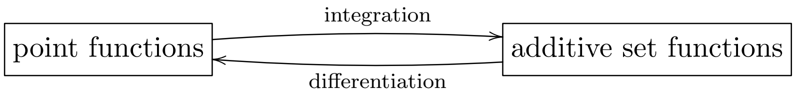
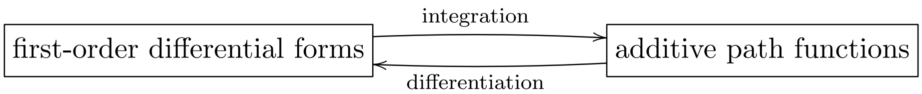
1 From path functions to differential forms

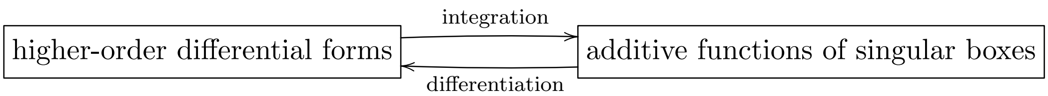
The relation



was treated in Analysis-III. A similar relation



is treated here, and generalized:



. . . this chapter may seem rather abstract and artificial . . . the best procedure for the moment is simply to regard differential forms as completely new mathematical objects. . .
Corwin and Szczarba, p. 487

. . . a k-form ω is some sort of mapping
$\omega : \{k\text{-surfaces in } A\} \to \mathbb{R}$.
Shurman, p. 404.

1a Why path functions

Life is a path function. You begin life, you end life–that's not so interesting, right? But quality of life is a path function. It's the path that you take from the beginning to the end, the integral of that path, that's the special part. Christopher Edwards

By a *path* (in $\mathbb{R}^n$) we mean a function $\gamma : [t_0, t_1] \to \mathbb{R}^n$ (real numbers $t_0 < t_1$ may depend on the path) of class C^1; that is, continuous on $[t_0, t_1]$, differentiable on (t_0, t_1), with uniformly continuous derivative $\gamma'(\cdot)$.

But sometimes we admit piecewise C^1 paths. A path is called *closed* if $\gamma(t_0) = \gamma(t_1)$.

A path may describe the motion of a body (a car, aircraft, ship, submarine, planet, particle etc); $\gamma(t)$ is the position of the body at time t.

For a car, the fuel consumption is roughly proportional to the energy required to overcome resistance, namely, air resistance and rolling resistance. This energy is a function Ω of a path;

$$\Omega(\gamma) = \int_{t_0}^{t_1} |F(t)| v(t) \, \mathrm{d}t \, ,$$

where $v(t) = |\gamma'(t)|$ is the speed of the car, and $F(t)$ is the resistance force. In a reasonable approximation,[1] the air resistance is of the form $c_2 v^2 + c_1 v$ (viscous and wind resistance), and the rolling resistance is a constant, c_0. Thus,

$$\Omega(\gamma) = \int_{t_0}^{t_1} \big(c_2 |\gamma'(t)|^2 + c_1 |\gamma'(t)| + c_0 \big) |\gamma'(t)| \, \mathrm{d}t \, .$$

For a planet or a particle resistance is usually negligible, but external fields (usually gravitational and/or electromagnetic) do a work (energy exchange)

$$\Omega(\gamma) = \int_{t_0}^{t_1} \langle F_\gamma(t), \gamma'(t) \rangle \, \mathrm{d}t$$

where $F_\gamma(t)$ is the force vector. Its dependence on γ is often of the form $F_\gamma(t) = F(\gamma(t))$ for a given vector field F; that is, $F : \mathbb{R}^n \to \mathbb{R}^n$.

And the most famous path function is, of course, the length,

$$\Omega(\gamma) = \int_{t_0}^{t_1} |\gamma'(t)| \, \mathrm{d}t \, .$$

1a1 Exercise. [2] Derive the energy conservation

$$\frac{1}{2} m |\gamma'(t_1)|^2 - \frac{1}{2} m |\gamma'(t_0)|^2 = \int_{t_0}^{t_1} \langle F_\gamma(t), \gamma'(t) \rangle \, \mathrm{d}t$$

from the Newton's second law of motion

$$m \gamma''(t) = F_\gamma(t) \, .$$

[1] Wikipedia, "Fuel economy in automobiles" and "Drag (physics)".

[2] Shifrin, Sect. 8.3.

1b Some properties of path functions

Path functions may be roughly classified according to presence or absence of the following properties.

ADDITIVITY: for every path $\gamma : [t_0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma|_{[t_0,t]}) + \Omega(\gamma|_{[t,t_1]}) = \Omega(\gamma) \quad \text{for all } t \in (t_0, t_1)\,. \tag{1b1}$$

All path functions mentioned in Sect. 1a are additive.

STATIONARITY: for every path $\gamma : [t_0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma(\cdot - s)) = \Omega(\gamma) \quad \text{for all } s \in \mathbb{R}\,; \tag{1b2}$$

here $\gamma(\cdot - s)$ is the time shifted path $t \mapsto \gamma(t - s)$ for $t \in [t_0 + s, t_1 + s]$.

Non-examples: for an aircraft, a night flight may differ in fuel consumption from a similar day flight; for a particle, external field sources may change in time.

For a stationary Ω we may restrict ourselves to the case $t_0 = 0$.

SYMMETRY AND ANTISYMMETRY (FOR STATIONARY Ω ONLY): for every path $\gamma : [0, t_1] \to \mathbb{R}^n$,

$$\Omega(\gamma_{-1}) = \Omega(\gamma)\,; \qquad \text{symmetry; or} \tag{1b3}$$

$$\Omega(\gamma_{-1}) = -\Omega(\gamma)\,; \qquad \text{antisymmetry} \tag{1b4}$$

here the inverse path $\gamma_{-1} : t \mapsto \gamma(t_1 - t)$ for $t \in [0, t_1]$.

