1e10 Definition. A *differential form* of order k and of class C^m on $\mathbb{R}^n$ is a function $\omega : \mathbb{R}^n \times (\mathbb{R}^n)^k \to \mathbb{R}$ of class C^m such that for every $x \in \mathbb{R}^n$ the function $\omega(x, \cdot, \dots, \cdot)$ is an antisymmetric multililear k-form on $\mathbb{R}^n$.

$$\int_\Gamma \omega = \int_B \omega\big(\Gamma(u), (D_1\Gamma)_u, \dots, (D_k\Gamma)_u\big)\, \mathrm{d}u\,. \tag{1e12}$$

Antisymmetric multililear k-forms on $\mathbb{R}^n$ are a vector space of dimension $\binom{n}{k}$.

2b4 Proposition. The following three conditions on a set $M \subset \mathbb{R}^N$ and a point $x_0 \in M$ are equivalent:

(a) there exists an n-chart of M around x_0;
(b) there exists an n-cochart of M around x_0;
(c) there exists a local diffeomorphism $h : \mathbb{R}^N \to \mathbb{R}^N$ near x_0 such that

$$(u, v) \in M \quad\Longleftrightarrow\quad h(u, v) \in \mathbb{R}^n \times \{0_{N-n}\}$$

for all $(u, v) \in \mathbb{R}^n \times \mathbb{R}^{N-n}$ near x_0.

2b8 Definition. A nonempty set $M \subset \mathbb{R}^N$ is an n-dimensional *manifold* (or n-manifold) if for every $x_0 \in M$ there exists an n-chart of M around x_0.

2b9 Exercise. Let M_1 be an n_1-manifold in $\mathbb{R}^{N_1}$, and M_2 an n_2-manifold in $\mathbb{R}^{N_2}$; then $M_1 \times M_2$ is an $(n_1 + n_2)$-manifold in $\mathbb{R}^{N_1+N_2}$.

2b10 Definition. Let $M \subset \mathbb{R}^N$ be an n-manifold; a function $f : M \to \mathbb{R}$ is *continuously differentiable* if for every chart (G, ψ) of M the function $f \circ \psi$ is continuously differentiable on G.

2b19 Exercise. Let (G, ψ) be a chart around $x_0 = \psi(u_0)$ and (U, φ) a co-chart around x_0. The following three conditions on a vector $h \in \mathbb{R}^N$ are equivalent:

(a) h is a tangent vector (at x_0);
(b) h belongs to the image of the linear operator $(D\psi)_{u_0} : \mathbb{R}^n \to \mathbb{R}^N$;
(c) h belongs to the kernel of the linear operator $(D\varphi)_{x_0} : \mathbb{R}^N \to \mathbb{R}^{N-n}$.

2c1 Definition. A *differential form* of order k (or k-form) on an n-manifold $M \subset \mathbb{R}^N$ is a continuous function ω on the set $\{(x, h_1, \dots, h_k) : x \in M,\ h_1, \dots, h_k \in T_xM\}$ such that for every $x \in M$ the function $\omega(x, \cdot, \dots, \cdot)$ is an antisymmetric multililear k-form on T_xM.

$$\int_{(G,\psi)} \omega = \int_G \omega\big(\psi(u), (D_1\psi)_u, \dots, (D_n\psi)_u\big)\, \mathrm{d}u\,. \tag{2c2}$$

2c3 Proposition. Let (G_1, ψ_1), (G_2, ψ_2) be two charts of an oriented manifold $(M, \mathcal{O})$. If $\psi_1(G_1) = \psi_2(G_2)$ then

$$\int_{(G_1,\psi_1)} \omega = \int_{(G_2,\psi_2)} \omega$$

for every n-form ω on M; that is, either these two integrals converge and are equal, or both integrals diverge.

2c6 Definition. An n-form μ on an oriented n-manifold $(M, \mathcal{O})$ in $\mathbb{R}^N$ is the *volume form,* if for every $x \in M$ the antisymmetric multililear n-form $\mu(x, \cdot, \dots, \cdot)$ on T_xM is normalized and $\mathcal{O}_x$-positive.

$$J_\psi(u) = \sqrt{\det\big(\langle (D_i\psi)_u, (D_j\psi)_u\rangle\big)_{i,j}} \qquad \text{(The (generalized) Jacobian)}$$

$$\int_U f = \int_G f(\psi(u))J_\psi(u)\, \mathrm{d}u \qquad \text{where } U = \psi(G) \text{ and } (G, \psi) \text{ is an } n\text{-chart.} \tag{2c17}$$

