+1}(b^{n+1} - a^{n+1})v(M_1)\,.$
(13a18)	revolution	$v(M) = 2\pi \int_{M_1} \|y\|\,.$

(13b3) $$\int_{\mathbb{R}^n} \nabla f = 0 \quad \text{if } f \in C^1(\mathbb{R}^n) \text{ has a bounded support.}$$

(13b6) $$\mathbf{n}_x = \frac{1}{\sqrt{1 + |\nabla g|^2}} \big(-(D_1 g), \dots, -(D_n g), 1 \big)\,.$$

(13b7) $$\nabla_{\rm sng} f(x) = \big(f(x + 0\mathbf{n}_x) - f(x - 0\mathbf{n}_x) \big) \mathbf{n}_x\,.$$

13b9 Theorem. Let $M \subset \mathbb{R}^{n+1}$ be an n-manifold, $K \subset M$ a compact subset, and $f : \mathbb{R}^{n+1} \setminus K \to \mathbb{R}$ a function such that
(a) f is continuously differentiable (on $\mathbb{R}^{n+1} \setminus K$);
(b) $f|_{\mathbb{R}^{n+1} \setminus \overline{M}}$ is continuous up to M;
(c) f has a bounded support, and ∇f is bounded (on $\mathbb{R}^{n+1} \setminus K$).
Then
$$\int_{\mathbb{R}^{n+1} \setminus K} \nabla f + \int_M \nabla_{\rm sng} f = 0\,.$$

13b11 Lemma. Let $(U_1, \dots, U_\ell)$ be an open covering of a compact set $K \subset \mathbb{R}^N$. Then there exist functions $\rho_1, \dots, \rho_i \in C^1(\mathbb{R}^N)$ such that $\rho_1 + \dots + \rho_i = 1$ on K and each ρ_j has a compact support within some U_m.

(13b13) $$\int_{\mathbb{R}^N \setminus K} u \nabla v = -\int_{\mathbb{R}^N \setminus K} v \nabla u - \int_M \nabla_{\rm sng}(uv)\,.$$

(13b14) $$\int_{\mathbb{R}^N} u \nabla v = -\int_{\mathbb{R}^N} v \nabla u \qquad \text{for } u, v \in C^1(\mathbb{R}^N),\ uv \text{ compactly supported.}$$

(13b15) $$\int_G \nabla f = \int_M f \mathbf{n}\,.$$

13c1 Theorem. Let $G \subset \mathbb{R}^{n+1}$ be an open set, $\varphi \in C^1(G)$, $\forall x \in G\ \nabla\varphi(x) \ne 0$, and $f \in C(G)$ compactly supported. Then for every $c \in \varphi(G)$ the set $M_c = \{ x \in G : \varphi(x) = c \}$ is an n-manifold in $\mathbb{R}^{n+1}$, the function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous and compactly supported, and
$$\int_{\varphi(G)} {\rm d}c \int_{M_c} f = \int_G f |\nabla\varphi|\,.$$

(13c8) $$\int_0^\infty {\rm d}r \int_{|\cdot|=r} f = \int_{|\cdot|>0} f\,;$$ (13c9) sphere: $$v(S_1) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,.$$

(14a4) $$\operatorname{div} F = \operatorname{tr}(DF) = D_1 F_1 + \dots + D_n F_n = (\nabla F_1)_1 + \dots + (\nabla F_n)_n\,.$$

(14a5) $$\int_{\mathbb{R}^n} \operatorname{div} F = 0 \qquad \text{if } F \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support.}$$

(14b1) $$\operatorname{div}_{\rm sng} F(x) = \langle F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x), \mathbf{n}_x \rangle\,.$$

(14b2) $$\operatorname{div}_{\rm sng} F = \sum_{k=1}^N (\nabla_{\rm sng} F_k)_k\,.$$

14b3 Theorem. Let $M \subset \mathbb{R}^{n+1}$ be an n-manifold, $K \subset M$ a compact subset, and $F : \mathbb{R}^{n+1} \setminus K \to \mathbb{R}^{n+1}$ a mapping such that
(a) F is continuously differentiable (on $\mathbb{R}^{n+1} \setminus K$);
(b) $F|_{\mathbb{R}^{n+1} \setminus \overline{M}}$ is continuous up to M;
(c) F has a bounded support, and DF is bounded (on $\mathbb{R}^{n+1} \setminus K$).
Then
$$\int_{\mathbb{R}^{n+1} \setminus K} \operatorname{div} F + \int_M \operatorname{div}_{\rm sng} f = 0\,.$$

(14c2) $$\int_G \operatorname{div} F = \int_{\partial G} \langle F, \mathbf{n} \rangle\,. \qquad \text{(flux of } F \text{ through } \partial G)$$

14c3 Theorem (*Divergence theorem*)**.** Let $G \subset \mathbb{R}^{n+1}$ be a bounded regular open set, ∂G an n-manifold, $F : \overline{G} \to \mathbb{R}^{n+1}$ continuous, $F|_G \in C^1(G \to \mathbb{R}^{n+1})$, with DF bounded on G.

