14 Sensitivity and superconcentration

14a Variance and gradient; proving (3a4)

14a1 Exercise. ("Gaussian integration by parts") Prove that

$$\int xf(x)\,\gamma^1(\mathrm{d}x) = \int f'(x)\,\gamma^1(\mathrm{d}x)$$

for every continuously differentiable, compactly supported $f : \mathbb{R} \to \mathbb{R}$.

14a2 Exercise. Prove that

$$\iint f(x\cos\varphi+y\sin\varphi, -x\sin\varphi+y\cos\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) = \iint f(x,y)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y)$$

for all bounded continuous $f : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi \in \mathbb{R}$.

14a3 Exercise. Prove that

$$\iint f(x,y)g(x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) = \\ = \iint f(x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi)g(x,y)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y)$$

for all bounded continuous $f, g : \mathbb{R}^2 \to \mathbb{R}$ and $\varphi \in \mathbb{R}$.

14a4 Exercise. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}\varphi}\iint f(x)g(x\cos\varphi - y\sin\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) = \\ = -\sin\varphi \iint f'(x)g'(x\cos\varphi - y\sin\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y)$$

for all continuously differentiable, compactly supported $f, g : \mathbb{R} \to \mathbb{R}$ and $\varphi \in \mathbb{R}$.

14a5 Exercise. Prove that

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint f(x)g(y)\,\gamma_t^1(\mathrm{d}x\mathrm{d}y) = -\mathrm{e}^{-t} \iint f'(x)g'(y)\,\gamma_t^1(\mathrm{d}x\mathrm{d}y)$$

for all continuously differentiable, compactly supported $f, g : \mathbb{R} \to \mathbb{R}$ and $t \in (0, \infty)$.

14a6 Exercise. Prove that

$$\int fg\,\mathrm{d}\gamma^1 - \Big(\int f\,\mathrm{d}\gamma^1\Big)\Big(\int g\,\mathrm{d}\gamma^1\Big) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t} \iint f'(x)g'(y)\,\gamma_t^1(\mathrm{d}x\mathrm{d}y)$$

for all continuously differentiable, compactly supported $f, g : \mathbb{R} \to \mathbb{R}$.

14a7 Exercise. (Generalization of 14a1 to $x \in \mathbb{R}^d$)

$$\int \nabla f(x)\,\gamma^d(\mathrm{d}x) = \int x f(x)\,\gamma^d(\mathrm{d}x)$$

for every continuously differentiable, compactly supported $f : \mathbb{R}^d \to \mathbb{R}$. That is,

$$\int \frac{\partial}{\partial x_k} f(x)\,\gamma^d(\mathrm{d}x) = \int x_k f(x)\,\gamma^d(\mathrm{d}x)$$

for $k = 1, \dots, d$.

Prove it.

14a8 Exercise. Generalize 14a3 to $x, y \in \mathbb{R}^d$.

14a9 Exercise. (Generalization of 14a4 to $\mathbb{R}^d$)

$$\frac{\mathrm{d}}{\mathrm{d}\varphi} \iint f(x)g(x\cos\varphi - y\sin\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) =$$
$$= -\sin\varphi \iint \langle \nabla f(x), \nabla g(x\cos\varphi - y\sin\varphi)\rangle\,\gamma^d(\mathrm{d}x)\gamma^d(\mathrm{d}y)$$

for all continuously differentiable, compactly supported $f, g : \mathbb{R}^d \to \mathbb{R}$ and $\varphi \in \mathbb{R}$.

Prove it.

Similarly to 14a5, 14a6 we get

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint f(x)g(y)\,\gamma_t^d(\mathrm{d}x\mathrm{d}y) = -\mathrm{e}^{-t} \iint \langle \nabla f(x), \nabla g(y)\rangle\,\gamma_t^d(\mathrm{d}x\mathrm{d}y) \tag{14a10}$$

and finally,

(14a11)

$$\int fg\,\mathrm{d}\gamma^d - \Big(\int f\,\mathrm{d}\gamma^d \Big) \Big(\int g\,\mathrm{d}\gamma^d \Big) = \int_0^\infty \mathrm{d}t\,\mathrm{e}^{-t} \iint \langle \nabla f(x), \nabla g(y) \rangle\, \gamma_t^d(\mathrm{d}x\mathrm{d}y)\,,$$

which may also be thought of as $\iint \langle \nabla f, \nabla g \rangle\,\mathrm{d}\nu$ where $\nu = \int \mathrm{e}^{-t} \gamma_t^d\,\mathrm{d}t$.

If f, g are Lip(1) functions then $|\langle \nabla f, \nabla g \rangle| \le 1$ and so, $|\int fg\,\mathrm{d}\gamma^d - \big(\int f\,\mathrm{d}\gamma^d\big)\big(\int g\,\mathrm{d}\gamma^d\big)| \le 1$. In particular,

$$\int f^2\,\mathrm{d}\gamma^d - \Big(\int f\,\mathrm{d}\gamma^d \Big)^2 \le 1 \quad \text{for } f \in \mathrm{Lip}(1)\,. \tag{14a12}$$

14a13 Exercise. Deduce (14a12) from Theorem 1a2.