2c20 Lemma. $J_\psi = \sqrt{1 + |\nabla f|^2}$. (Jacobian for a graph)

2d3 Lemma. Let $M \subset \mathbb{R}^N$ be an n-manifold and $K \subset M$ a compact set. Then there exist single-chart continuous functions $\rho_1, \dots, \rho_i : M \to [0, 1]$ such that $\rho_1 + \dots + \rho_i = 1$ on K.

$$\int_M f = \int_{(G,\psi)} f\mu_{(G,\psi)} = \int_G (f \circ \psi)J_\psi\,. \tag{2d8}$$

(2d14)	product	$v(M_1 \times M_2) = v(M_1)v(M_2)\,.$
(2d15)	scaling	$v(sM) = s^n v(M)\,.$
(2d16)	motion	$v(T(M)) = v(M)\,;\quad \int_{T(M)} f \circ T^{-1} = \int_M f\,.$
(2d17)	cylinder	$v(M) = (b - a)\lvert h\rvert v(M_1)\,.$
(2d18)	cone	$v(M) = \frac{c}{n+1}(b^{n+1} - a^{n+1})v(M_1)\,.$
(2d19)	revolution	$v(M) = 2\pi \int_{M_1} \lvert y\rvert\,.$

$$\int_{\mathbb{R}^n} \nabla f = 0 \qquad \text{if } f \in C^1(\mathbb{R}^n) \text{ has a bounded support.} \tag{3b1}$$

$$\mathbf{n}_x = \frac{1}{\sqrt{1 + |\nabla g|^2}}\big(-(D_1g), \dots, -(D_ng), 1\big)\,. \qquad \text{(Unit normal vector to a graph)} \tag{3b4}$$

$$\nabla_{\text{sng}} f(x) = \big(f(x + 0\mathbf{n}_x) - f(x - 0\mathbf{n}_x)\big)\mathbf{n}_x\,. \tag{3b5}$$

3b8 Theorem. Let $M \subset \mathbb{R}^N$ be an $(N - 1)$-manifold, $K \subset M$ a compact subset, and $f : \mathbb{R}^N \setminus K \to \mathbb{R}$ a continuously differentiable function with a bounded support and bounded gradient ∇f (on $\mathbb{R}^N \setminus K$). Then

$$\int_{\mathbb{R}^N\setminus K} \nabla f + \int_M \nabla_{\text{sng}} f = 0\,.$$

3b10 Lemma. Let $(U_1, \dots, U_\ell)$ be an open covering of a compact set $K \subset \mathbb{R}^N$. Then there exist functions $\rho_1, \dots, \rho_i \in C^1(\mathbb{R}^N)$ such that $\rho_1 + \dots + \rho_i = 1$ on K and each ρ_j has a compact support within some U_m.

$$\int_{\mathbb{R}^N\setminus K} u\nabla v = -\int_{\mathbb{R}^N\setminus K} v\nabla u - \int_M \nabla_{\text{sng}}(uv)\,. \tag{3b12}$$

$$\int_{\mathbb{R}^N} u\nabla v = -\int_{\mathbb{R}^N} v\nabla u \qquad \text{for } u, v \in C^1(\mathbb{R}^N),\ uv \text{ compactly supported.} \tag{3b13}$$

3b14 Definition. A bounded regular open set $G \subset \mathbb{R}^N$ whose boundary ∂G is a (necessarily compact) hypersurface (that is, $(N - 1)$-manifold) will be called a *smooth set.*

(3b15) $$\int_G \nabla f = \int_M f\mathbf{n}\,. \qquad \text{(Smooth } G\text{, bounded } \nabla f\text{)}$$

3c1 Theorem. Let $G \subset \mathbb{R}^{n+1}$ be an open set, $\varphi \in C^1(G)$, $\forall x \in G\ \nabla\varphi(x) \ne 0$, and $f \in C(G)$ compactly supported. Then for every $c \in \varphi(G)$ the set $M_c = \{x \in G : \varphi(x) = c\}$ is an n-manifold in $\mathbb{R}^{n+1}$, the function $c \mapsto \int_{M_c} f$ on $\varphi(G)$ is continuous and compactly supported, and

$$\int_{\varphi(G)} \mathrm{d}c \int_{M_c} f = \int_G f|\nabla\varphi|\,.$$

(3c8) $$\int_0^\infty \mathrm{d}r \int_{|\cdot|=r} f = \int_{|\cdot|>0} f\,;$$ (3c9) sphere: $$v(S_1) = \frac{2\pi^{N/2}}{\Gamma(N/2)}\,.$$