Every stationary path function Ω is the sum of its symmetric part $\gamma \mapsto \big(\Omega(\gamma) + \Omega(\gamma_{-1})\big)/2$ and antisymmetric part $\gamma \mapsto \big(\Omega(\gamma) - \Omega(\gamma_{-1})\big)/2$; and if Ω is additive then its symmetric part and antisymmetric part are also additive (think, why).

NO WAITING CHARGE:

$$\gamma(\cdot) = \text{const (that is, } \gamma'(\cdot) = 0) \quad \text{implies} \quad \Omega(\gamma) = 0\,. \tag{1b5}$$

PARAMETRIZATION INVARIANCE:

$$\Omega(\gamma \circ \varphi) = \Omega(\gamma) \tag{1b6}$$

whenever $\gamma : [t_0, t_1] \to \mathbb{R}^n$ is a path and $\varphi : [s_0, s_1] \to [t_0, t_1]$ an increasing diffeomorphism (sometimes, only piecewise). In this case the path $\gamma \circ \varphi : [s_0, s_1] \to \mathbb{R}^n$ is called *equivalent* to γ.

Clearly, parametrization invariance implies stationarity.

1b7 Exercise. Consider path functions of the form

$$\Omega : \gamma \mapsto \int_{t_0}^{t_1} f\big(t, \gamma(t), \gamma'(t)\big)\, \mathrm{d}t \tag{1b8}$$

for arbitrary continuous functions $f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$.

(a) For each of the properties defined above give a sufficient condition in terms of f.

(b) Are your conditions necessary?

1b9 Exercise. [1] Determine the work $\int \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t$ done on a particle moving along γ in $\mathbb{R}^3$ through the force field $F(x, y, z) = (1, -x, z)$, where γ is

(a) the line segment from $(0, 0, 0)$ to $(1, 2, 1)$;

(b) the unit circle in the plane $z = 1$ with center $(0, 0, 1)$ beginning and ending at $(1, 0, 1)$ and starting toward $(0, 1, 1)$.

1b10 Exercise. [2] The same for $F(x, y, z) = (x^2, y^2, z^2)$ and $\gamma(t) = (\cos t, \sin t, at)$, $t \in [0, t_1]$ (the arc of helix).

The following property holds for a very restricted but very important class of path functions.

Given paths $\gamma, \gamma_1, \gamma_2, \dots : [t_0, t_1] \to \mathbb{R}^n$, we define convergence, $\gamma_k \to \gamma$, as follows:

$$\begin{gathered} \forall t \in [t_0, t_1] \;\; \gamma_k(t) \to \gamma(t) \,, \\ \exists L \; \forall k \;\; \gamma_k \in \operatorname{Lip}(L) \,, \end{gathered} \tag{1b11}$$

The condition $\gamma_k \in \operatorname{Lip}(L)$ is equivalent to $\forall t \;\; |\gamma'(t)| \le L$ (with one-sided derivatives when needed). Note that this convergence is stronger than the uniform convergence.

CONTINUITY:

$$\gamma_k \to \gamma \quad \text{implies} \quad \Omega(\gamma_k) \to \Omega(\gamma) \,. \tag{1b12}$$

Significantly, the length is a discontinuous path function. A counterexample: $\gamma_k(t) = \big(t, \frac{1}{k} \sin kt\big)$ (or just $\gamma_k(t) = \frac{1}{k} \sin kt$).

All path functions mentioned in Sect. 1a become continuous if one stipulates convergence in C^1 for paths, that is, $\max_t |\gamma_k'(t) - \gamma'(t)| \to 0$. But we do not!

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[1] Corwin, Szczarba Sect. 13.3.

[2] Hubbard, Sect. 6.5.

1c First-order differential forms emerge

1c1 Definition. Let Ω be a stationary additive path function, and $f : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ a continuous function. We say that f is the *derivative* of Ω (symbolically, $f = D\Omega$) if

$$\Omega(\gamma) = \int_{t_0}^{t_1} f(\gamma(t), \gamma'(t))\, \mathrm{d}t \tag{1c2}$$

for every path γ.

Such f is unique (if exists), since

$$f(\gamma(t), \gamma'(t)) = \frac{\mathrm{d}}{\mathrm{d}t}\Omega\big(\gamma|_{[t_0,t]}\big) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\Omega\big(\gamma|_{[t,t+\varepsilon]}\big)\,.$$

If such f exists, we say that Ω is continuously differentiable (or that $D\Omega$ exists), and denote $f(x,h)$ by $(D_h\Omega)_x$.[1]

1c3 Proposition. If a stationary additive path function Ω is continuous and $D\Omega$ exists then for every x the function $h \mapsto (D_h\Omega)_x$ is affine (that is, the function $h \mapsto (D_h\Omega)_x - (D_0\Omega)_x$ is linear).

1c4 Lemma. The following two conditions on a function $f : \mathbb{R}^n \to \mathbb{R}$ are equivalent:

(a) $f(\theta x + (1-\theta)y) = \theta f(x) + (1-\theta) f(y)$ for all $x, y \in \mathbb{R}^n$ and $\theta \in (0,1)$;

(b) f is affine; that is, the function $x \mapsto f(x) - f(0)$ is linear.

Proof. We define $g(x) = f(x) - f(0)$.

(b)$\Longrightarrow$(a): $f(\theta x+(1-\theta)y)-f(0) = g(\theta x+(1-\theta)y) = \theta g(x)+(1-\theta)g(y) = \theta(f(x) - f(0)) + (1-\theta)(f(y) - f(0)) = \theta f(x) + (1-\theta) f(y) - f(0)$.

