7 Against the stream: the Freidlin-Wentzell theory

7a Constant drift by spatial bias

We return to the rescaled random walk treated in 5a12–5a14, $X_n\big(\frac{k}{n}\big) = \frac{\beta_1+\dots+\beta_k}{n}$, $\mathbb{P}\big(\beta_k = 0\big) = \mathbb{P}\big(\beta_k = 1\big) = \frac12$; here $k = 0,1,2,\dots$ (not just $0,1,\dots,n$). The corresponding LDP follows from (5b7), like (5a11) from Theorem 5a9; the rate function is $w \mapsto \int_0^\infty I_{0.5}(w'(t))\,\mathrm{d}t$.

We modify the random walk as follows:

$$X_n\Big(\frac{k}{n}\Big) = \frac{s_1+\dots+s_k}{n} + v\frac{k}{n}\,, \quad \mathbb{P}\big(s_k=-1\big) = \mathbb{P}\big(s_k=+1\big) = \frac12\,;$$

here $v \in (0,1)$ is a parameter (and s_k are independent, of course). Now X_n is not monotone, but tends to increase with the given speed v (the drift). It satisfies LDP with the rate function

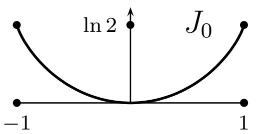
(7a1) $$J(w) = \int_0^\infty J_0(w'(t)-v)\,\mathrm{d}t\,.$$

Here w runs over all functions $[0,\infty) \to \mathbb{R}$ satisfying

$$(v-1)(t-s) \le w(t)-w(s) \le (v+1)(t-s) \text{ for } 0 \le s \le t < \infty\,, \text{ and } w(0)=0$$

(recall (5a10)), with the topology of uniform convergence on compacta (say, $\operatorname{dist}(w_1,w_2) = \max_{t\in[0,\infty)} \frac{1}{t^2+1}|w_1(t)-w_2(t)|$); and

$$J_0(x) = I_{0.5}\Big(\frac{1+x}{2}\Big) =$$

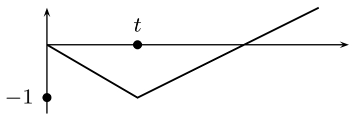


$$= \frac12(1-x)\ln(1-x) + \frac12(1+x)\ln(1+x) \quad \text{for } x\in[-1,1]\,.$$

We consider the (rare) event

$$A_n : \quad \exists k \quad X_n\Big(\frac{k}{n}\Big) \le -1 \, .$$

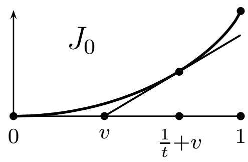
Similarly to 5a13 we guess the extremal function $w(\cdot)$,



$$w'(s) = \begin{cases} -1/t & \text{for } s \in (0,t), \\ v & \text{for } s \in (t,\infty), \end{cases}$$

but we have to find the optimal t by minimizing

$$J(w) = tJ_0\Big(-\frac{1}{t} - v \Big) = tJ_0\Big(\frac{1}{t} + v \Big) \, .$$



The minimum exists and is unique due to strict convexity of J_0.

Conditionally, given A_n, the (random) time of the first hit of $(-\infty,-1]$ is close to t with high probability. Moreover, all k/n such that $X_n\big(\frac{k}{n}\big) \le -1$ are close to t (with high probability).

For small v we get $1/t \approx v$, since $J_0(\varepsilon) \approx \varepsilon^2/2$ for small ε.

7a2 Exercise. Prove all said above.

You see, the stronger the stream, the faster one should move against it!