(3d3) $$\operatorname{div} F = \operatorname{tr}(DF) = D_1F_1 + \dots + D_nF_n = (\nabla F_1)_1 + \dots + (\nabla F_n)_n\,.$$

(3d4) $$\int_{\mathbb{R}^n} \operatorname{div} F = 0 \qquad \text{if } F \in C^1(\mathbb{R}^n \to \mathbb{R}^n) \text{ has a bounded support.}$$

(3e1) $$\operatorname{div}_{\mathrm{sng}} F(x) = \langle F(x + 0\mathbf{n}_x) - F(x - 0\mathbf{n}_x), \mathbf{n}_x\rangle\,.$$

(3e2) $$\operatorname{div}_{\mathrm{sng}} F = \sum_{k=1}^N (\nabla_{\mathrm{sng}} F_k)_k\,.$$

3e3 Theorem. Let $M \subset \mathbb{R}^N$ be an $(N-1)$-manifold, $K \subset M$ a compact subset, and $F : \mathbb{R}^N \setminus K \to \mathbb{R}^N$ a continuously differentiable mapping with a bounded support and bounded derivative (on $\mathbb{R}^N \setminus K$). Then

$$\int_{\mathbb{R}^N\setminus K} \operatorname{div} F + \int_M \operatorname{div}_{\mathrm{sng}} f = 0\,.$$

4a3 Theorem (*Divergence theorem*)**.** Let $G \subset \mathbb{R}^N$ be a smooth set, $F \in C^1(G \to \mathbb{R}^N)$, with DF bounded on G. Then the integral of $\operatorname{div} F$ over G is equal to the (outward) flux of F through ∂G: $$\int_G \operatorname{div} F = \int_{\partial G} \langle F, \mathbf{n}\rangle.$$

4a4 Exercise. $\operatorname{div}(fF) = f \operatorname{div} F + \langle \nabla f, F\rangle$ whenever $f \in C^1(G)$ and $F \in C^1(G \to \mathbb{R}^N)$.

(4a5) $$\int_G \langle \nabla f, F\rangle = \int_{\partial G} f\langle F, \mathbf{n}\rangle - \int_G f \operatorname{div} F\,.$$

4a6 Exercise. (a) If $f(x) = g(|x|)$, then $\nabla f(x) = \frac{g'(|x|)}{|x|}x$;

(b) if $F(x) = g(|x|)x$, then $\operatorname{div} F(x) = |x|g'(|x|) + Ng(|x|)$;

(c) if $F(x) = g(|x|)x$, then the (outward) flux of F through the boundary of the ball $\{x : |x| < r\}$ is $cr^N g(r)$, where $c = \frac{2\pi^{N/2}}{\Gamma(N/2)}$ is the area of the unit sphere.

4b4 Def. Divergence theorem holds for G and $\partial G \setminus Z$, if $G \subset \mathbb{R}^N$ is bounded regular open, $Z \subset \partial G$ closed, $\partial G \setminus Z$ n-manifold of finite volume, and $\int_G \operatorname{div} F = \int_{\partial G\setminus Z} \langle F, \mathbf{n}\rangle$ for all $F \in C(\overline{G} \to \mathbb{R}^N)$ such that $F|_G \in C^1(G \to \mathbb{R}^N)$ and DF is bounded on G.

(4c1) $$\Delta f = \operatorname{div} \nabla f\,; \quad f \text{ is } \textit{harmonic,} \text{ if } \Delta f = 0.$$

(4c2) $$\int_G \Delta f = \int_{\partial G} \langle \nabla f, \mathbf{n}\rangle = \int_{\partial G} D_{\mathbf{n}} f\,, \qquad \text{first Green formula}$$

(4c3) $$\int_G (u\Delta v + \langle \nabla u, \nabla v\rangle) = \int_{\partial G} \langle u\nabla v, \mathbf{n}\rangle = \int_{\partial G} uD_{\mathbf{n}}v\,, \quad \text{second Green formula}$$

(4c4) $$\int_G (u\Delta v - v\Delta u) = \int_{\partial G} (uD_{\mathbf{n}}v - vD_{\mathbf{n}}u)\,, \qquad \text{third Green formula}$$

4d1 Lemma. For every $N > 2$ and $f \in C^2(\mathbb{R}^N)$ with a compact support,

$$\int_{\mathbb{R}^N} \frac{\Delta f(x)}{|x|^{N-2}}\,\mathrm{d}x = -(N-2)\frac{2\pi^{N/2}}{\Gamma(N/2)} f(0)\,.$$

4d2 Remark. For $N = 2$, $$\int_{\mathbb{R}^2} \Delta f(x) \log|x|\,\mathrm{d}x = 2\pi f(0).$$

4d3 Proposition (*Mean value property*)**.** For every harmonic function on a ball, with bounded second derivatives, its value at the center of the ball is equal to its mean value on the boundary of the ball.