(a)$\Longrightarrow$(b):

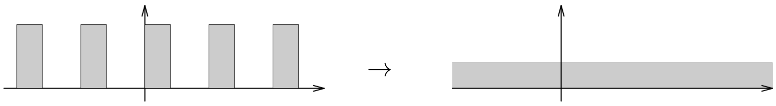
First, we have $g(\theta x) + f(0) = f(\theta x) = f(\theta x + (1-\theta)0) = \theta f(x) + (1-\theta) f(0) = \theta g(x) + f(0)$, that is, $g(\theta x) = \theta g(x)$ for $\theta \in (0,1)$ and therefore for $\theta \in (0,\infty)$ (since $(1/\theta)g(x) = g((1/\theta)x)$).

Second, $g(\frac12 x + \frac12 y) + f(0) = f(\frac12 x + \frac12 y) = \frac12 f(x) + \frac12 f(y) = \frac12 g(x) + \frac12 g(y) + f(0)$, and we get additivity: $g(x+y) = g(x) + g(y)$.

Third, $g(x) + g(-x) = g(0) = 0$, thus $g(\theta x) = \theta g(x)$ also for negative θ. $\square$

1c5 Lemma. Let $\theta \in (0,1)$ and $T_k = \biguplus_{i=-\infty}^{\infty} [\frac{i}{k}, \frac{i+\theta}{k}]$. Then $\int_{T_k} f \to \theta \int_{\mathbb{R}} f$ (as $k \to \infty$) for every Riemann integrable $f : \mathbb{R} \to \mathbb{R}$.

[1] The same condition may be imposed on an arbitrary path function, and then it may be called "additivity, stationarity and continuous differentiability".



Proof. The claim holds when f is the indicator of an interval, since in this case $|\int_{T_k} f - \theta \int_{\mathbb{R}} f| \le \frac{\theta(1-\theta)}{k}$. By linearity the claim holds for all step functions. By sandwich, it holds for all integrable functions. $\square$

Now we prove the proposition admitting piecewise C^1 paths. For the other case see Remark 1c7 afterwards.

Proof *of Prop. 1c3.*
First,

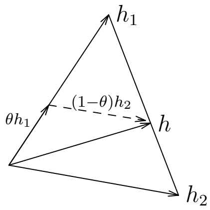
$$\text{(1c6)} \qquad \gamma_k \to \gamma \quad \text{implies} \quad \int_{t_0}^{t_1} f\big(\gamma(t), \gamma_k'(t)\big)\,\mathrm{d}t \to \int_{t_0}^{t_1} f\big(\gamma(t), \gamma'(t)\big)\,\mathrm{d}t\,,$$

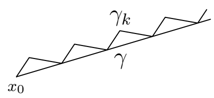
since $\Omega(\gamma_k) \to \Omega(\gamma)$ by continuity of Ω, and $\sup_t |f\big(\gamma(t), \gamma_k'(t)\big) - f\big(\gamma_k(t), \gamma_k'(t)\big)| \to 0$ due to uniform continuity of f on bounded sets. By 1c4 it is sufficient to prove that

$$(D_h\Omega)_{x_0} = \theta (D_{h_1}\Omega)_{x_0} + (1-\theta)(D_{h_2}\Omega)_{x_0}$$

whenever $h = \theta h_1 + (1-\theta)h_2$, $\theta \in (0,1)$, and $x_0 \in \mathbb{R}^n$. We construct paths $\gamma, \gamma_k : [0, t_1] \to \mathbb{R}^n$ such that

$$\gamma(0) = \gamma_k(0) = x_0\,,$$
$$\gamma'(t) = h \quad \text{for all } t \in (0, t_1)\,,$$
$$\gamma_k'(t) = \begin{cases} h_1 & \text{for } t \in (0,t_1) \cap T_k^\circ\,, \\ h_2 & \text{for } t \in (0,t_1) \setminus T_k\,, \end{cases}$$





T_k being as in Lemma 1c5.

We have $\gamma_k(\frac{i}{k}) = \gamma(\frac{i}{k})$ (for integer i such that $\frac{i}{k} \in [0, t_1]$), since $\int_{i/k}^{(i+1)/k} \gamma_k'(t)\,\mathrm{d}t = \int_{i/k}^{(i+1)/k} \gamma'(t)\,\mathrm{d}t$; thus, $\sup_t |\gamma_k(t) - \gamma(t)| \le \theta |h_1|/k \to 0$; and $\gamma_k \in \operatorname{Lip}(\max(|h_1|, |h_2|)$. Thus, $\gamma_k \to \gamma$.

By (1c6),

$$\int_0^{t_1} f\big(x_0 + th, \gamma_k'(t)\big)\,\mathrm{d}t \to \int_0^{t_1} f(x_0 + th, h)\,\mathrm{d}t\,.$$

We have

$$\int_0^{t_1} f\big(x_0 + th, \gamma_k'(t)\big)\,\mathrm{d}t = \int_{[0,t_1]\cap T_k} f(x_0 + th, h_1)\,\mathrm{d}t + \int_{[0,t_1]\setminus T_k} f(x_0 + th, h_2)\,\mathrm{d}t\,.$$

By Lemma 1c5, in the limit $k \to \infty$ we get

$$\int_0^{t_1} f(x_0 + th, h)\,\mathrm{d}t = \theta \int_0^{t_1} f(x_0 + th, h_1)\,\mathrm{d}t + (1-\theta) \int_0^{t_1} f(x_0 + th, h_2)\,\mathrm{d}t\,.$$

We see that the continuous function

$$x \mapsto f(x,h) - \theta f(x,h_1) - (1-\theta) f(x,h_2)$$

has zero integral on every straight interval of direction h. It follows easily that this function vanishes everywhere. □