7b Constant drift by biased probabilities

An unfair coin leads to another random walk with drift,

$$X_n\Big(\frac{k}{n}\Big) = \frac{s_1 + \dots + s_k}{n} \, , \quad \mathbb{P}\big(s_k = -1\big) = 1-p \, , \quad \mathbb{P}\big(s_k = +1\big) = p \, ;$$

here $p \in (0,1)$ is a parameter. Now X_n tends to move with the speed $v = \mathbb{E}\, s_k = 2p-1 \in (-1,1)$ (the drift). It satisfies LDP with the rate function

$$J(w) = \int_0^\infty J_v(w'(t)) \, \mathrm{d}t \, .$$

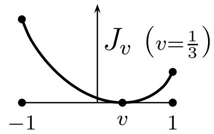
Here w runs over all functions $[0,\infty) \to \mathbb{R}$ satisfying

(7b1) $\quad |w(t) - w(s)| \le t - s \quad$ whenever $0 \le s \le t < \infty \, , \quad$ and $w(0) = 0$

(with the topology uniform on compacta); and (letting $p = (1+v)/2$)

$$J_v(x) = I_p\Big(\frac{1+x}{2}\Big) = \frac{1-x}{2}\ln\frac{1-x}{2(1-p)} + \frac{1+x}{2}\ln\frac{1+x}{2p} =$$

$$= J_0(x) - \frac{x}{2}\ln\frac{1+v}{1-v} - \frac12\ln(1-v^2) \quad \text{for } x\in[-1,1]\,.$$

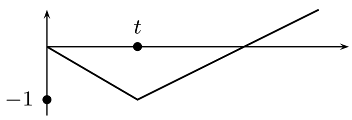


7b2 Exercise. Prove the LDP above.
Hint: Theorem 5a9 and 2c1; recall also the end of 3a.

Let $p\in(0.5,1)$, then $v>0$, and the event

$$A_n: \quad \exists k \quad X_n\Big(\frac{k}{n}\Big) \le -1$$

is rare. Still, the extremal function $w(\cdot)$ is of the form

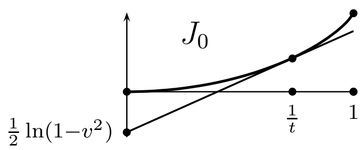


$$w'(s) = \begin{cases} -1/t & \text{for } s\in(0,t),\\ v & \text{for } s\in(t,\infty),\end{cases}$$

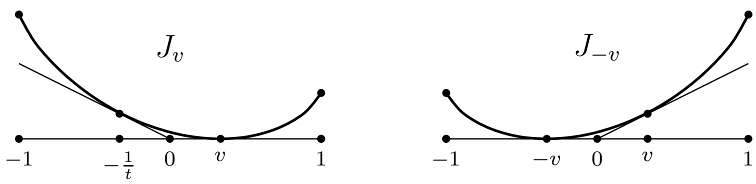
but this time the optimal t minimizes

$$J(w) = tJ_v\Big(-\frac1t\Big) = \tag{7b3}$$

$$= tJ_0\Big(\frac1t\Big) - \frac{t}{2}\ln(1-v^2) + \frac12\ln\frac{1+v}{1-v}\,.$$



As before, the minimum exists and is unique due to strict convexity of J_0. But now we are able to calculate it explicitly: $t = 1/v$ is optimal!



Indeed, $J_v(-x) = J_{-v}(x) = J_v(x) + \text{const}\cdot x$.

Conditionally, given A_n, the (random) time of the first hit of $(-\infty,-1]$ is close to $1/v$ with high probability. Moreover, all k/n such that $X_n\big(\frac{k}{n}\big) \le -1$ are close to $1/v$ (with high probability).

7b4 Exercise. Prove all said above.

You see, against the stream v one should move with the speed $(-v)$!

7c Variable drift by biased probabilities

Let $v : \mathbb{R} \to (-1,1)$ be a continuous function. We define processes X_n by

$$\text{(7c1)}\qquad \mathbb{P}\Big(X_n\Big(\frac{k+1}{n}\Big) = X_n\Big(\frac{k}{n}\Big) \pm \frac{1}{n} \Big) = \frac{1}{2}\Big(1 \pm v\Big(X_n\Big(\frac{k}{n}\Big)\Big)\Big) \,.$$

Such process tends to move with the variable speed $v(X_n(\cdot))$ (the drift). It satisfies LDP with the rate function

(7c2)

$$J(w) = \int_0^\infty \Big(\underbrace{J_0(w'(t)) - \frac{w'(t)}{2} \ln \frac{1+v(w(t))}{1-v(w(t))} - \frac{1}{2}\ln\big(1-v^2(w(t))\big)}_{J_{v(w(t))}(w'(t))} \Big) \, \mathrm{d}t \,.$$

Here w runs over all functions $[0,\infty) \to \mathbb{R}$ satisfying (7b1) (with the topology uniform on compacta).

