8 Nonlinear change of variables

Change of variables is the most powerful tool for calculating multidimensional integrals (in particular, volumes). Integration, differentiation (diffeomorphism, its derivative) and linear algebra (the determinant) are all relevant.

8a Introduction

The area of a disk $\{(x,y) : x^2 + y^2 < 1\} \subset \mathbb{R}^2$ may be calculated by iterated integral,

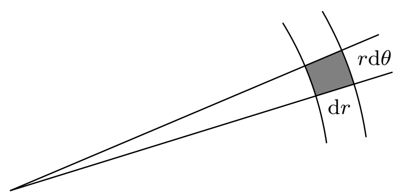
$$\int_{-1}^{1} \mathrm{d}x \int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} \mathrm{d}y = \int_{-1}^{1} 2\sqrt{1-x^2}\,\mathrm{d}x = \dots$$

or alternatively, in polar coordinates,

$$\int_0^1 r\,\mathrm{d}r \int_0^{2\pi} \mathrm{d}\theta = \int_0^1 2\pi r\,\mathrm{d}r = \pi\,;$$

the latter way is much easier! Note "$r\mathrm{d}r$" rather than "$\mathrm{d}r$" (otherwise we would get 2π instead of π).

Why the factor r? In analogy to the one-dimensional theory we may expect something like $\frac{\mathrm{d}x\,\mathrm{d}y}{\mathrm{d}r\,\mathrm{d}\theta}$; is it r? Well, basically, it is r because an infinitesimal rectangle $[r, r+\mathrm{d}r] \times [\theta, \theta+\mathrm{d}\theta]$ of area $\mathrm{d}r \cdot \mathrm{d}\theta$ on the (r,θ)-plane corresponds to an infinitesimal rectangle or area $\mathrm{d}r \cdot r\mathrm{d}\theta$ on the (x,y)-plane.



Here we use the mapping $\varphi : (r,\theta) \mapsto (r\cos\theta, r\sin\theta)$, and r is $|\det(D\varphi)_{(r,\theta)}|$ (see Exer. 8b2). Some authors[1] denote $\det(D\varphi)$ by $J\varphi$ and call it the Jacobian of φ. Some[2] denote $\det(D\varphi)$ by Δ_φ and call it the Jacobian determinant (of the Jacobian matrix J_φ). Others[3] leave $\det(D\varphi)$ as is. Here is a general result, to be proved in Sect. 8f.

8a1 Theorem. Let $U, V \subset \mathbb{R}^n$ be admissible open sets, $\varphi : U \to V$ a diffeomorphism, and $f : V \to \mathbb{R}$ a bounded function such that the function $(f \circ \varphi)|\det D\varphi| : U \to \mathbb{R}$ is also bounded. Then[4]

(a) (f is integrable on V) $\iff$ ($f \circ \varphi$ is integrable on U) $\iff$ ($(f \circ \varphi)|\det D\varphi|$ is integrable on U);

(b) if they are integrable, then

$$\int_V f = \int_U (f \circ \varphi)|\det D\varphi| \,.$$

8a2 Remark. Applying Th. 8a1 to a linear $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ we get Th. 7c1 (for integrable functions). On the other hand, Th. 7c1 is instrumental in the proof of Th. 8a1.

8a3 Remark. Applying Th. 8a1 to indicator functions f, we get:

(a) if $\det D\varphi$ is bounded, then $v(V) = \int_U |\det D\varphi|$;

(b) if $\det D\varphi$ is bounded on an admissible set $E \subset U$, then $\varphi(E)$ is admissible, and $v(\varphi(E)) = \int_E |\det D\varphi|$.

8a4 Remark. (a) If $\det D\varphi$ is bounded, then boundedness of f implies boundedness of $(f \circ \varphi)|\det D\varphi|$;

(b) if $\det D\varphi$ is bounded away from 0, then boundedness of $(f\circ\varphi)|\det D\varphi|$ implies boundedness of f;

(c) f has a compact support within V[5] if and only if $(f \circ \varphi)|\det D\varphi|$ has a compact support within U, and in this case boundedness of f is equivalent to boundedness of $(f \circ \varphi)|\det D\varphi|$ (since $\det D\varphi$ is bounded, and bounded away from 0, on the support).

Unbounded functions will be treated (in Sect. 9) by improper integral.

The proof of Theorem 8a1, rather complicated, occupies Sections 8c–8f. Some authors[6] decompose an arbitrary diffeomorphism (locally) into the

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[1] Burkill.