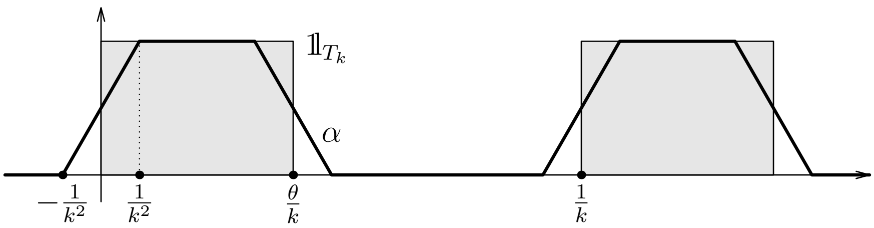
1c7 Remark. If paths are required to be C^1 (rather than piecewise C^1), the proposition still holds; here is why. Instead of

$$\gamma_k' = 1\!\!1_{T_k} h_1 + (1 - 1\!\!1_{T_k}) h_2$$

we take

$$\tilde\gamma_k' = \alpha h_1 + (1-\alpha) h_2$$

where α is such a piecewise linear approximation of $1\!\!1_{T_k}$:



Still, $\tilde\gamma_k\bigl(\frac{i}{k}\bigr) = \gamma\bigl(\frac{i}{k}\bigr)$, since the integral of α over the period is equal to θ/k. As before, $\tilde\gamma_k \to \gamma$. And $\tilde\gamma_k$ is of class C^1. It remains to check that

$$\biggl| \int_{t_0}^{t_1} f\bigl(\gamma(t), \tilde\gamma_k'(t)\bigr)\,\mathrm{d}t - \int_{t_0}^{t_1} f\bigl(\gamma(t), \gamma_k'(t)\bigr)\,\mathrm{d}t \biggr| \to 0 \quad \text{as } k \to \infty\,.$$

We note that $\tilde\gamma_k' = \gamma_k'$ (and therefore the difference vanishes) outside a set of 1-dimensional volume $\mathcal{O}\bigl(\frac{1}{k}\bigr)$. On this set, the difference is $\mathcal{O}(1)$, since both $|\gamma_k'|$ and $|\tilde\gamma_k'|$ never exceed $\max(|h_1|,|h_2|)$, and f is bounded on a bounded set.

1c8 Exercise. Assume that an additive path function Ω is continuous, and satisfies

$$\Omega(\gamma) = F\bigl(|\gamma(t_1)|\bigr) - F\bigl(|\gamma(t_0)|\bigr)$$

(where F is a given function) in two cases: first, for all γ of the form $\gamma(t) = \varphi(t)x$ ("radial"), and second, for all γ such that $|\gamma(\cdot)| = \text{const}$ ("tangential"). Prove that the same formula holds for all γ.

1c9 Definition. A *first-order differential form* of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x, \cdot)$ is linear.

For brevity we say just "1-form".

Every 1-form ω leads to an additive stationary path function Ω,

$$\Omega(\gamma) = \int_{t_0}^{t_1} \omega\big(\gamma(t), \gamma'(t)\big)\,\mathrm{d}t = \int_\gamma \omega\,; \tag{1c10}$$

note the convenient notation $\int_\gamma \omega$. This Ω satisfies the "no waiting charge" condition (1b5).

Now Proposition 1c3 may be reformulated: if an additive path function Ω is continuous and $D\Omega$ exists then

$$\forall \gamma \quad \Omega(\gamma) = \int_\gamma \omega + \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$$

for some 1-form ω of class C^0 and some continuous function $f : \mathbb{R}^n \to \mathbb{R}$.

Indeed, $f(x) = (D_0\Omega)_x$ and $\omega(x, h) = (D_h\Omega)_x - (D_0\Omega)_x$.

1c11 Exercise. Prove that the symmetric part of Ω is $\gamma \mapsto \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$ and the antisymmetric part is $\gamma \mapsto \int_\gamma \omega$.

Note that the symmetric part (if not identically zero) violates the "no waiting charge" condition (1b5), while the antisymmetric part satisfies this condition.

1c12 Exercise. The path function $\gamma \mapsto \int_{t_0}^{t_1} f\big(\gamma(t)\big)\,\mathrm{d}t$ is continuous for arbitrary continuous $f : \mathbb{R}^n \to \mathbb{R}$.

Prove it.

The path function $\gamma \mapsto \int_\gamma \omega$ is continuous for arbitrary 1-form ω; we'll prove it much later.

We have $\omega(x, h) = \omega(x, h_1e_1 + \dots + h_ne_n) = \omega(x, e_1)h_1 + \dots + \omega(x, e_n)h_n = f_1(x)h_1 + \dots + f_n(x)h_n$. Traditionally one denotes the coordinates $h_1, \dots, h_n$ of the vector h by $dx_1, \dots, dx_n$ and writes

$$\omega = f_1\,dx_1 + \dots + f_n\,dx_n\,, \quad \text{or}$$
$$\omega(x) = \omega(x_1, \dots, x_n) = f_1(x_1, \dots, x_n)\,dx_1 + \dots + f_n(x_1, \dots, x_n)\,dx_n$$

rather than

$$\omega(x_1, \dots, x_n; dx_1, \dots, dx_n) = f_1(x_1, \dots, x_n)\,dx_1 + \dots + f_n(x_1, \dots, x_n)\,dx_n\,.$$

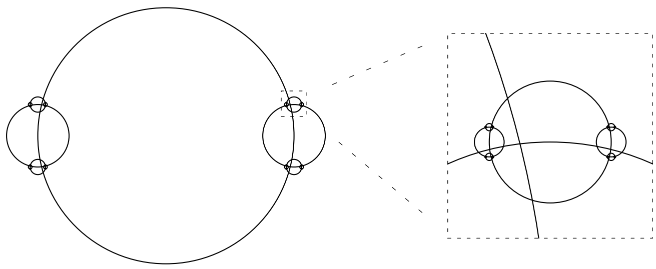
In this notation,

$$\int_\gamma \big(f_1(x)\,dx_1+\dots+f_n(x)\,dx_n\big) = \int_{t_0}^{t_1} \big(f_1(\gamma(t))\,\mathrm{d}\gamma_1(t)+\dots+f_n(\gamma(t))\,\mathrm{d}\gamma_n(t)\big)$$

for $\gamma(t)=(\gamma_1(t),\dots,\gamma_n(t))$.