In order to prove this LDP we generalize Theorem 2c1 as follows.

7c3 Theorem. Let $(\mu_n)_n$, $(\nu_n)_n$ be two sequences of probability measures on a compact metrizable space K, satisfying

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n} = c_n \mathrm{e}^{-nh_n} \quad \text{for all } n$$

for some $c_1, c_2, \dots \in (0,\infty)$ and a convergent sequence $(h_n)_n$, $h_n \in C(K)$; thus, $\|h_n - h\|_{C(K)} \to 0$ for some $h \in C(K)$.

(a) If $(\mu_n)_n$ is LD-convergent then $(\nu_n)_n$ is LD-convergent.

(b) If $(\mu_n)_n$ satisfies LDP with a rate function I, then $(\nu_n)_n$ satisfies LDP with the rate function

$$J = (I+h) - \min_K (I+h) = I + h - \lim_{n\to\infty} \frac{1}{n} \ln c_n \,.$$

7c4 Exercise. Prove Theorem 7c3.

Hint: recall 2c2 and note that

$$\min_K \frac{\mathrm{e}^{-nh_n}}{\mathrm{e}^{-nh}} \le \frac{\int |f|^n \mathrm{e}^{-nh_n} \, \mathrm{d}\mu_n}{\int |f|^n \mathrm{e}^{-nh} \, \mathrm{d}\mu_n} \le \max_K \frac{\mathrm{e}^{-nh_n}}{\mathrm{e}^{-nh}} \,.$$

Let us prove (7c2) for a finite time interval, namely, $k \le n$ and $t \le 1$; generalization to a longer time interval is straightforward, and LDP for intinite time follows via Theorem 5b2 as before.

We compare the distribution ν_n of the process X_n given by (7c1) with the distribution μ_n for $v = 0$. Denoting for convenience $w\big(\frac{k}{n}\big)$ by $w_{n,k}$ and

$n\big(w\big(\frac{k+1}{n}\big)-w\big(\frac{k}{n}\big)\big)$ by $w'_{n,k}$ we have

$$\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}(w)=\prod_{k=0}^{n-1}\big(1+w'_{n,k}v(w_{n,k})\big)\,;$$

$$\frac1n\ln\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}(w)=\frac1n\sum_{k=0}^{n-1}\ln\big(1+w'_{n,k}v(w_{n,k})\big)=$$

$$=\frac{1}{2n}\sum_{k=0}^{n-1}w'_{n,k}\ln\frac{1+v(w_{n,k})}{1-v(w_{n,k})}+\frac{1}{2n}\sum_{k=0}^{n-1}\ln(1-v^2(w_{n,k}))$$

(when checking it do not forget that $w'_{n,k}=\pm1$). We define h_n,h by

$$h_n(w)=\frac{1}{2n}\sum_{k=0}^{n-1}w'_{n,k}\ln\frac{1+v(w_{n,k})}{1-v(w_{n,k})}\,,$$

$$h(w)=\frac12\int_0^1 w'(t)\ln\frac{1+v(w(t))}{1-v(w(t))}\,\mathrm{d}t\,.$$

Functions $w:[0,1]\to\mathbb{R}$ of the given class may be described by

$$w(t)=\int_0^t\varphi(s)\,\mathrm{d}s\,,\quad \varphi\in L_\infty(0,1)\,,\quad \|\varphi\|_{L_\infty}\le1\,;$$

the given topology (uniform on compacta) on the functions w corresponds to the weak* topology on the functions φ, recall (5a3)–(5a5). (No matter that now $-1\le\varphi(\cdot)\le1$ rather than $0\le\varphi(\cdot)\le1$.)