Moreover,

$$\begin{aligned} \Big| \iint \langle \nabla f(x), \nabla f(y) \rangle\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \Big| &\le \\ \le \Big(\iint |\nabla f(x)|^2\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \Big)^{1/2} &\Big(\iint |\nabla f(y)|^2\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \Big)^{1/2} = \\ &= \int |\nabla f|^2\,\mathrm{d}\gamma^d\,; \end{aligned}$$

in combination with (14a11) (for $f = g$) it gives

$$\int f^2\,\mathrm{d}\gamma^d - \Big(\int f\,\mathrm{d}\gamma^d \Big)^2 \le \int |\nabla f|^2\,\mathrm{d}\gamma^d\,, \tag{14a14}$$

the Poincare inequality[1] for Gaussian measure. It is evidently stronger than (14a12).

We cannot just apply (14a11) to $f = g = \xi$, $\xi(x) = \max_{a \in A} \langle x, a \rangle$, since ξ is neither continuously differentiable nor compactly supported. However, the needed generalizations are easy. First, 14a1 holds for a piecewise continuously differentiable Lipschitz function $f : \mathbb{R} \to \mathbb{R}$ (think, why).[2] Second, 14a7 holds for $f = \xi$, since the restriction of ξ to a straight line is the maximum of finitely many linear functions. Thus, 14a11 applies to $f = g = \xi$; taking into account that $\nabla \xi = \alpha$ we get (3a4).

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[1] The simplest classical Poincare inequality: $\int_0^1 f^2(x)\,\mathrm{d}x - \big(\int_0^1 f(x)\,\mathrm{d}x\big)^2 \le \frac{1}{\pi^2} \int_0^1 f'^2(x)\,\mathrm{d}x$; the equality holds for $f(x) = \cos \pi x$.

[2] Wider generalization is possible, but we do not need it.

14b Proving Lemma 3a2

14b1 Exercise. Prove that

$$\iint f(x, y, x\cos\varphi + y\sin\varphi, -x\sin\varphi + y\cos\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) = \\ = \iint f(x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi, x, y)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y)$$

for all bounded continuous $f : \mathbb{R}^4 \to \mathbb{R}$ and $\varphi \in \mathbb{R}$.

But do not think that

$$\iint f(x, y, x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y) = \\ = \iint f(x\cos\varphi - y\sin\varphi, x\sin\varphi + y\cos\varphi, x, y)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}y)\,,$$

this is generally wrong (think, why).

14b2 Exercise. The measure γ^1_t is symmetric. That is,

$$\iint f(x, y)\gamma^1_t(\mathrm{d}x\mathrm{d}y) = \iint f(y, x)\gamma^1_t(\mathrm{d}x\mathrm{d}y)$$

for all bounded continuous $f : \mathbb{R}^2 \to \mathbb{R}$ and $t \in [0, \infty)$.
Prove it.

Thus,

$$\iint f(x, \mathrm{e}^{-t}x + \sqrt{1 - \mathrm{e}^{-2t}}u)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}u) = \\ = \iint f(\mathrm{e}^{-t}y + \sqrt{1 - \mathrm{e}^{-2t}}v, y)\,\gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}v)\,.$$

14b3 Lemma. For every bounded continuous $f : \mathbb{R} \to \mathbb{R}$ and $t \in [0, \infty)$,

$$\iint f(x)f(z)\,\gamma^1_{2t}(\mathrm{d}x\mathrm{d}z) = \int \Big(\int f(\mathrm{e}^{-t}y + \sqrt{1 - \mathrm{e}^{-2t}}u)\,\gamma^1(\mathrm{d}u) \Big)^2 \gamma^1(\mathrm{d}y)\,.$$

Proof.

$$I_2 = \int \Big(\int f(\mathrm{e}^{-t}y + \sqrt{1 - \mathrm{e}^{-2t}}u)\,\gamma^1(\mathrm{d}u) \Big)^2 \gamma^1(\mathrm{d}y) = \\ = \iiint f(\mathrm{e}^{-t}y + \sqrt{1 - \mathrm{e}^{-2t}}u) f(\mathrm{e}^{-t}y + \sqrt{1 - \mathrm{e}^{-2t}}v)\,\gamma^1(\mathrm{d}y)\gamma^1(\mathrm{d}u)\gamma^1(\mathrm{d}v)\,;$$

for every v we have

$$\iint f(\mathrm{e}^{-t}y + \sqrt{1-\mathrm{e}^{-2t}}u) f(\mathrm{e}^{-t}y + \sqrt{1-\mathrm{e}^{-2t}}v)\, \gamma^1(\mathrm{d}y)\gamma^1(\mathrm{d}u) = \\ = \iint f(x) f\big(\mathrm{e}^{-t}(\mathrm{e}^{-t}x + \sqrt{1-\mathrm{e}^{-2t}}w) + \sqrt{1-\mathrm{e}^{-2t}}v\big)\, \gamma^1(\mathrm{d}x)\gamma^1(\mathrm{d}w)\,;$$