1c13 Exercise. Prove that the path function $\gamma\mapsto\int_\gamma\omega$ is parametrization invariant.

A *curve* is often defined as an equivalence class of paths. Then, by 1c13, a 1-form may be integrated over a curve. But be warned: such "curve" need not be piecewise smooth (since $\gamma'(\cdot)$ may vanish on an infinite set) even if paths are C^1. On the picture below you see what may happen to the set $\gamma([t_0,t_1])$ for $\gamma\in C^1$.

(1c14) 

1c15 Exercise. [1] Prove that the following pairs of paths are equivalent:
(a) $\gamma_1(t)=(\sin t,\cos t)$, $t\in[0,2\pi]$; $\gamma_2(t)=(-\cos t,\sin t)$, $t\in[\frac{\pi}{2},\frac{5\pi}{2}]$;
(b) $\gamma_1(t)=(2\cos t,2\sin t)$, $t\in[0,\frac{\pi}{2}]$; $\gamma_2(t)=\big(\frac{2-2t^2}{1+t^2},\frac{4t}{1+t^2}\big)$, $t\in[0,1]$.

1c16 Exercise. [2] Compute $\int_\gamma\omega$ for $\omega(x,y)=x\,\mathrm{d}x-y\,\mathrm{d}y$ over the following paths:
(a) $\gamma(t)=(\cos\pi t,\sin\pi t)$, $t\in[0,1]$;
(b) $\gamma(t)=(1-t,0)$, $t\in[0,2]$;
(c) $\gamma(t)=(1-t,1-|1-t|)$, $t\in[0,2]$.

1c17 Exercise. [3] The same for $\omega(x,y,z)=yz\,\mathrm{d}x+xz\,\mathrm{d}y+xy\,\mathrm{d}z$ and
(a) $\gamma(t)=(\cos 2\pi t,\sin 2\pi t,2t)$, $t\in[0,3]$;
(b) $\gamma(t)=(1,0,t)$, $t\in[0,6]$.

1c18 Exercise. [4] The same for $\omega(x,y)=y\,\mathrm{d}x+xy\,\mathrm{d}y$ and a closed curve that traverses the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ once in the "counterclockwise" direction.

[1] Corwin, Szczarba Sect. 13.1.
[2] Devinatz, Sect. 9.1.
[3] Devinatz, Sect. 9.1.
[4] Devinatz, Sect. 9.1.

1c19 Exercise. [1] Integrate the 1-form $y\,\mathrm{d}x$ on $\mathbb{R}^3$ along the intersection of the unit sphere and the plane $x+y+z=0$, oriented counterclockwise as viewed from high above the xy-plane.[2]

1d Example: winding number

Every point $(x,y) \in \mathbb{R}^2 \setminus \{(0,0)\}$ is $(r\cos\theta, r\sin\theta)$ for $r = \sqrt{x^2+y^2}$ and some θ, but θ is not unique. We note that

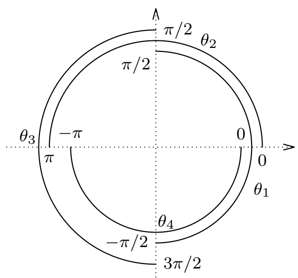
$$\begin{gathered}\mathbb{R}^2 \setminus \{(0,0)\} = U_1 \cup U_2 \cup U_3 \cup U_4\,, \\ U_1 = \{(x,y) : x > 0\}\,, \quad U_2 = \{(x,y) : y > 0\}\,, \\ U_3 = \{(x,y) : x < 0\}\,, \quad U_4 = \{(x,y) : y < 0\}\end{gathered}$$

and define functions $\theta_i : U_i \to \mathbb{R}$ for $i = 1,2,3,4$ by

$$\begin{gathered}\theta_1(x,y) = \arcsin\frac{y}{\sqrt{x^2+y^2}}\,, \quad \theta_2(x,y) = \arccos\frac{x}{\sqrt{x^2+y^2}}\,, \\ \theta_3(x,y) = \pi - \arcsin\frac{y}{\sqrt{x^2+y^2}}\,, \quad \theta_4(x,y) = -\arccos\frac{x}{\sqrt{x^2+y^2}}\,,\end{gathered}$$

then

$$\begin{gathered}\theta_1 = \theta_2 \text{ on } U_1 \cap U_2\,, \quad \theta_2 = \theta_3 \text{ on } U_2 \cap U_3\,, \\ \theta_3 = \theta_4 + 2\pi \text{ on } U_3 \cap U_4\,, \quad \theta_4 = \theta_1 \text{ on } U_4 \cap U_1\,.\end{gathered}$$



They conform only up to a constant; but their derivatives (or gradients) do conform,

$$D\theta_i = D\theta_j \text{ on } U_i \cap U_j\,.$$

A calculation gives

$$\forall (x,y) \in U_i \quad \nabla\theta_i(x,y) = \frac{1}{x^2+y^2}(-y,x)\,,$$

that is, for all $x = (x_1,x_2) \in U_i$, $h = (h_1,h_2) \in \mathbb{R}^2$,

$$(D_h\theta_i)_x = \frac{\det(x,h)}{|x|^2} = \frac{1}{x_1^2+x_2^2}\begin{vmatrix} x_1 & h_1 \\ x_2 & h_2 \end{vmatrix}\,.$$

---

[1] Shifrin, Sect. 8.3.