The functions h_n are evidently continuous. It is less evident that the function h is continuous. On one hand, $w\mapsto\int w'\eta$ is continuous for every $\eta\in L_1(0,1)$, and the function $\eta_w(\cdot)=\ln\frac{1+v(w(\cdot))}{1-v(w(\cdot))}$ belongs to $L_1(0,1)$ (moreover, to $C[0,1]$). On the other hand, $\int w'\eta_w$ is more dangerous than just $\int w'\eta$.

Let $w_n\to w$ (uniformly) as $n\to\infty$, then $\eta_{w_n}\to\eta_w$ in $L_1(0,1)$ (and moreover, in $C[0,1]$). We have, first, $\int w'_n\eta_w\to\int w'\eta_w$ and second, $|\int w'_n\eta_{w_n}-\int w'_n\eta_w|\le\|w'_n\|_\infty\|\eta_{w_n}-\eta_w\|_1\to0$. Therefore $\int w'_n\eta_{w_n}\to\int w'\eta_w$ as $n\to\infty$, whenever $w_n\to w$. It means that h is continuous.

7c5 Exercise. Prove that $h_n(w)\to h(w)$ as $n\to\infty$, for every w of the given class.

Hint: $h(w)=\int w'\eta_w$ and $h_n(w)=\int w'\eta_{w,n}$, where $\eta_{w,n}(t)=\eta_w\big(\frac kn\big)$ for $t\in\big(\frac kn,\frac{k+1}{n}\big)$.

7c6 Exercise. Prove that the convergence in 7c5 is uniform in w.

Hint: the function $\ln\frac{1+v(\cdot)}{1-v(\cdot)}$ is uniformly continuous on $[-1,1]$, therefore all $\eta_w(\cdot)$ are equicontinuous.

(Of course, continuity of h can be deduced from 7c6.)

7c7 Exercise. Prove the LDP with the rate function (7c2) for the process (7c1).

Hint: use 7c3 and 7c6, and do not forget the last term $\frac{1}{2n}\sum \ln(\dots)$ in $\frac{1}{n}\ln\frac{\mathrm{d}\nu_n}{\mathrm{d}\mu_n}$.

Assuming $v(x) > 0$ for all $x \in [-1,0]$ we consider the rare event

$$A_n : \quad \exists k \quad X_n\Big(\frac{k}{n}\Big) \le -1 \, .$$

How to find the extremal function $w(\cdot)$? We guess that it is a smooth monotone function on $[0,t]$, $w(t) = -1$, $w'(\cdot) < 0$ on $[0,t]$; and, as before, $[t,\infty)$ does not contribute to $J(w)$. Assuming all that, we introduce the function $\tau : [-1,0] \to [0,t]$ inverse to $w|_{[0,t]}$ and note that

$$\begin{aligned} J(w) &= \int_0^t \Big(J_0(w'(s)) - \frac{w'(s)}{2} \ln\frac{1+v(w(s))}{1-v(w(s))} - \frac12 \ln\big(1 - v^2(w(s))\big) \Big) \, \mathrm{d}s = \\ &= \int_{-1}^0 \Big(-\tau'(x) J_0\Big(\frac{1}{\tau'(x)}\Big) + \frac12 \ln\frac{1+v(x)}{1-v(x)} + \frac12 \tau'(x) \ln\big(1 - v^2(x)\big) \Big) \, \mathrm{d}x \, . \end{aligned}$$

We have to minimize it in $\tau(\cdot)$. This task boils down to minimization for each x separately, in the variable $(-\tau'(x))$. We get (7b3) with $(-\tau'(x))$ substituted for t. The minimum is reached at $-\tau'(x) = 1/v(x)$, which means

$$w'(s) = -v(w(s)) \quad \text{for } s \in [0,t] \, ,$$

just the opposite to the drift![1]

7d Extremal functions as mechanical trajectories

Minimization of the functional $J(w)$ (defined by (7c2)) over smooth functions w is a usual task for the calculus of variations. If w minimizes J among all smooth functions satisfying boundary conditions $w(0) = 0$, $w(t) = a$ (for given t and a), then necessarily

$$\text{(7d1)} \qquad \frac{\mathrm{d}}{\mathrm{d}\varepsilon}\Big|_{\varepsilon=0} J(w + \varepsilon w_1) = 0$$