Then the integral of $\operatorname{div} F$ over G is equal to the (outward) flux of F through ∂G.

14c5 Exercise. $\operatorname{div}(fF) = f \operatorname{div} F + \langle \nabla f, F \rangle$ whenever $f \in C^1(G)$ and $F \in C^1(G \to \mathbb{R}^N)$.

(14c6) $$\int_G \langle \nabla f, F \rangle = \int_{\partial G} f \langle F, \mathbf{n} \rangle - \int_G f \operatorname{div} F\,.$$

(14d1) $\Delta f = \operatorname{div} \nabla f\,;$ f is *harmonic,* if $\Delta f = 0$.

(14d2) $$\int_G \Delta f = \int_{\partial G} \langle \nabla f, \mathbf{n} \rangle = \int_{\partial G} D_{\mathbf{n}} f\,, \quad \text{first Green formula}$$

(14d3) $$\int_G (u \Delta v + \langle \nabla u, \nabla v \rangle) = \int_{\partial G} \langle u \nabla v, \mathbf{n} \rangle = \int_{\partial G} u D_{\mathbf{n}} v\,, \quad \text{second Green formula}$$

(14d4) $$\int_G (u \Delta v - v \Delta u) = \int_{\partial G} (u D_{\mathbf{n}} v - v D_{\mathbf{n}} u)\,, \quad \text{third Green formula}$$

14e1 Lemma.
$$\int_{\mathbb{R}^N} \frac{\Delta f(x)}{|x|^{N-2}}\,{\rm d}x = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0)$$
for every $N > 2$ and $f \in C^2(\mathbb{R}^N)$ with a compact support.

14e2 Remark. For $N = 2$ the situation is similar:
$$\int_{\mathbb{R}^2} \Delta f(x) \log |x|\,{\rm d}x = 2\pi f(0)$$
for every compactly supported $f \in C^2(\mathbb{R}^2)$.

14e3 Remark. Let $G \subset \mathbb{R}^N$ be a bounded regular open set, ∂G an n-manifold, $f \in C^2(G)$ with bounded second derivatives, and $0 \in G$. Then

$$\int_G \frac{\Delta f(x)}{|x|^{N-2}} \, \mathrm{d}x = -(N-2) \frac{2\pi^{N/2}}{\Gamma(N/2)} f(0) - \\ - \int_{\partial G} \Big(x \mapsto f(x) D_{\mathbf{n}} \frac{1}{|x|^{N-2}} \Big) + \int_{\partial G} \Big(x \mapsto (D_{\mathbf{n}} f(x)) \frac{1}{|x|^{N-2}} \Big) .$$

The case $N = 2$ is similar to 14e2, of course.

14e4 Proposition (*Mean value property*)**.** For every harmonic function on a ball, with bounded second derivatives, its value at the center of the ball is equal to its mean value on the boundary of the ball.

14e7 Exercise (*Maximum principle for harmonic functions*)**.**
Let u be a harmonic function on a connected open set $G \subset \mathbb{R}^N$. If $\sup_{x \in G} u(x) = u(x_0)$ for some $x_0 \in G$ then u is constant.

(14e8) $$\Delta f(x) = 2N \lim_{\varepsilon \to 0} \frac{1}{\varepsilon^2} \Big(\big(\text{mean of } f \text{ on } \{ y : |y-x| = \varepsilon \} \big) - f(x) \Big) .$$

14e10 Exercise. (a) For every f integrable (properly) on $\{x : |x| < R\}$,

$$\frac{\int_{|\cdot|<R} f}{\int_{|\cdot|<R} 1} = \int_0^R \frac{\int_{|\cdot|=r} f}{\int_{|\cdot|=r} 1} \, \frac{\mathrm{d}r^N}{R^N} .$$

(b) For every bounded harmonic function on a ball, its value at the center of the ball is equal to its mean value on the ball.

14e11 Proposition. *(Liouville's theorem for harmonic functions)*
Every harmonic function $\mathbb{R}^N \to [0, \infty)$ is constant.