[2] Lang.

[3] Hubbard, Shifrin, Shurman, Zorich.

[4] Recall Def. 4d5.

[5] It means existence of a compact $K \subset V$ such that $f(\cdot) = 0$ on $V \setminus K$.

[6] Shurman, Zorich.

composition of diffeomorphisms that preserve a part of the coordinates, and use the iterated integral. Some[1] introduce the derivative of a set function and prove that $|\det D\varphi|$ is the derivative of $E \mapsto v(\varphi(E))$. Others[2] reduce the general case to indicators of small cubes and use the linear approximation. We do it this way, too.

8b Examples

In this section we take for granted Theorem 8a1 (to be proved in Sect. 8f).

8b1 Exercise. Show that 5d4 and 5d5 are special cases of 8a1.

8b2 Exercise (polar coordinates in $\mathbb{R}^2$)**.** (a) Prove that

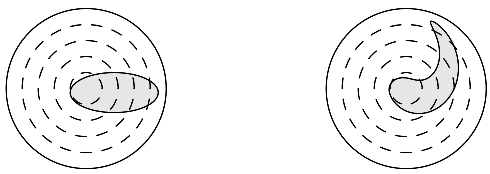
$$\int_{x^2+y^2<R^2} f(x,y)\,\mathrm{d}x\mathrm{d}y = \int_{0<r<R,0<\theta<2\pi} f(r\cos\theta, r\sin\theta)\,r\,\mathrm{d}r\mathrm{d}\theta$$

for every integrable function f on the disk $x^2+y^2<R^2$;[3]

(b) it can happen that the function $(r,\theta) \mapsto rf(r\cos\theta, r\sin\theta)$ is integrable on $(0,R)\times(0,2\pi)$, but f is not integrable on the disk; find a counterexample;

(c) however, (b) cannot happen if f is bounded on the disk; prove it.[4]

In particular, we have now the "curvilinear Cavalieri principle for concentric circles" promised in 5e9.



8b3 Exercise (spherical coordinates in $\mathbb{R}^3$)**.** Consider the mapping $\Psi : \mathbb{R}^3 \to \mathbb{R}^3$, $\Psi(r,\varphi,\theta) = (r\cos\varphi\sin\theta, r\sin\varphi\sin\theta, r\cos\theta)$.

(a) Draw the images of the planes $r = \text{const}$, $\varphi = \text{const}$, $\theta = \text{const}$, and of the lines $(\varphi,\theta) = \text{const}$, $(r,\theta) = \text{const}$, $(r,\varphi) = \text{const}$.

(b) Show that Ψ is surjective but not injective.

(c) Show that $|\det D\Psi| = r^2\sin\theta$. Find the points (r,φ,θ), where the operator $D\Psi$ is invertible.

(d) Let $V = (0,\infty)\times(-\pi,\pi)\times(0,\pi)$. Prove that $\Psi|_V$ is injective. Find $U = \Psi(V)$.

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[1] Burkill.

[2] Hubbard, Lang, Shifrin.

[3] Do you use a diffeomorphism between $(0,R)\times(0,2\pi)$ and the disk? (Look closely!)

[4] Do not forget: Theorem 8a1 is taken for granted.

8b4 Exercise. Compute the integral $\iiint_{x^2+y^2+(z-2)^2\le 1} \frac{\mathrm{d}x\mathrm{d}y\mathrm{d}z}{x^2+y^2+z^2}$.

Answer: $\pi\left(2-\frac{3}{2}\log 3\right)$. [1]

8b5 Exercise. Compute the integral $\iint \frac{\mathrm{d}x\mathrm{d}y}{(1+x^2+y^2)^2}$ over one loop of the lemniscate $(x^2+y^2)^2 = x^2-y^2$. [2]

8b6 Exercise. Compute the integral over the four-dimensional unit ball: $\iiiint_{x^2+y^2+u^2+v^2\le 1} e^{x^2+y^2-u^2-v^2}\,\mathrm{d}x\mathrm{d}y\mathrm{d}u\mathrm{d}v$. [3]

8b7 Exercise. Compute the integral $\iiint |xyz|\,\mathrm{d}x\mathrm{d}y\mathrm{d}z$ over the ellipsoid $\{x^2/a^2+y^2/b^2+z^2/c^2\le 1\}$.