[1] It can be proved that this function is indeed the unique global minimizer of J on the set of all functions $w : [0,\infty) \to \mathbb{R}$ satisfying (7b1) and such that $\exists t \;\; w(t) = -1$.

for all smooth w_1 satisfying $w_1(0) = 0$, $w_1(t) = 0$. For J of the form

(7d2) $$J(w) = \int_0^t L\big(w(s), w'(s)\big)\,\mathrm{d}s$$

the necessary condition becomes the Euler-Lagrange equation (see also [1, Appendix G])

(7d3) $$\frac{\mathrm{d}}{\mathrm{d}s} L_{,2}\big(w(s), w'(s)\big) = L_{,1}\big(w(s), w'(s)\big)$$

where $L_{,1}$ and $L_{,2}$ are the partial derivatives of L (in the first and the second argument, respectively).

In our case

(7d4) $$L(q, r) = J_0(r) - \frac{1}{2} \ln\big(1 - v^2(q)\big)\,;$$

the middle term in (the integrand of) (7c2) is omitted, since

$$\int_0^t \frac{w'(s)}{2} \ln \frac{1 + v(w(s))}{1 - v(w(s))}\,\mathrm{d}s = \int_0^a \frac{1}{2} \ln \frac{1 + v(q)}{1 - v(q)}\,\mathrm{d}q$$

does not depend on w. We have

$$L_{,1}(q, r) = \frac{v(q)v'(q)}{1 - v^2(q)}\,, \quad L_{,2}(q, r) = J_0'(r) = \frac{1}{2} \ln \frac{1 + r}{1 - r}\,;$$

thus, the Euler-Lagrange equation becomes

(7d5) $$\underbrace{\frac{\mathrm{d}}{\mathrm{d}s} \frac{1}{2} \ln \frac{1 + w'(s)}{1 - w'(s)}}_{\frac{w''(s)}{1 - w'^2(s)}} = \frac{v(w(s))v'(w(s))}{1 - v^2(w(s))}\,,$$

a nonlinear ordinary differential equation of second order.

A paradox: (7d5) is intact if v is replaced with $(-v)$. The direction of the drift does not matter! Well, it does not matter as long as the boundary conditions are fixed ($w(0) = 0$, $w(t) = a$).

Here is a nice mechanical interpretation. Imagine a massive particle with a coordinate $q = q(s)$, speed $\dot q = \frac{\mathrm{d}q}{\mathrm{d}s}$ and Lagrangian $L = J_0(\dot q) - \frac{1}{2} \ln(1 - v^2(q))$. Its action (on the time interval $[0, t]$) is $\int_0^t L\,\mathrm{d}s$, and the least action principle leads just to the minimization problem treated above. The momentum is $p = \frac{\partial L}{\partial \dot q} = J_0'(\dot q) = \frac{1}{2} \ln \frac{1 + \dot q}{1 - \dot q}$, the force is $F = \frac{\partial L}{\partial q} = \frac{v(q)v'(q)}{1 - v^2(q)}$, and the Euler-Lagrange equation turns into the motion equation $\dot p = F$, that is,

$$\frac{1}{1 - \dot q^2} \ddot q = \frac{v(q)v'(q)}{1 - v^2(q)}\,,$$

the same as (7d5).

Mechanics tells us also, that the potential energy is $U = \frac{1}{2}\ln(1-v^2(q))$ (its gradient being the force F), and the kinetic energy is $T = p\dot q - L = -\frac{1}{2}\ln(1-\dot q^2)$ (check it). The total energy $T + U = \frac{1}{2}\ln\frac{1-v^2(q)}{1-\dot q^2}$ is constant along each trajectory. It means that

$$\frac{1-v^2(w(s))}{1-w'^2(s)} = \text{const}$$

for every extremal function.