[2] Hint: find an orthonormal basis for the plane.

We introduce a 1-form ω on $\mathbb{R}^2 \setminus \{0\}$ by

$$\omega(x,h) = (D_h \theta_i)_x \quad \text{whenever } x \in U_i \, .$$

That is,

$$\omega(x_1,x_2) = \frac{1}{x_1^2+x_2^2} \begin{vmatrix} x_1 & dx_1 \\ x_2 & dx_2 \end{vmatrix} ; \qquad \omega = \frac{-y\,dx + x\,dy}{x^2+y^2} \, .$$

It is easy to guess that $\int_\gamma \omega$ is the angle of rotation (around the origin), and therefore

$$\int_\gamma \omega \in 2\pi\mathbb{Z} \quad \text{for all closed paths } \gamma \text{ in } \mathbb{R}^2 \setminus \{0\} \, .$$

Here is a way to the proof.

1d1 Exercise. (a) If $\gamma : [t_0,t_1] \to U_i$ then $\int_\gamma \omega = \theta_i(\gamma(t_1)) - \theta_i(\gamma(t_0))$;

(b) for every $\gamma : [t_0,t_1] \to \mathbb{R}^2 \setminus \{0\}$ there exists a partition $t_0 < s_1 < \dots < s_k < t_1$ of $[t_0,t_1]$ and $i_0,\dots,i_k \in \{1,2,3,4\}$ such that $\gamma([t_0,s_1]) \subset U_{i_0}$, $\gamma([s_1,s_2]) \subset U_{i_1}$, ..., $\gamma([s_{k-1},s_k]) \subset U_{i_{k-1}}$, $\gamma([s_k,t_1]) \subset U_{i_k}$;[1]

(c) every $\gamma : [t_0,t_1] \to \mathbb{R}^2 \setminus \{0\}$ satisfies $\theta_{i_1}(\gamma(t_1)) - \theta_{i_0}(\gamma(t_0)) - \int_\gamma \omega \in 2\pi\mathbb{Z}$ whenever $\gamma(t_0) \in U_{i_0}$, $\gamma(t_1) \in U_{i_1}$;

(d) if $\gamma(t_0) = \gamma(t_1)$ then $\int_\gamma \omega \in 2\pi\mathbb{Z}$.

Prove it.

The integer $\frac{1}{2\pi}\int_\gamma \omega$ is called the *winding number* (or index) of a close path γ on $\mathbb{R}^2 \setminus \{0\}$ around 0. The winding number of γ around another point $x_0 \in \mathbb{R}^2 \setminus \gamma([t_0,t_1])$ may be defined as the winding number of the shifted path $t \mapsto \gamma(t) - x_0$ around 0. This is an integer-valued continuous function of x_0 defined on the open set $\mathbb{R}^2 \setminus \gamma([t_0,t_1])$; therefore it is constant on each connected component of this open set. The proof of the continuity is simple: if $x_k \to x_0$ then

$$\int_{t_0}^{t_1} \omega(\gamma(t)-x_k, \gamma'(t))\mathrm{d}t \to \int_{t_0}^{t_1} \omega(\gamma(t)-x_0, \gamma'(t))\mathrm{d}t$$

since $\omega(x,h) = \frac{\det(x,h)}{|x|^2}$ is continuous in x (for a given h), uniformly outside a neighborhood of 0.

It would be interesting to integrate over all $x_0 \in \mathbb{R}^2$ the winding number around x_0. This could give us a formula for calculating the area of a planar domain via integral over the boundary of this domain. The function $x \mapsto \frac{\det(x,h)}{|x|^2}$ is unbounded (near 0), with unbounded support, which leads to an

[1] Hint: continuity of γ is enough, differentiability does not help.

improper integral. It converges near 0, but diverges on infinity (try polar coordinates). Thus, the right choice of exhaustion is important. It is futile to nullify $\omega(x,h)$ for large x, but it is wise to integrate $\omega(\gamma(t)-x_0,\gamma'(t))$ over not too large x_0. It appears that[1]

$$\int_{|x_0|\le R} \omega(x-x_0,h) \to \pi \det(x,h) \quad \text{as } R\to\infty\,;$$

thus, the integrated winding number is $\frac12 \int_{t_0}^{t_1} \det\big(\gamma(t),\gamma'(t)\big)\mathrm{d}t$, the half of the integral over γ of the 1-form $(-y\mathrm{d}x + x\mathrm{d}y)$. We'll return to this form in the end of Sect. 4.

1d2 Exercise. [2] Compute $\int_\gamma \omega$ for $\omega(x,y) = \frac{-y\,dx + x\,dy}{2}$ and γ that bounds the triangle with vertices $(0,0),(a,0),(b,c)$ $(a,b,c>0)$ and traverses its boundary once in the "counterclockwise" direction.

1e Higher-order differential forms

1e1 Definition. A *singular k-cube* in $\mathbb{R}^n$ is a mapping $\Gamma : [0,1]^k \to \mathbb{R}^n$ of class C^1; that is, Γ is continuous on $[0,1]^k$, differentiable on $(0,1)^k$, and its derivative $D\Gamma$ is uniformly continuous (that is, extends by continuity to the boundary of the cube).

Similarly we may use any closed box in $\mathbb{R}^k$, not just the cube; then we have a singular k-box.

1e2 Example. A singular 2-box in $\mathbb{R}^2$: [Sh:Sect.9.13]

$$\Gamma(r,\theta) = (r\cos\theta, r\sin\theta) \quad \text{for } (r,\theta)\in[0,1]\times[0,2\pi]\,.$$

Note that this is not a homeomorphism.