Answer: $\frac{a^2b^2c^2}{6}$. [4]

The *centroid*[5] of an admissible set $E\subset\mathbb{R}^n$ of non-zero volume is the point $C_E\in\mathbb{R}^n$ such that for every linear (or affine) $f:\mathbb{R}^n\to\mathbb{R}$ the mean of f on E (recall the end of Sect. 4d) is equal to $f(C_E)$. That is,

$$C_E = \frac{1}{v(E)}\left(\int_E x_1\,\mathrm{d}x,\dots,\int_E x_n\,\mathrm{d}x\right),$$

which is often abbreviated to $C_E=\frac{1}{v(E)}\int_E x\,\mathrm{d}x$.

8b8 Exercise. Find the centroids of the following bodies in $\mathbb{R}^3$:

(a) The cone $\{(x,y,z): h\sqrt{x^2+y^2}<z<h\}$ for a given $h>0$.

(b) The tetrahedron bounded by the three coordinate planes and the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(c) The hemispherical shell $\{a^2\le x^2+y^2+z^2\le b^2,\ z\ge 0\}$.

(d) The octant of the ellipsoid $\{x^2/a^2+y^2/b^2+z^2/c^2\le 1,\ x,y,z\ge 0\}$.

The *solid torus* in $\mathbb{R}^3$ with minor radius r and major radius R (for $0<r<R<\infty$) is the set

$$\tilde\Omega = \left\{(x,y,z): \left(\sqrt{x^2+y^2}-R\right)^2+z^2\le r^2\right\}\subset\mathbb{R}^3$$

generated by rotating the disk

$$\Omega = \left\{(x,z): (x-R)^2+z^2\le r^2\right\}\subset\mathbb{R}^2$$

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[1]Hint: $1<r<3$; $\cos\theta > \frac{r^2+3}{4r}$.

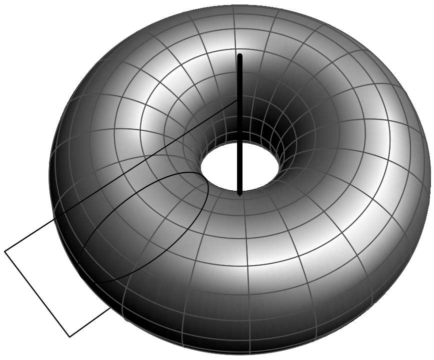
[2]Hints: use polar coordinates; $-\frac{\pi}{4}<\varphi<\frac{\pi}{4}$; $0<r<\sqrt{\cos 2\varphi}$; $1+\cos 2\varphi = 2\cos^2\varphi$; $\int\frac{\mathrm{d}\varphi}{\cos^2\varphi}=\tan\varphi$.

[3]Hint: The integral equals $\iint_{x^2+y^2\le 1} e^{x^2+y^2}\left(\iint_{u^2+v^2\le 1-(x^2+y^2)} e^{-(u^2+v^2)}\,\mathrm{d}u\mathrm{d}v\right)\mathrm{d}x\mathrm{d}y$. Now use the polar coordinates.

[4]Hint: 4h3 can help.

[5]In other words, the barycenter of (the uniform distribution on) E.

on the (x,z) plane (with the center $(R,0)$ and radius r) about the z axis.



Interestingly, the volume $2\pi^2 Rr^2$ of $\tilde\Omega$ is equal to the area πr^2 of Ω multiplied by the distance $2\pi R$ traveled by the center of Ω. (Thus, it is also equal to the volume of the cylinder $\{(x,y,z) : (x,z) \in \Omega,\ y \in [0,2\pi R].\}$) Moreover, this is a special case of a general property of all solids of revolution.

8b9 Proposition (the second Pappus's centroid theorem).[1,2] Let $\Omega \subset (0,\infty)\times\mathbb{R} \subset \mathbb{R}^2$ be an admissible set and $\tilde\Omega = \{(x,y,z) : \big(\sqrt{x^2+y^2}, z\big) \in \Omega\} \subset \mathbb{R}^3$. Then $\tilde\Omega$ is admissible, and

$$v_3(\tilde\Omega) = v_2(\Omega)\cdot 2\pi x_{C_\Omega}\,;$$

here $C_\Omega = (x_{C_\Omega}, z_{C_\Omega})$ is the centroid of Ω.