Especially, if $\dot q$ is small then the kinetic energy $T \approx \frac{1}{2}\dot q^2$ (as usual…). Also, if the function $v(\cdot)$ is small then the potential energy $U \approx -\frac{1}{2}v^2(q)$. Accordingly, the motion equation becomes $\ddot q \approx v(q)v'(q)$, and the energy conservation law becomes $\dot q^2 - v^2(q) \approx \text{const}$.

In other words, if $v(\cdot)$ and $w'(\cdot)$ are small then (7d5) becomes $w''(s) \approx v(w(s))v'(w(s))$; also, $w'^2(s) - v^2(w(s)) \approx \text{const}$.

7d6 Example. Let $v(q) = 1 + (q-0.5)^2$ for $q \in [0,1]$. We consider the (rare) event

$$A_n: \quad \forall k \le 10n \quad 0 \le X_n\Big(\frac{k}{n}\Big) \le 1\,.$$

The boundary conditions are $w(0) = 0$, $w(10) = 1$. We guess that the extremal function $w(\cdot)$ increases from 0 to (nearly) 0.5 in a time < 1; then it remains near 0.5 till a time > 9; afterwards it increases and arrives to 1 at 10.

The mechanical interpretation refines the guess. The potential energy is maximal at 0.5. The particle starts from 0 with a kinetic energy higher than the potential wall, but only a little. After climbing to the top of the potential wall, the particle slows down (almost stops). After a time it falls down on the other side of the wall.

A wonder: under some condition, the (inertialess) Markov process behaves like a mechanical particle endowed with inertia!

7e Variable drift by spatial bias

Let $v: \mathbb{R} \to (-1,1)$ be a continuous function. We define processes X_n by

$$\begin{gathered} X_n(0) = 0\,, \\ X_n\Big(\frac{k+1}{n}\Big) = X_n\Big(\frac{k}{n}\Big) + \frac{1}{n}s_{k+1} + \frac{1}{n}v\Big(X_n\Big(\frac{k}{n}\Big)\Big)\,; \end{gathered} \tag{7e1}$$

as in 7a, s_k are independent equiprobable ± 1. Such process tends to move with the variable speed $v(X_n(\cdot))$ (the drift). By analogy with (7a1) it is easy to guess that $(X_n)_n$ satisfies LDP with the rate function

$$J(w) = \int_0^\infty J_0\big(w'(t) - v(w(t)\big)\,\mathrm{d}t\,. \tag{7e2}$$

How to prove it? And what is the set of functions w?

We consider the corresponding driftless processes

$$Y_n\Big(\frac{k}{n}\Big) = \frac{s_1 + \dots + s_k}{n}$$

and note that

$$\frac{Y_n\big(\frac{k+1}{n}\big) - Y_n\big(\frac{k}{n}\big)}{\frac1n} = \frac{X_n\big(\frac{k+1}{n}\big) - X_n\big(\frac{k}{n}\big)}{\frac1n} - v\bigg(X_n\Big(\frac{k}{n}\Big)\bigg)\,;$$

$$Y_n\Big(\frac{k}{n}\Big) = X_n\Big(\frac{k}{n}\Big) - \frac1n\sum_{j=0}^{k-1} v\bigg(X_n\Big(\frac{j}{n}\Big)\bigg)\,. \tag{7e3}$$

We define a map F of the space of continuous functions on $[0,\infty)$ (vanishing at 0) to itself,

$$F(w)(t) = w(t) - \int_0^t v(w(s))\,\mathrm{d}s\,. \qquad (w(0)=0) \tag{7e4}$$

In general, F is not one-to-one (a counterexample: $v(x) = \sqrt{|x|}$, $w_1(t) = t^2/4$, $w_2(t) = 0$). However, we assume that v is a Lipschitz function, that is,

$$|v(x) - v(y)| \le C_v|x-y| \quad \text{for all } x,y \tag{7e5}$$

and some constant $C_v < \infty$.[1]

7e6 Exercise. The map F is one-to-one.

Prove it.

Hint: if $F(w_1) = F(w_2)$ then $|w_1(t) - w_2(t)| \le C_v \int_0^t |w_1(s) - w_2(s)|\,\mathrm{d}s$; show that $\mathrm{e}^{-C_v t}|w_1(t) - w_2(t)|$ is decreasing in t.