4d7 Exercise (*Maximum principle for harmonic functions*)**.**
Let u be a harmonic function on a connected open set $G \subset \mathbb{R}^N$. If $\sup_{x\in G} u(x) = u(x_0)$ for some $x_0 \in G$ then u is constant.

(4d8) $$\Delta f(x) = 2N \lim_{\varepsilon\to 0} \frac{1}{\varepsilon^2}\Big(\big(\text{mean of } f \text{ on } \{y : |y-x| = \varepsilon\}\big) - f(x)\Big)\,.$$

4d10 Exercise. (a) For every f integrable (properly) on $\{x : |x| < R\}$,

$$\frac{\int_{|\cdot|<R} f}{\int_{|\cdot|<R} 1} = \int_0^R \frac{\int_{|\cdot|=r} f}{\int_{|\cdot|=r} 1}\,\frac{\mathrm{d}r^N}{R^N}\,.$$

(b) For every bounded harmonic function on a ball, its value at the center of the ball is equal to its mean value on the ball.

4d11 Proposition. *(Liouville's theorem for harmonic functions)*
Every harmonic function $\mathbb{R}^N \to [0,\infty)$ is constant.

(4e5) $\forall h\ \langle h, h_1 \times \dots \times h_n\rangle = \det(h, h_1, \dots, h_n)$ (Cross-product, orthogonal to $h_1, \dots, h_n$)

(4e6) $$\omega(x, h_1, \dots, h_n) = \langle F(x), h_1 \times \dots \times h_n\rangle = \det(F(x), h_1, \dots, h_n)\,,$$

a linear one-to-one correspondence between $(N-1)$-forms ω on $\mathbb{R}^N$ and (continuous) vector fields F on $\mathbb{R}^N$;

(4e7) $$\int_M \langle F, \mathbf{n}\rangle = \int_{(M,\mathcal{O})} \omega \qquad \text{for } \omega \text{ of (4e6) and } \mathcal{O} \text{ conforming to } \mathbf{n}.$$

(4e8) $$\int_M \langle F, \mathbf{n}\rangle = \int_G \det\big(F(\psi(u)), (D_1\psi)_u, \dots, (D_n\psi)_u\big)\,\mathrm{d}u \quad \text{if } \det(\mathbf{n}, D_1\psi, \dots, D_n\psi) > 0.$$

4e10 Proposition. For every $(N-1)$-form ω of class C^1 on $\mathbb{R}^N$ there exists an N-form ω' on $\mathbb{R}^N$ such that for every smooth set $U \subset \mathbb{R}^N$, $$\int_{\partial U} \omega = \int_U \omega'.$$

(4e17) $$\oint_{\partial U} (L\,dx + M\,dy) = \iint_U \Big(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \Big)\,\mathrm{d}x\mathrm{d}y\,.$$ (Green's theorem)

5a1 Definition. (a) Let $M \subset \mathbb{R}^N$ be a manifold (of some dimension n). A mapping $\varphi : M \to \mathbb{R}^{N_2}$, $\varphi(x) = \big(\varphi_1(x),\dots,\varphi_{N_2}(x)\big)$, is *continuously differentiable*, in symbols $\varphi \in C^1(M \to \mathbb{R}^{N_2})$, if $\varphi_1,\dots,\varphi_{N_2} \in C^1(M)$.

(b) Let $M_1 \subset \mathbb{R}^{N_1}$, $M_2 \subset \mathbb{R}^{N_2}$ be manifolds (of some dimensions n_1, n_2). A mapping $\varphi : M_1 \to M_2$ is *continuously differentiable*, in symbols $\varphi \in C^1(M_1 \to M_2)$, if φ is continuously differentiable as a mapping $M_1 \to \mathbb{R}^{N_2}$. If, in addition, φ is invertible and $\varphi^{-1} \in C^1(M_2 \to M_1)$, then φ is a *diffeomorphism* $M_1 \to M_2$.