8b10 Exercise. Prove Prop. 8b9.[3]

8c Measure 0 is preserved

8c1 Proposition. Let $U,V \subset \mathbb{R}^n$ be open sets, and $\varphi : U \to V$ diffeomorphism. Then, for every set $Z \subset U$,

$$(Z \text{ has measure } 0) \iff (\varphi(Z) \text{ has measure } 0)\,.$$

Recall Def. 6c1.

8c2 Lemma. The following three conditions on a set $Z \subset \mathbb{R}^n$ are equivalent:

(a) for every $\varepsilon > 0$ there exist pixels $Q_i = 2^{-N_i}\big([0,1]^n + k_i\big)$ such that $Z \subset \bigcup_{i=1}^\infty Q_i$ and $\sum_{i=1}^\infty v(Q_i) \le \varepsilon$;

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[1] Pappus of Alexandria (≈ 0290–0350) was one of the last great Greek mathematicians of Antiquity.

[2] The first Pappus's centroid theorem, about surface area, has to wait for Analysis 4.

[3] Hint: use cylindrical coordinates: $\Psi(r,\varphi,z) = (r\cos\varphi, r\sin\varphi, z)$.

(b) Z has measure 0;

(c) for every $\varepsilon > 0$ there exist admissible sets $E_1, E_2, \dots \subset \mathbb{R}^n$ such that $Z \subset \bigcup_{i=1}^\infty E_i$ and $\sum_{i=1}^\infty v(E_i) \le \varepsilon$.

Proof. Clearly, (a)$\Longrightarrow$(b)$\Longrightarrow$(c); we'll prove that (c)$\Longrightarrow$(a).

First, recall Sect. 4d: for every admissible E we have $v(E) = v^*(E) = \lim_N U_N(\mathbb{1}_E)$, and $U_N(\mathbb{1}_E)$ is the total volume of all N-pixels that intersect E. Given $\varepsilon > 0$, we take N such that $U_N(\mathbb{1}_E) \le v(E) + \varepsilon$, denote the N-pixels that intersect E by $Q_1, \dots, Q_j$ and get $E \subset Q_1 \cup \dots \cup Q_j$ and $v(Q_1) + \dots + v(Q_j) \le v(E) + \varepsilon$.

Now we prove that (c)$\Longrightarrow$(a). Given E_i as in (c) and $\varepsilon > 0$, we take $\varepsilon_i > 0$ such that $\sum_i \varepsilon_i \le \varepsilon$, and for each i we take pixels $Q_{i,1}, \dots, Q_{i,j_i}$ such that $E_i \subset Q_{i,1} \cup \dots \cup Q_{i,j_i}$ and $v(Q_{i,1}) + \dots + v(Q_{i,j_i}) \le v(E_i) + \varepsilon_i$. Then $Z \subset \bigcup_i E_i \subset \bigcup_i (Q_{i,1} \cup \dots \cup Q_{i,j_i})$ and $\sum_i \big(v(Q_{i,1}) + \dots + v(Q_{i,j_i}) \big) \le \sum_i (v(E_i) + \varepsilon_i) \le 2\varepsilon$. It remains to enumerate all these $Q_{i,j}$ by a single index. $\square$

Euclidean metric is convenient when working with balls, not cubes. Another norm (called "cubical norm" or "sup-norm"),

$$\|x\|_\square = \max(|x_1|, \dots, |x_n|) \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

becomes more convenient, since its "ball" $\{x : \|x\|_\square \le r\}$ is a cube (of volume $(2r)^n$), and is equivalent to the Euclidean norm, since $\frac{1}{\sqrt{n}}|x| \le \|x\|_\square \le |x|$. (Some authors[1] use the cubic norm; others,[2] using Euclidean norm, complain about "pesky $\sqrt{n}$".) The corresponding operator norm (recall 1f11),

$$\|A\|_\square = \sup_{x \in \mathbb{R}^n} \frac{\|Ax\|_\square}{\|x\|_\square} = \max_{\|x\|_\square \le 1} \|Ax\|_\square \,,$$

is also equivalent to the usual operator norm.

8c3 Exercise. Prove the cubical-norm counterpart of (1f31):[3]

$$\|f(b) - f(a)\|_\square \le C \|b - a\|_\square \quad \text{if } \|Df(\cdot)\|_\square \le C \text{ on } [a, b] \,.$$

***Proof** of Prop. 8c1.* It is sufficient to prove "$\Rightarrow$"; applied to φ^{-1} it gives "$\Leftarrow$".

We consider the pixels $Q = 2^{-N}([0,1]^n + k)$ for all N and all $k \in \mathbb{Z}^n$ such that $Q \subset U$. They are a countable set,[4] and their union is the whole U. Thus, Z is the union of countably many sets $Z \cap Q$ of measure 0, and $\varphi(Z)$ is the

[1] Shifrin, Sect. 7.6 (explicitly); Lang, p. 590 (implicitly).