thus,

$$I_2 = \int \gamma^1(\mathrm{d}x) f(x) \iint \gamma^1(\mathrm{d}v)\gamma^1(\mathrm{d}w) f(\mathrm{e}^{-2t}x + \mathrm{e}^{-t}\sqrt{1-\mathrm{e}^{-2t}}w + \sqrt{1-\mathrm{e}^{-2t}}v) = \\ = \int \gamma^1(\mathrm{d}x) f(x) \int \gamma^1(\mathrm{d}u) f(\mathrm{e}^{-2t}x + \sqrt{1-\mathrm{e}^{-4t}}u) = I_1\,,$$

since $\mathrm{e}^{-2t}(1-\mathrm{e}^{-2t}) + 1 - \mathrm{e}^{-2t} = 1 - \mathrm{e}^{-4t}$. □

The same holds for γ^d, and we get

$$\iint f(x) f(y)\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \ge 0$$

for every bounded continuous $f : \mathbb{R}^d \to \mathbb{R}$. By approximation it holds for all $f \in L_2(\gamma^d)$, which proves a half of Lemma 3a2.

The same holds for vector-functions (think, why). In particular,

$$\iint \langle \nabla f(x), \nabla f(y) \rangle\, \gamma_t^d(\mathrm{d}x\mathrm{d}y) \ge 0$$

whenever $\int |\nabla f|^2 \,\mathrm{d}\gamma^d < \infty$.

By (14a10), $\iint f(x) f(y)\, \gamma_t^d(\mathrm{d}x\mathrm{d}y)$ decreases in t for good functions f. By approximation it holds for all $f \in L_2(\gamma^d)$, which completes the proof of Lemma 3a2.[1]

14c Proving Theorem 3a3

Correction. Item (a) of Theorem 3a3 should be: assumption D_{2n^2} implies assumption E_n.

The function

$$\varphi(t) = \mathbb{E}\,\langle \alpha(X), \alpha(X_t) \rangle$$

[1]In fact, Lemma 3a2 is not "Gaussian"; it holds for every time-symmetric Markov process. Here is its translation into the language of functional analysis. Let $(U_t)_{t\ge0}$ be a one-parameter semigroup of Hermitian operators in a Hilbert space, satisfying $\|U_t\| \le 1$ for all t. Then the function $t \mapsto \langle U_t\psi, \psi \rangle$ is nonnegative and decreasing on $[0,\infty)$ for every vector ψ of the Hilbert space. (The proof is quite simple.)

satisfies

$$\forall t \;\; 0 \le \varphi(t) \le 1\,, \qquad \varphi \text{ is decreasing on } [0,\infty) \tag{14c1}$$

(think, why).

14c2 Exercise. For every φ satisfying (14c1) and every $x \in (0,\infty)$,

(a) $\displaystyle\int_0^\infty e^{-t}\varphi(t)\,dt \le x + \varphi(x)$;

(b) $\displaystyle\varphi(x) \le \frac{e^x}{x}\int_0^\infty e^{-t}\varphi(t)\,dt$.

Prove it.

By (3a4), $\int_0^\infty e^{-t}\varphi(t)\,dt = \operatorname{Var}(\xi)$. Thus, D_{2n^2} means $\int e^{-t}\varphi(t)\,dt \le \frac{1}{2n^2}$ and implies $\varphi(1/n) \le \frac{e^{1/n}}{1/n}\cdot\frac{1}{2n^2} = \frac{e^{1/n}}{2n} \le \frac{1}{n}$ for $n \ge 2$, which proves 3a3(a).

On the other hand, E_{2n} means $\varphi(\frac{1}{2n}) \le \frac{1}{2n}$ and implies $\int e^{-t}\varphi(t)\,dt \le \frac{1}{2n} + \frac{1}{2n} = \frac{1}{n}$, which is D_n; 3a3(b) is thus proved.

14d Hints to exercises

14a3: this is a generalization of 14a2, and nevertheless, it is a special case of 14a2!

14a4: $\int f(x\cos\varphi + y\sin\varphi)y\,\gamma^1(dy) = \sin\varphi\int f'(x\cos\varphi + y\sin\varphi)\,\gamma^1(dy)$.

14a5: $e^{-t} = \cos\varphi$.

14a13: $\frac{1}{2}\iint |f(x)-f(y)|^2\,\mu(dx)\mu(dy) = \int f^2\,d\mu - \big(\int f\,d\mu\big)^2$.

14b1: this is, again, a generalization of 14a2, and nevertheless, a special case of 14a2!

14b2: apply 14b1 to $f(x,y,u,v) = g(x,u)$.

14c2: $\int_0^\infty = \int_0^x + \int_x^\infty$.