5a2 Exercise. If (G,ψ) is a chart of an n-dimensional manifold $M \subset \mathbb{R}^N$, then ψ is a diffeomorphism between the n-dimensional manifold $G \subset \mathbb{R}^n$ and the n-dimensional manifold $\psi(G) \subset M \subset \mathbb{R}^N$.

5a3 Exercise. Let $U,V \subset \mathbb{R}^N$ be open sets, $\varphi : U \to V$ a diffeomorphism, and $M \subset U$ a manifold. Then $\varphi(M) \subset V$ is a manifold, and $\varphi|_M : M \to \varphi(M)$ is a diffeomorphism.

5a4 Exercise. Let $\varphi \in C^1(M_1 \to M_2)$, $\psi \in C^1(M_2 \to M_3)$, then $\psi\circ\varphi \in C^1(M_1 \to M_3)$.

(5a5) $$(\psi\circ\varphi)_* = \psi_* \circ \varphi_*\,, \quad (\psi\circ\varphi)^* = \varphi^* \circ \psi^*\,.$$ (Always)

5a6 Definition. The *tangent bundle* TM of an n-manifold $M \subset \mathbb{R}^N$ is the set
$$TM = \{(x,h) : x \in M,\ h \in T_x M\} \subset \mathbb{R}^{2N}\,.$$

5a9 Lemma. Let $M_1 \subset \mathbb{R}^{N_1}$, $M_2 \subset \mathbb{R}^{N_2}$ be manifolds (of some dimensions n_1, n_2), and $\varphi \in C^1(M_1 \to M_2)$. Then there exists one and only one mapping $D\varphi \in C(TM_1 \to TM_2)$ such that
$$\big((\varphi\circ\gamma)(t), (\varphi\circ\gamma)'(t)\big) = (D\varphi)\big(\gamma(t),\gamma'(t)\big)$$
whenever $\gamma \in C^1([t_0,t_1] \to M_1)$ is a path, and $t \in [t_0,t_1]$.

(5a11) $$D(\psi\circ\varphi)_x h = (D\psi)_{\varphi(x)} (D\varphi)_x h\,.$$ (The chain rule of Analysis-4)

$$\big(\varphi^*(\omega)\big)(x,h) = \omega\big(\varphi_*(x),\varphi_*(h)\big) = \omega\big(\varphi(x),(D\varphi)_x h\big)\,.$$ (Pullback of 1-form)

(5a12) $$\varphi^*(f\omega) = \varphi^*(f)\varphi^*(\omega)\,;$$ (5a13) $$D(\varphi^* f) = \varphi^*(Df)\,.$$

(5a14) $$\int_\gamma \omega = \int_{(t_0,t_1)} \gamma^*(\omega)\,;$$ (5a15) $$\int_\gamma \varphi^*(\omega) = \int_{\varphi_*(\gamma)} \omega\,.$$

Pullback of n-forms:
$$\big(\varphi^*(\omega)\big)(x,h_1,\dots,h_n) = \omega\big(\varphi_*(x),\varphi_*(h_1),\dots,\varphi_*(h_n)\big) = \omega\big(\varphi(x),(D\varphi)_x h_1,\dots,(D\varphi)_x h_n\big).$$

(5a17) $$\int_\Gamma \omega = \int_{B^\circ} \Gamma^*(\omega)\,;$$ (5a18) $$\int_\Gamma \varphi^*(\omega) = \int_{\varphi_*(\Gamma)} \omega\,.$$

(5a19) $$\int_{(M_1,\mathcal{O}_1)} \varphi^*\omega = \int_{(M_2,\mathcal{O}_2)} \omega\,.$$ (For orientation preserving diffeomorphism $\varphi : M_1 \to M_2$.)

5b6 Lemma. Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets; $\varphi \in C^1(U_1 \to U_2)$; and $F_1 \in C(U_1 \to \mathbb{R}^N)$, $F_2 \in C(U_2 \to \mathbb{R}^N)$ the vector fields that correspond (by (4e6)) to $(N-1)$-forms ω_1, ω_2 such that $\omega_1 = \varphi^*\omega_2$. Then $F_1 = (\operatorname{adj} D\varphi)(F_2\circ\varphi)$.

5b7 Exercise (polar coordinates)**.** $F_1 \begin{pmatrix} r \\ \theta \end{pmatrix} = \begin{pmatrix} r\cos\theta & r\sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} F_2 \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix}$.