[2] Hubbard, after Prop. A19.3.

[3] Surprisingly, this is simpler than (1f31).

[4] Many of them are redundant, but this is harmless.

union of countably many sets $\varphi(Z \cap Q)$. By 6c2 it is sufficient to prove that each $\varphi(Z \cap Q)$ has measure 0.

By compactness, the exists M such that $\|D\varphi(x)\|_{\square} \le M$ for all $x \in Q$. By 8c3, $\|\varphi(x) - \varphi(y)\|_{\square} \le M\|x - y\|_{\square}$ for all $x, y \in Q$.

Given $\varepsilon > 0$, using 8c2 we take pixels $Q_i = 2^{-N_i}([0,1]^n + k_i)$ such that $Z \cap Q \subset \bigcup_i Q_i$ and $\sum_i v(Q_i) \le \varepsilon$. WLOG, $Q_i \subset Q$.

For all $x \in Q_i$ we have $\|\varphi(x) - \varphi(2^{-N_i}k_i)\|_{\square} \le M\|x - 2^{-N_i}k_i\|_{\square} \le 2^{-N_i}M$, thus, $\varphi(Q_i)$ is contained in a cube of volume $(2 \cdot 2^{-N_i}M)^n = (2M)^n v(Q_i)$,[1] and therefore $\varphi(Z \cap Q)$ is contained in the union of cubes of total volume $\le (2M)^n\varepsilon$, which shows that $\varphi(Z \cap Q)$ has measure 0. □

Here is a lemma needed (in addition to 8c1) in order to prove Th. 8a1(a).

8c4 Lemma. Let $E \subset \mathbb{R}^n$ be an admissible set, and $f : E \to \mathbb{R}$ a bounded function. Then f is integrable on E if and only if the discontinuity points of f on E° are a set of measure 0.

Proof. Denote by Z the set of all discontinuity points of $f \cdot 1\!\!1_E$; then $Z \cap E^\circ$ is the set of all discontinuity points of f on E°. The difference $Z \setminus (Z \cap E^\circ) \subset \partial E$ has volume 0 (see 6b8(b)), therefore, measure 0. Using Lebesgue criterion 6d2,

$$\big(f \text{ is integrable on } E\big) \iff \big(Z \text{ has measure } 0\big) \iff \big(Z \cap E^\circ \text{ has measure } 0\big)\,.$$

□

***Proof** of Item (a) of Th. 8a1.* Denote by Z the set of all discontinuity points of f (on V); then $\varphi^{-1}(Z)$ is the set of all discontinuity points of $f \circ \varphi$ (on U), since φ is a homeomorphism, and of $(f \circ \varphi)|\det D\varphi|$ as well, since $\det D\varphi$ is continuous and never 0. By 8c1, if one of these three functions is continuous almost everywhere, then the other two are. It remains to apply 8c4. □

8c5 Corollary. A set $E \subset U$ is admissible if and only if $\varphi(E) \subset V$ is admissible.

8d Approximation from within

Here we reduce Item (b) of Theorem 8a1 to such a special case (to be proved later).

8d1 Proposition. Let U, V, φ, f be as in Th. 8a1, and in addition, f be compactly supported within V. Then 8a1(b) holds.

[1] Moreover, of volume $M^n v(Q_i)$; never mind.

8d2 Lemma. For every $\varepsilon > 0$ there exists admissible compact $K \subset U$ satisfying

$$v(K) \ge v(U) - \varepsilon\,, \quad v(\varphi(K)) \ge v(V) - \varepsilon\,.$$

Proof. Recall Sect. 4d:[1] $v(U) = v_*(U) = \lim_N L_N(1\!\!1_U)$, and $L_N(1\!\!1_U)$ is the total volume of all N-pixels contained in U; denoting the union of these pixels by E_N we have $v(E_N) \to v(U)$, and each E_N is an admissible compact subset of U.