1e3 Example. A singular 2-box in $\mathbb{R}^3$:

$$\Gamma(\varphi,\theta) = (\sin\theta\sin\varphi, \sin\theta\cos\varphi, \cos\theta) \quad \text{for } (\varphi,\theta)\in[0,2\pi]\times[0,\pi]\,.$$

Also, not a homeomorphism.

A singular 1-box is nothing but a path.

A singular 2-box may be thought of as a path in the space of paths. Even in two ways. Or, as a parametrized surface. But this "surface" may be

[1] Try to check it, if you are ambitious enough.
[2] Fleming, Sect. 6.4.

rather strange (recall the one-dimensional example (1c14)) and/or degenerated (even to a single point).

A function Ω of a singular k-box is called *additive* if

$$\Omega(\Gamma) = \sum_{C \in P} \Omega(\Gamma|_C)$$

for every partition P of a box B. For $k = 1$ this is (1b1).

Similarly to (1c2) we consider Ω of the form

$$\Omega(\Gamma) = \int_B f\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\, \mathrm{d}u\,; \tag{1e4}$$

here $(D_1\Gamma)_x, \dots, (D_k\Gamma)_x \in \mathbb{R}^n$ are partial derivatives of Γ, and $f : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ is a continuous function.

Again, we wonder what can be said about f if Ω is continuous in the following sense:

$$\Gamma_j \to \Gamma \quad \text{implies} \quad \Omega(\Gamma_j) \to \Omega(\Gamma)\,, \tag{1e5}$$

where convergence of singular k-cubes (or boxes) $\Gamma, \Gamma_1, \Gamma_2, \dots : [0,1]^k \to \mathbb{R}^n$ is defined by

$$\begin{gathered} \forall u \in [0,1]^k \;\; \Gamma_j(u) \to \Gamma(u)\,, \\ \exists L \; \forall j \;\; \Gamma_j \in \mathrm{Lip}(L)\,. \end{gathered}$$

(For $k = 1$ this is (1b11)).

We consider first the case $k = 2$. Similarly to Prop. 1c3 we have the following.

1e6 Proposition. If Ω satisfies (1e4) and is continuous then for all $x, h_1 \in \mathbb{R}^n$ the function $h_2 \mapsto f(x, h_1, h_2)$ is affine.

Proof. Similarly to (1c6),

(1e7)

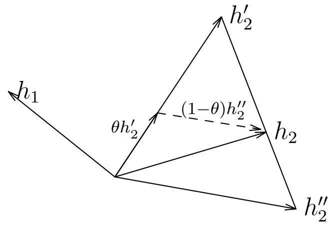
$$\Gamma_j \to \Gamma \quad \text{implies} \quad \int_B f\big(\Gamma(u), (D_1\Gamma_j)_u, (D_2\Gamma_j)_u\big)\, \mathrm{d}u \to \int_B f\big(\Gamma(u), (D_1\Gamma)_u, (D_2\Gamma)_u\big)\, \mathrm{d}u\,.$$

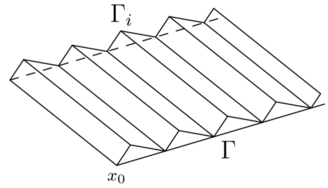
Again, by 1c4 it is sufficient to prove that

$$f(x_0, h_1, h_2) = \theta f(x_0, h_1, h_2') + (1-\theta) f(x_0, h_1, h_2'')$$

whenever $h_2 = \theta h_2' + (1-\theta)h_2''$, $\theta \in (0,1)$, and $x_0 \in \mathbb{R}^n$. Given a box $B = [0,U_1] \times [0,U_2] \subset \mathbb{R}^2$, we construct $\Gamma, \Gamma_j : B \to \mathbb{R}^n$ such that

$$
\begin{gathered}
\Gamma(0,0) = \Gamma_j(0,0) = x_0 \,, \\
(D_1\Gamma)_u = (D_1\Gamma_j)_u = h_1 \quad \text{for all } u \in B^\circ \,, \\
(D_2\Gamma)_u = h_2 \quad \text{for all } u \in B^\circ \,, \\
(D_2\Gamma_j)_u = \begin{cases} h_2' & \text{for } u \in B^\circ \cap (\mathbb{R} \times T_j^\circ) \,, \\ h_2'' & \text{for } u \in B^\circ \setminus (\mathbb{R} \times T_j) \,, \end{cases}
\end{gathered}
$$





T_j being as in Lemma 1c5. These Γ_j are not singular boxes (since they are only piecewise C^1), but still, (1e7) applies to Γ_j, since there exist (by the argument of 1c7) singular boxes $\tilde\Gamma_j$ such that $\tilde\Gamma_j \to \Gamma$ and

$$
\biggl| \int_B f\bigl(\Gamma(u), (D_1\tilde\Gamma_j)_u, (D_2\tilde\Gamma_j)_u\bigr)\,\mathrm{d}u - \int_B f\bigl(\Gamma(u), (D_1\Gamma_j)_u, (D_2\Gamma_j)_u\bigr)\,\mathrm{d}u \biggr| \to 0 \,.
$$

Similarly to the proof of 1c3 we get

$$
\begin{aligned}
\int_0^{U_2} \Bigl(\int f(x_0 + u_1h_1 + u_2h_2, h_1, (D_2\Gamma_j)_{x_0+u_1h_1+u_2h_2})\,\mathrm{d}u_1 \Bigr)\,\mathrm{d}u_2 \to \\
\to \theta \int_0^{U_2} \Bigl(\int f(x_0 + u_1h_1 + u_2h_2, h_1, h_2')\,\mathrm{d}u_1 \Bigr)\,\mathrm{d}u_2 + \\
+ (1-\theta) \int_0^{U_2} \Bigl(\int f(x_0 + u_1h_1 + u_2h_2, h_1, h_2'')\,\mathrm{d}u_1 \Bigr)\,\mathrm{d}u_2 \,.
\end{aligned}
$$

We conclude that the continuous function

$$
x \mapsto f(x,h_1,h_2) - \theta f(x,h_1,h_2') - (1-\theta) f(x,h_1,h_2'')
$$

has zero integral on every parallelepiped, and therefore vanishes everywhere.