5b8 Exercise (rotation)**.** Let $\varphi = L : \mathbb{R}^N \to \mathbb{R}^N$ be a linear transformation such that $\forall x \in \mathbb{R}^N\ |Lx| = |x|$, and $\det L = +1$. Then the relation $F_1 = (\operatorname{adj} D\varphi)(F_2\circ\varphi)$ becomes
$$F_1 = L^{-1} \circ F_2 \circ L\,.$$

5b9 Proposition. For the constant vector field $F_2(x) = (1,0,\dots,0)$ and a mapping $\varphi : x \mapsto \big(\varphi_1(x),\dots,\varphi_N(x)\big)$ of class C^1, the vector field $(\operatorname{adj} D\varphi)(F_2\circ\varphi)$ is $\nabla\varphi_2\times\dots\times\nabla\varphi_N$.

5b11 Corollary. For the vector field $F_2(x_1,\dots,x_N) = (x_1,0,\dots,0)$ and a mapping $\varphi : x \mapsto \big(\varphi_1(x),\dots,\varphi_N(x)\big)$ of class C^1 we have $(\operatorname{adj} D\varphi)(F_2\circ\varphi) = \varphi_1\nabla\varphi_2\times\dots\times\nabla\varphi_N$.

5b12 Corollary. Let $U_1, U_2 \subset \mathbb{R}^N$ be open sets, $\varphi : U_1 \to U_2$ a diffeomorphism, $\det D\varphi > 0$, F_2 a continuous vector field on U_2, and V_2 a smooth set such that $\overline{V}_2 \subset U_2$. Then $V_1 = \varphi^{-1}(V_2)$ is a smooth set such that $\overline{V}_1 \subset U_1$, $F_1 : x \mapsto \operatorname{adj}(D\varphi)_x F_2(\varphi(x))$ is a continuous vector field on U_1, and $\displaystyle\int_{\partial V_1} \langle F_1, \mathbf{n}_1\rangle = \int_{\partial V_2} \langle F_2, \mathbf{n}_2\rangle$.

5c10 Proposition. $\displaystyle\int_{\mathbb{R}^n} \det Df = 0$ if $f \in C^1(\mathbb{R}^n \to \mathbb{R}^n)$ has a bounded support.

(A digression to topology: no-retraction theorem and Brouwer fixed point theorem.)

In the rest of Sect. 5 we define the pullback of vector fields by $\varphi^* F = (\operatorname{adj} D\varphi)(F\circ\varphi)$, redefine the pullback of functions by $\varphi^* f = (\det D\varphi)(f\circ\varphi)$, and get for C^2-diffeomorphisms
$$f = \operatorname{div} F \iff \varphi^* f = \operatorname{div}(\varphi^* F)\,, \qquad \text{that is,} \qquad \varphi^*(\operatorname{div} F) = \operatorname{div}(\varphi^* F)\,.$$

5c19 Definition. Let $U \subset \mathbb{R}^N$ be an open set, $F \in C(U \to \mathbb{R}^N)$ a vector field, and $f \in C(U)$ a function. We say that f is the *generalized divergence* of F and write $f = \operatorname{div} F$, if $\displaystyle\int_V f = \int_{\partial V} \langle F, \mathbf{n}\rangle$ for all smooth sets V such that $\overline{V} \subset U$.

5c20 Remark. (a) The generalized divergence is unique;

(b) if $F \in C^1$, then $\operatorname{tr}(DF)$ is the generalized divergence of F.

5c24 Exercise. Let $U \subset \mathbb{R}^N$ be an open set and $\varphi_1(x),\dots,\varphi_N \in C^1(U \to \mathbb{R}^N)$. Then $\operatorname{div}\big(\varphi_1\nabla\varphi_2\times\dots\times\nabla\varphi_N\big) = \det(D\varphi_1,\dots,D\varphi_N)\big)$.

5d1 Proposition. Let $U,V \subset \mathbb{R}^N$ be open sets, $\overline{V} \subset U$, and $Z \subset \partial V$. If the divergence theorem holds (see 4b4) for V and $\partial V \setminus Z$, and a vector field $F \in C(U \to \mathbb{R}^N)$ has the generalized divergence, then $\displaystyle\int_V \operatorname{div} F = \int_{\partial V \setminus Z} \langle F, \mathbf{n}\rangle$.