For every $\varepsilon > 0$ there exists admissible compact $E \subset U$ such that $v(E) \ge v(U) - \varepsilon$. Similarly, there exists an admissible compact $F \subset V$ such that $v(F) \ge v(V) - \varepsilon$. By 8c5, $\varphi^{-1}(F)$ and $\varphi(E)$ are admissible; we take $K = E \cup \varphi^{-1}(F)$. □

Proof *that Prop. 8d1 implies Th. 8a1(b).* We take M such that $|f(y)| \le M$ for all $y \in V$, and $|f(\varphi(x)) \det(D\varphi)_x| \le M$ for all $x \in U$.

We take $\varepsilon_i \to 0$; Lemma 8d2 gives K_i for ε_i; we introduce functions $f_i = f \cdot 1\!\!1_{\varphi(K_i)}$, then $f_i \circ \varphi = (f \circ \varphi) 1\!\!1_{K_i}$.

We use the integral norm (recall Sect. 4e): $\|f - f_i\| = {}^*\!\!\int |f - f_i| \le \int M \cdot 1\!\!1_{V \setminus \varphi(K_i)} \le M\varepsilon_i$, which gives the integral convergence: $f_i \to f$ as $i \to \infty$. Similarly, $(f_i \circ \varphi)|\det D\varphi| \to (f \circ \varphi)|\det D\varphi|$.

We apply 8d1 to each f_i and get 8a1(b) in the limit $i \to \infty$, since the integral convergence implies convergence of integrals. □

8e All we need is small volume

Now we reduce Proposition 8d1, getting rid of the function f.

8e1 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $K \subset U$ a compact set. Then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that for all $\delta \in (0, \delta_\varepsilon]$ and $h \in \mathbb{R}^n$, if $\delta(Q + h) \cap K \ne \varnothing$, where $Q = [0,1]^n$, then $\delta(Q + h) \subset U$ and

$$(8e2) \qquad 1 - \varepsilon \le \frac{v(\varphi(\delta(Q + h)))}{\delta^n |\det(D\varphi)_x|} \le 1 + \varepsilon \quad \text{for all } x \in \delta(Q + h)\,.$$

Note that $\varphi(\delta(Q + h))$ is admissible by 8c5.

Proof *that Prop. 8e1 implies Prop. 8d1 (and therefore Th. 8a1).* We have a compact $K \subset U$ such that $f = 0$ on $V \setminus \varphi(K)$. Given $\varepsilon > 0$, we'll show that the two integrals are ε-close. Prop. 8e1 gives δ_ε, and we take N such that $2^{-N} \le \delta_\varepsilon$

[1] See also the proof of 8c2.

and $U_N\big((f\circ\varphi)|\det D\varphi|\big)-L_N\big((f\circ\varphi)|\det D\varphi|\big)\le\varepsilon$. By 8e1, for every N-pixel Q such that $Q\cap K\ne\varnothing$,

$$1-\varepsilon\le\frac{v(\varphi(Q))}{v(Q)|\det(D\varphi)_x|}\le 1+\varepsilon\quad\text{for all } x\in Q\,.$$

That is,

$$(1-\varepsilon)v(Q)\Big(\sup_{x\in Q}|\det(D\varphi)_x|\Big)\le v(\varphi(Q))\le(1+\varepsilon)v(Q)\Big(\inf_{x\in Q}|\det(D\varphi)_x|\Big)\,.$$

WLOG, $f\ge 0$ (otherwise, take $f=f^+-f^-$). Denoting for convenience $g=(f\circ\varphi)|\det(D\varphi)_x|$ we have (below, Q runs over all N-pixels that intersect K)

$$\begin{aligned}(1-\varepsilon)L_N(g)&=(1-\varepsilon)\sum_Q v(Q)\inf_{x\in Q}g(x)=\\&=(1-\varepsilon)\sum_Q v(Q)\inf_{x\in Q}\big(f(\varphi(x))|\det(D\varphi)_x|\big)\le\\&\le(1-\varepsilon)\sum_Q v(Q)\Big(\inf_{x\in Q}f(\varphi(x))\Big)\Big(\sup_{x\in Q}|\det(D\varphi)_x|\Big)\le\\&\le\sum_Q v(\varphi(Q))\inf_{y\in\varphi(Q)}f(y)\le\sum_Q\int_{\varphi(Q)}f=\int_V f\le\sum_Q v(\varphi(Q))\sup_{y\in\varphi(Q)}f(y)\le\\&\le(1+\varepsilon)\sum_Q v(Q)\Big(\sup_{x\in Q}f(\varphi(x))\Big)\Big(\inf_{x\in Q}|\det(D\varphi)_x|\Big)\le\\&\le(1+\varepsilon)\sum_Q v(Q)\sup_{x\in Q}\big(f(\varphi(x))|\det(D\varphi)_x|\big)=(1+\varepsilon)U_N(g)\,.\end{aligned}$$