□

Assuming in addition that $\Gamma(\cdot) = \text{const}$ implies $\Omega(\Gamma) = 0$ we get $f(x,0,0) = 0$, but still, $f(x,h_1,0)$ need not vanish. Here is an appropriate generalization of the “no waiting charge” condition (1b5):

(1e8) if $\Gamma(B)$ is contained in a $(k-1)$-dimensional affine subspace of $\mathbb{R}^n$ then $\Omega(\Gamma) = 0$.

Taking $\Gamma(u_1, u_2) = x_0 + u_1 h_1$ we see that (1e8) implies $f(x, h_1, 0) = 0$. Thus, for every x, $f(x, h_1, h_2)$ is linear in h_2 for each h_1; similarly it is linear in h_1 for each h_2; that is,

condition (1e8) implies that $f(x, \cdot, \cdot)$ is a bilinear form;

$$f(x, h_1, h_2) = \sum_{i,j=1}^{n} c_{i,j}(x)(h_1)_i(h_2)_j \,.$$

Further, taking $\Gamma(u_1, u_2) = x_0 + u_1 h + u_2 h$ we see that $f(x, h, h) = 0$ for all h (and x). It means that the bilinear form is antisymmetric,

$$f(x, h_2, h_1) = -f(x, h_1, h_2)\,;$$

indeed,

$$\underbrace{f(x, h_1 + h_2, h_1 + h_2)}_{=0} = \underbrace{f(x, h_1, h_1)}_{=0} + f(x, h_1, h_2) + f(x, h_2, h_1) + \underbrace{f(x, h_2, h_2)}_{=0} \,.$$

Generalization to $k = 3, 4, \dots$ is straightforward.

First, recall a notion from linear algebra: a (multililear) k-form[1] on $\mathbb{R}^n$ is a function $L : (\mathbb{R}^n)^k \to \mathbb{R}$ such that $L(x_1, \dots, x_k)$ is separately linear in each of the k variables $x_1, \dots, x_k \in \mathbb{R}^n$. Further, L is called antisymmetric[2] if it changes its sign under exchange of any pair of arguments.

1e9 Exercise. The following three conditions on a multililear k-form L on $\mathbb{R}^n$ are equivalent:

(a) L is antisymmetric;

(b) $L(x_1, \dots, x_k) = 0$ whenever $x_i = x_j$ for some $i \ne j$;

(c) $L(x_1, \dots, x_k) = 0$ whenever vectors $x_1, \dots, x_k$ are linearly dependent.

Now we generalize 1c9 and 1e6.

1e10 Definition. A *differential form* of order[3] k and of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x, \cdot, \dots, \cdot)$ is an antisymmetric multililear k-form on $\mathbb{R}^n$.

[1] Called also multililear form (or function) of degree (or order) k.

[2] Or “skew symmetric”, or “alternating”.

[3] Or “degree”.

For brevity we say "differential k-form" or just "k-form".

1e11 Proposition. If a function Ω of a singular k-box in $\mathbb{R}^n$ is of the form (1e4), satisfies (1e5) and (1e8), then the function f from (1e4) is a k-form (of class C^0).

Similarly to (1c10) we define the integral of a k-form ω over a singular k-box Γ,

$$\int_\Gamma \omega = \int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big) \, \mathrm{d}u \tag{1e12}$$

(recall (1e4)) and observe that $\Gamma \mapsto \int_\Gamma \omega$ is an additive function of a singular box. Now, Prop. 1e11 gives a sufficient condition for Ω to be the integral of some ω.

A k-form on $\mathbb{R}^n$ may be thought of as a mapping from $\mathbb{R}^n$ to the vector space of all antisymmetric multililear k-forms on $\mathbb{R}^n$. What is the dimension of this space?

First, $k = 1$. A linear form is uniquely determined by its values on the basis vectors $e_1, \dots, e_n$ of $\mathbb{R}^n$, and these values are arbitrary; thus, linear forms are an n-dimensional space.

Second, $k = 2$. An antisymmetric bilinear form is uniquely determined by its values on the pairs (e_i, e_j) for $i < j$, and these values are arbitrary; thus, bilinear forms are a space of dimension $\binom{n}{2} = \frac{n(n-1)}{2}$.

Similarly, antisymmetric multililear k-forms are a space of dimension $\binom{n}{k}$.

Differential 0-forms, as well as differential n-forms, are functions with 1-dimensional values, since $\binom{n}{0} = 1 = \binom{n}{n}$; basically, scalar functions. More exactly, a differential 0-form $\omega : \mathbb{R}^n \to \mathbb{R}$ is itself a scalar function, while a differential n-form ω corresponds to a scalar function $x \mapsto \omega(x, e_1, \dots, e_n)$.

1e13 Exercise. [1] Find $\int_\Gamma \omega$ where

$$\omega(x, e_2, e_3) = x_1 \, , \quad \omega(x, e_1, e_2) = \omega(x, e_1, e_3) = 0 \, ,$$

that is,

$$\omega(x, h, k) = x_1 \begin{vmatrix} h_2 & k_2 \\ h_3 & k_3 \end{vmatrix} \quad \text{for } x, h, k \in \mathbb{R}^3 \, ,$$

and $\Gamma(u, v) = (u^2, u + v, v^3)$ for $u, v \in [-1, 1]$.

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[1] Hubbard, Sect. 6.2.

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