5d6 Theorem. Let $U,V \subset \mathbb{R}^N$ be open sets, $\varphi : U \to V$ a mapping of class C^1, $F : V \to \mathbb{R}^N$ a vector field that has the generalized divergence. Then the generalized divergence of $\varphi^* F$ exists and is equal to $\varphi^*(\operatorname{div} F)$.

5d8 Theorem (divergence theorem for a singular box)**.** Let a vector field $F \in C(U \to \mathbb{R}^N)$ on an open set $U \subset \mathbb{R}^N$ have the generalized divergence, and $\Gamma \in C^1(\overline{B} \to \mathbb{R}^N)$, $\Gamma(\overline{B}) \subset U$. Then
$$\int_B \Gamma^*(\operatorname{div} F) = \int_{\partial B \setminus Z} \langle \Gamma^*(F), \mathbf{n}\rangle\,.$$

The *generalized exterior derivative* of an $(N-1)$-form ω on $\mathbb{R}^N$ is an N-form $d\omega$ such that

(6a1) $$\int_{\partial U} \omega = \int_U d\omega \quad \text{for all smooth sets } U \, ,$$

if such $d\omega$ exists.

In terms of the function f that corresponds to $d\omega$ according to $d\omega = f \det$ and the vector field F that corresponds to ω according to (4e6) we have

(6a2) $$\omega' = d\omega \quad \Longleftrightarrow \quad f = \operatorname{div} F \, .$$

6a3 Definition. Let $U \subset \mathbb{R}^N$ be an open set, $n \in \{1,\dots,N\}$, ω an $(n-1)$-form on U. We say that an n-form ω' on U is the *generalized exterior derivative* of ω, and write $\omega' = d\omega$, if $\varphi^*\omega'$ is the generalized exterior derivative of $\varphi^*\omega$ (as defined by (6a1)) whenever $\varphi : V \to U$ is a map of class C^1, and $V \subset \mathbb{R}^n$ is an open set.

6a4 Exercise. A function $f \in C^1(U)$, treated as a 0-form, has the generalized exterior derivative $df : (x,h) \mapsto (D_h f)_x$.

6a5 Remark. In the special case $n = N-1$ Definition 6a3 conforms to (6a1).

6a6 Lemma. If $d\omega$ exists, then $d(\varphi^*\omega)$ exists and is equal to $\varphi^*(d\omega)$. (Here $\varphi \in C^1(\mathbb{R}^M \to \mathbb{R}^N)$.)

6a8 Theorem. Every differential form of class C^1 has the exterior derivative.

(6b3) $$\big(dx_{i_1} \wedge \dots \wedge dx_{i_n}\big)(h_1,\dots,h_n) = \begin{vmatrix} h_{1,i_1} & \dots & h_{n,i_1} \\ \dots & \dots & \dots \\ h_{1,i_n} & \dots & h_{n,i_n} \end{vmatrix} \qquad \text{(Notation)}$$

where $h_{i,j}$ is the j-th coordinate of h_i.

(6b5) $$\omega = \sum_{1 \le m_1 < \dots < m_n \le N} f_{m_1,\dots,m_n}(x)\, dx_{m_1} \wedge \dots \wedge dx_{m_n} \, , \qquad \text{(For every } n\text{-form } \omega \text{ on } \mathbb{R}^N)$$
$$f_{m_1,\dots,m_n}(x) = \omega(x, e_{m_1}, \dots, e_{m_n}) \, .$$

In particular, the volume form on $\mathbb{R}^n$ is $\det = dx_1 \wedge \dots \wedge dx_n$.

(6b6) $$d\varphi_1 \wedge \dots \wedge d\varphi_n = \varphi^*(dx_1 \wedge \dots \wedge dx_n) \qquad \text{(Notation)}$$

for $\varphi : x \mapsto \big(\varphi_1(x),\dots,\varphi_n(x)\big)$, $\varphi \in C^1(\mathbb{R}^N \to \mathbb{R}^n)$.

6b10 Proposition. $d(\det) = 0$.

6b13 Corollary. $d(d\varphi_1 \wedge \dots \wedge d\varphi_n) = 0$ for all $\varphi_1,\dots,\varphi_n \in C^1(\mathbb{R}^N)$.