For any $C \in (0,\infty)$ we introduce the compact metrizable space $\mathrm{Lip}(C)$ of all $w : [0,\infty) \to \mathbb{R}$ such that

$$w(0) = 0\,, \quad |w(s) - w(t)| \le C|s-t| \quad \text{for all } s,t \in \mathbb{R}\,,$$

[1] Also 'locally Lipschitz' would be enough; that is, Lipschitz on every bounded interval.

endowed with the topology uniform on bounded time intervals. Note that

$$X_n(\cdot) \in \mathrm{Lip}(2)\,; \quad Y_n(\cdot) \in \mathrm{Lip}(1)\,;$$
$$w \in \mathrm{Lip}(C) \quad \Longrightarrow \quad F(w) \in \mathrm{Lip}(C+1)\,.$$

It follows from 7e6 that the restriction

$$F|_{\mathrm{Lip}(C)} : \mathrm{Lip}(C) \to F(\mathrm{Lip}(C)) \quad \text{is a homeomorphism}\,.$$

7e7 Lemma.

$$\max_{[0,t]} |F(X_n(\cdot)) - Y_n(\cdot)| \le \frac{2C_v t + 4}{n}$$

for all $t > 0$, n and $s_1, \dots, s_n$.

Proof. Using (7e3), (7e4) and (7e5),

$$\Big| F\big(X_n(\cdot)\big)\Big(\frac{k}{n}\Big) - Y_n\Big(\frac{k}{n}\Big) \Big| = \bigg| \int_0^{k/n} v\big(X_n(s)\big)\,\mathrm{d}s - \frac1n \sum_{j=0}^{k-1} v\Big(X_n\Big(\frac{j}{n}\Big)\Big) \bigg| \le$$
$$\le \frac{k}{n} \max_j \bigg| n \int_{j/n}^{(j+1)/n} v\big(X_n(s)\big)\,\mathrm{d}s - v\Big(X_n\Big(\frac{j}{n}\Big)\Big) \bigg| \le \frac{k}{n} C_v \cdot 2\,,$$

since $X_n(\cdot) \in \mathrm{Lip}(2)$. It remains to note that $F(X_n(\cdot)) - Y_n(\cdot) \in \mathrm{Lip}(4)$. $\square$

It follows that the distribution of $F(X_n(\cdot))$ and the distribution of $Y_n(\cdot)$ come together (recall 5d) as measures on Lip(3). By Prop. 5d1, LDP for $(Y_n)_n$ implies LDP for $(F(X_n))_n$ with the same rate function.

We know that the rate function for $(Y_n)_n$ is $w \mapsto \int_0^\infty J_0(w'(t))\,\mathrm{d}t$, but this happens on Lip(1). What happens on Lip(3)? We just extend the rate function from Lip(1) to Lip(3) by $+\infty$ on $\mathrm{Lip}(3) \setminus \mathrm{Lip}(1)$.

7e8 Exercise. Formulate and prove the corresponding general result.

We may still write $\int_0^\infty J_0(w'(t))\,\mathrm{d}t$, provided that J_0 is extended from $[-1,1]$ to $\mathbb{R}$ by $+\infty$ on $\mathbb{R} \setminus [-1,1]$.

Having the rate function for $(F(X_n))_n$ we get the rate function for $(X_n)_n$ via the homeomorphism: $J(w) = \int_0^\infty J_0\big((F(w))'(t)\big)\,\mathrm{d}t$.

7e9 Exercise. Formulate and prove the corresponding general result.

Hint: if you like, treat it as a trivial case of the contraction principle.

We got (7e2), with two remarks. First, w rans over Lip(2). Second, $J_0(x) = +\infty$ for $x \in \mathbb{R} \setminus [-1,1]$.

See also [3, Sect. 6], [2, Sect. 5.6] and [1, Sect. 4C].

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[3] P. Dupuis, R.S. Ellis, *A weak convergence approach to the theory of large deviations,* Wiley, 1997.