We see that $\int_V f\in[(1-\varepsilon)L_N(g),(1+\varepsilon)U_N(g)]$; also $\int_U g\in[L_N(g),U_N(g)]$; thus,

$$\begin{aligned}\Big|\int_U g-\int_V f\Big|&\le(1+\varepsilon)U_N(g)-(1-\varepsilon)L_N(g)\le(1+\varepsilon)(L_N(g)+\varepsilon)-(1-\varepsilon)L_N(g)=\\&=2\varepsilon L_N(g)+\varepsilon+\varepsilon^2\to 0\quad\text{as }\varepsilon\to 0\,.\end{aligned}$$

□

Now we reduce the proposition further, making it local, and formulated in terms of the cubic norm.

For convenience we say that a cube $Q_0\subset\mathbb{R}^n$ is ε-good, if $Q_0\subset U$, and every sub-cube $Q\subset Q_0$ satisfies

$$\text{(8e3)}\qquad 1-\varepsilon\le\frac{v(\varphi(Q))}{v(Q)|\det(D\varphi)_x|}\le 1+\varepsilon\quad\text{for all } x\in Q\,.$$

Clearly, every sub-cube of an ε-good cube is also ε-good.

8e4 Proposition. Let $U, V \subset \mathbb{R}^n$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $x_0 \in U$. Then for every $\varepsilon > 0$ there exists $\delta_\varepsilon > 0$ such that the cube $Q_0 = \{x \in \mathbb{R}^n : \|x - x_0\|_\square \le \delta_\varepsilon\}$ is ε-good.

***Proof** that Prop. 8e4 implies Prop. 8e1 (and therefore Th. 8a1).* A compact set $K \subset U$ is given, and $\varepsilon > 0$. For every $x_0 \in K$, 8e4 gives an ε-good cube $Q_0(x_0)$. Open cubes $Q_0^\circ(x_0)$ cover K. Applying 6b5 (in the cubic norm, equivalent to the Euclidean norm) to a finite subcovering we get a covering number, denote it $2\delta_\varepsilon$, such that for every $x_0 \in K$ the cube $Q_1(x_0) = \{y : \|y - x_0\|_\square < 2\delta_\varepsilon\}$ is covered by a single $Q_0^\circ(x)$ and therefore is ε-good. For every $\delta \in (0, \delta_\varepsilon]$ every cube $\delta([0,1]^n + h)$ that intersects K at some x_0 is contained in $Q_1(x_0)$, which proves 8e1. $\square$

8f Small volume in the linear approximation

Now we prove Prop. 8e4. We have $\varphi : U \to V$, $x_0 \in U$, and $\varepsilon > 0$. We rewrite (8e3), using the linear change of variables Th. 7b3:

$$1 - \varepsilon \le \frac{v(\varphi(Q))}{v((D\varphi)_x(Q))} \le 1 + \varepsilon \quad \text{for all } x \in Q\,; \tag{8f1}$$

here $(D\varphi)_x(Q) = \{(D\varphi)_x h : h \in Q\}$. Treating $\varphi : U \to \mathbb{R}^n$ as $\varphi : U \to W$ where W is an n-dimensional vector space, we note that (8f1), being about the ratio of two volumes in W, is insensitive to (arbitrary) change of basis in W (recall the framed phrase before (7b4)). Changing the basis (similarly to Sect. 2c, 2d) we ensure, WLOG, that[1] $(D\varphi)_{x_0} = \mathrm{id}$.

Thus, $|\det(D\varphi)_{x_0}| = 1$. WLOG,

$$1 - \varepsilon \le |\det(D\varphi)_x| \le 1 + \varepsilon \quad \text{for all } x \in U\,; \tag{8f2}$$

otherwise we replace U with a small neighborhood of x_0 (using continuity of $x \mapsto |\det(D\varphi)_x|$).