(6b17) $$d\omega = \sum_{1 \le m_1 < \dots < m_{n-1} \le N} df_{m_1,\dots,m_{n-1}} \wedge dx_{m_1} \wedge \dots \wedge dx_{m_{n-1}} \, . \qquad \text{(See (6b5))}$$

6b19 Corollary. (a) $d(\varphi^*\omega) = \varphi^*(d\omega)$ whenever $d\omega$ exists and $\varphi \in C^1$;

(b) if $\omega, \varphi \in C^1$, then $d\omega$ is classical, but $d(\varphi^*\omega)$ is (generally) not;

(c) if $\omega \in C^1$ and $\varphi \in C^2$, then $d\omega$ and $d(\varphi^*\omega)$ are classical.

6b20 Corollary. $d(d\omega) = 0$ for all n-forms ω of class C^1.

6c2 Theorem (Stokes' theorem).

$$\int_\Gamma d\omega = \int_{\partial\Gamma} \omega$$

for every $(n-1)$-form ω of class C^1 on $\mathbb{R}^N$ and singular n-box Γ in $\mathbb{R}^N$.

6c3 Remark. The theorem still holds when $d\omega$ is the generalized exterior derivative of an $(n-1)$-form ω of class C^0.

$$\int_\gamma d\varphi = \int_{t_0}^{t_1} (\varphi \circ \gamma)'(t)\, \mathrm{d}t = \varphi(\gamma(t_1)) - \varphi(\gamma(t_0)) = \int_{\partial\gamma} \varphi \, .$$
$$\int_\gamma \omega = \int_{t_0}^{t_1} \langle F(\gamma(t)), \gamma'(t) \rangle \, \mathrm{d}t \quad \text{when } \omega(x,h) = \langle F(x), h \rangle \, .$$
$$\big|\varphi\big(\gamma(t_1)\big) - \varphi\big(\gamma(t_0)\big)\big| \le \Big(\max_t |\nabla\varphi(\gamma(t))|\Big) \operatorname{length}(\gamma) \, .$$

$$d\omega = df_1 \wedge dx_1 + \dots + df_N \wedge dx_N = \sum_{i<j} (D_i f_j - D_j f_i)\, dx_i \wedge dx_j \, . \qquad \text{(For 1-form)}$$

Dimension 2: $\omega = E_1\, dx_1 + E_2\, dx_2 = \begin{vmatrix} H_1 & dx_1 \\ H_2 & dx_2 \end{vmatrix}$;

$$d\omega = (D_2 E_1)\, dx_2 \wedge dx_1 + (D_1 E_2)\, dx_1 \wedge dx_2 = (D_1 E_2 - D_2 E_1)\, dx_1 \wedge dx_2 \, .$$
$$\int_{\partial\Gamma} \omega = \int_\gamma (-H_2\, dx_1 + H_1\, dx_2) = \int_{t_0}^{t_1} (H_1 \gamma_2' - H_2 \gamma_1')\, \mathrm{d}t = \text{flux}$$
$$\oint_{\partial\Gamma} (E_1\, dx + E_2\, dy) = \int_\Gamma (D_1 E_2 - D_2 E_1)\, dxdy \, .$$

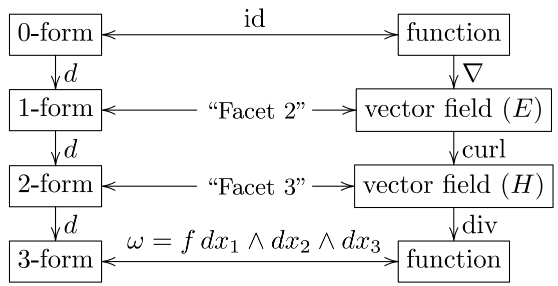
Dimension 3: $\omega = E_1\, dx_1 + E_2\, dx_2 + E_3\, dx_3$; $H = \operatorname{curl} E$;

$$d\omega = \underbrace{(D_1 E_2 - D_2 E_1)}_{H_3} dx_1 \wedge dx_2 + \underbrace{(D_2 E_3 - D_3 E_2)}_{H_1} dx_2 \wedge dx_3 + \underbrace{(D_3 E_1 - D_1 E_3)}_{H_2} dx_3 \wedge dx_1 \, .$$

(6e1) The circulation of E around $\partial\Gamma$ is equal to the flux of $\operatorname{curl} E$ through Γ.

(6e2) $$\left| \oint_{\partial\Gamma} E \right| \le \big(\max |\operatorname{curl} E|\big) \operatorname{area}(\Gamma) \, .$$

(6e3)



(6e4) $$\operatorname{curl}(\nabla f) = 0 \, , \qquad \operatorname{div}(\operatorname{curl} E) = 0 \, .$$