(15a1,2) $$C = c_1 \Gamma_1 + \dots + c_p \Gamma_p , \quad \int_C \omega = c_1 \int_{\Gamma_1} \omega + \dots + c_p \int_{\Gamma_p} \omega .$$

(15a3) $$C_1 \sim C_2 \quad \text{means} \quad \int_{C_1} \omega = \int_{C_2} \omega \quad \text{for all } k\text{-forms } \omega \text{ (of class } C^0) .$$

$$\int_{\partial \gamma} \omega = \omega(\gamma(t_1)) - \omega(\gamma(t_0)) \quad \text{for a 0-form } \omega .$$

(15b2) $$(d\omega)(x, h) = (D\omega)_x(h) = (D_h \omega)_x .$$

15b3 Proposition. (*Stokes' theorem for $k = 1$*)
Let C be a 1-chain in $\mathbb{R}^n$, and ω a 0-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega .$$

$$\partial \Gamma = \Gamma|_{AB} + \Gamma|_{BC} + \Gamma|_{CD} + \Gamma|_{DA} ; \quad \partial(\partial \Gamma) \sim 0 \quad \text{for a singular 2-box } \Gamma .$$

15c2 Definition. The *exterior derivative* of a 1-form ω of class C^1 is the 2-form $d\omega$ defined by

$$(d\omega)(\cdot, h, k) = D_h \omega(\cdot, k) - D_k \omega(\cdot, h) .$$

15c3 Theorem. (*Stokes' theorem for $k = 2$*)
Let C be a 2-chain in $\mathbb{R}^n$, and ω a 1-form of class C^1 on $\mathbb{R}^n$. Then

$$\int_C d\omega = \int_{\partial C} \omega .$$

15c4 Exercise. For a 1-form $\omega = f(x,y) \, dx + g(x,y) \, dy$ we have $d\omega = (D_1 g - D_2 f) \mu_2$, where μ_2 is the volume form on $\mathbb{R}^2$.

(15d1) $$\omega(x, h_1, \dots, h_n) = \langle F(x), h_1 \times \dots \times h_n \rangle . \qquad \omega|_M = \langle F, \mathbf{n} \rangle \mu_{(M, \mathcal{O})} .$$

(15d2) Flux of (vector field) F through (oriented hypersurface) $(M, \mathcal{O})$ is $\int_M \langle F, \mathbf{n} \rangle$.

(15d3) $$\int_{(M, \mathcal{O})} \omega = \int_M \langle F, \mathbf{n} \rangle$$

15d4 Exercise. For a 1-form $\omega = f(x,y) \, dx + g(x,y) \, dy$ on $\mathbb{R}^2$ (or an open subset of $\mathbb{R}^2$) the corresponding vector field is $F = (F_1, F_2) = (g, -f)$, and $d\omega = (\operatorname{div} F) \mu_2$.

(15e1) $$\int_{\partial B} f = \sum_{i=1}^N \sum_{x_i = 0,1} \int \cdots \int_{(0,1)^n} f(x_1, \dots, x_N) \prod_{j : j \ne i} \mathrm{d}x_j ,$$

(15e2) $$\int_{\partial B} \langle F, \mathbf{n} \rangle = \sum_{i=1}^N \sum_{x_i = 0,1} (2x_i - 1) \int \cdots \int_{(0,1)^n} F_i(x_1, \dots, x_N) \prod_{j : j \ne i} \mathrm{d}x_j .$$

15e3 Proposition. Let $F \in C^1\big((0,1)^N \to \mathbb{R}^N\big)$, with DF bounded. Then the integral of $\operatorname{div} F$ over $(0,1)^N$ is equal to the (outward) flux of F through the boundary.

$$\Delta_{i,a}(u_1, \dots, u_n) = (u_1, \dots, u_{i-1}, a, u_i, \dots, u_n) \quad \text{for } u \in (0,1)^n$$

(15e4) $$\partial B = \sum_{i=1}^N \sum_{a=0,1} (-1)^{i-1} (2a-1) \Delta_{i,a} .$$

(15e5) $$\int_{\partial B} \omega = \int_{\partial B} \langle F, \mathbf{n} \rangle$$

(15e6) $$\partial \Gamma = \sum_{i=1}^N \sum_{a=0,1} (-1)^{i-1} (2a-1) \Gamma \circ \Delta_{i,a} .$$

15f1 Definition. The *exterior derivative* of a $(k-1)$-form ω of class C^1 is the k-form $d\omega$ defined by

$$(d\omega)(\cdot, h_1, \dots, h_k) = \sum_{i=1}^k (-1)^{i-1} D_{h_i} \omega(\cdot, h_1, \dots, h_{i-1}, h_{i+1}, \dots, h_k) .$$

15f2 Theorem. (*Stokes' theorem*)
Let C be a k-chain in $\mathbb{R}^N$, and ω a $(k-1)$-form of class C^1 on $\mathbb{R}^N$. Then

$$\int_C d\omega = \int_{\partial C} \omega .$$