Now we may replace (8f1) with

$$1 - \varepsilon \le \frac{v(\varphi(Q))}{v(Q)} \le 1 + \varepsilon\,, \tag{8f3}$$

since $\frac{v(\varphi(Q))}{v((D\varphi)_x(Q))} = \frac{v(\varphi(Q))}{v(Q)} \frac{v(Q)}{v((D\varphi)_x(Q))} = \frac{v(\varphi(Q))}{v(Q)} \frac{1}{|\det(D\varphi)_x|}$, and so (8f3) implies (by (8f2))

$$\frac{1-\varepsilon}{1+\varepsilon} \le \frac{v(\varphi(Q))}{v((D\varphi)_x(Q))} \le \frac{1+\varepsilon}{1-\varepsilon}\,,$$

[1] Mind it: $(D\varphi)_{x_0}$, not $(D\varphi)_x$.

which is not quite (8f1), but we may change ε accordingly.

Similarly to (8f2), WLOG,

$$\|(D\varphi)_x - \mathrm{id}\|_{\square} \le \varepsilon \quad \text{for all } x \in U ,$$

and in addition, U is convex (just a ball or a cube). By 8c3,[1]

$$\|\big(\varphi(b) - \varphi(a)\big) - (b-a)\|_{\square} \le \varepsilon \|b-a\|_{\square} \quad \text{for all } a, b \in U . \tag{8f4}$$

We take $\delta_\varepsilon > 0$ such that, first, the cube $Q_0 = \{x \in \mathbb{R}^n : \|x - x_0\|_{\square} \le \delta_\varepsilon\}$ satisfies $Q_0 \subset U$, and second, $\{y \in \mathbb{R}^n : \|y - y_0\|_{\square} \le (1+\varepsilon)\delta_\varepsilon\} \subset V$, where $y_0 = \varphi(x_0)$; this is possible, since V is an (open) neighborhood of y_0.

It is sufficient to prove that

$$(1-\varepsilon)^n \le \frac{v(\varphi(Q))}{v(Q)} \le (1+\varepsilon)^n \quad \text{for every sub-cube } Q \subset Q_0 . \tag{8f5}$$

This is not quite (8f3), but again, we may change ε accordingly.

Given such Q, WLOG, the center of Q is 0, and $\varphi(0) = 0$ (since, as before, we may shift the origins in both copies of $\mathbb{R}^n$). Thus,

$$Q = \{x \in \mathbb{R}^n : \|x\|_{\square} \le r\}$$

for some $r \in (0, \delta_\varepsilon]$; it remains to prove that

$$(1-\varepsilon)Q \subset \varphi(Q) \subset (1+\varepsilon)Q . \tag{8f6}$$

By (8f4) for $a = 0$, $\|\varphi(x) - x\|_{\square} \le \varepsilon\|x\|_{\square}$ for all $x \in$ U; thus, $(1-\varepsilon)\|x\|_{\square} \le \|\varphi(x)\|_{\square} \le (1+\varepsilon)\|x\|_{\square}$. For $x \in Q$ we get $\|\varphi(x)\|_{\square} \le (1+\varepsilon)r$, thus, $\varphi(x) \in (1+\varepsilon)Q$, which proves the inclusion $\varphi(Q) \subset (1+\varepsilon)Q$. It remains to prove the other inclusion, $(1-\varepsilon)Q \subset \varphi(Q)$.

We note that $V \cap (1-\varepsilon)Q \subset \varphi(Q)$, since $\varphi(x) \in (1-\varepsilon)Q \implies \|\varphi(x)\|_{\square} \le (1-\varepsilon)r \implies (1-\varepsilon)\|x\|_{\square} \le (1-\varepsilon)r \implies \|x\|_{\square} \le r \implies x \in Q$.

It remains to prove that $(1-\varepsilon)Q \subset V$; we'll prove a bit more: that $Q \subset \{y \in \mathbb{R}^n : \|y - y_0\|_{\square} \le (1+\varepsilon)\delta_\varepsilon\}$ (and therefore $Q \subset V$).

The given inclusion $Q \subset Q_0$ means that $\|x_0\|_{\square} + r \le \delta_\varepsilon$ (think, why); similarly, the needed inclusion becomes $\|y_0\|_{\square} + r \le (1+\varepsilon)\delta_\varepsilon$. The latter follows from the former:

$$\|y_0\|_{\square} + r = \|\varphi(x_0)\|_{\square} + r \le (1+\varepsilon)\|x_0\|_{\square} + r \le (1+\varepsilon)\big(\|x_0\|_{\square} + r\big) \le (1+\varepsilon)\delta_\varepsilon ,$$

which completes the proof of Prop. 8e4, and therefore Theorem 8a1, at last!

[1] Recall the proof of 2c1 (and 2c3).