1 Basic ideas

1a From Cantor's uncountability theorem to Baire category theorem

By the famous Cantor's uncountability theorem, $\mathbb{R}$ is not countable. Here is one of the proofs. Let $a_1, a_2, \dots \in \mathbb{R}$; we need $x \in \mathbb{R}$ such that $\forall n \; x \ne a_n$. To this end we first take $b_1 < c_1$ such that $a_1 \notin [b_1, c_1]$. Then we take $b_2 < c_2$ such that $[b_2, c_2] \subset [b_1, c_1]$ and $a_2 \notin [b_2, c_2]$. And so on; $[b_1, c_1] \supset [b_2, c_2] \supset [b_3, c_3] \supset \dots$ Their intersection is not empty, and contains no a_n.

Can we generalize it to some sets $A_1, A_2, \dots \subset \mathbb{R}$ proving that $\cup_n A_n \ne \mathbb{R}$? Yes, provided that these sets satisfy the following.

1a1 Definition. A set $A \subset \mathbb{R}$ is *nowhere dense* if every nonempty open interval contains some nonempty open subinterval that does not intersect A.

1a2 Exercise. A set $A \subset \mathbb{R}$ is nowhere dense if and only if $\operatorname{Int}(\operatorname{Cl}(A)) = \emptyset$. Prove it. (Here "Int" stands for interior, and "Cl" for closure.)

1a3 Theorem (Baire)**.** If $A_1, A_2, \dots \subset \mathbb{R}$ are nowhere dense then $\operatorname{Int}(\cup_n A_n) = \emptyset$.

1a4 Exercise. Prove the theorem.

Equivalently: $\mathbb{R} \setminus \cup_n A_n$ is dense; that is, $\operatorname{Cl}(\mathbb{R} \setminus \cup_n A_n) = \mathbb{R}$.

In particular, $\cup_n A_n \ne \mathbb{R}$.

Clearly, a singleton is nowhere dense; therefore Cantor's uncountability theorem follows from Baire category theorem.

1b From Cantor's uncountability theorem to null sets

Here is another proof of Cantor's uncountability theorem. Let $a_1, a_2, \dots \in \mathbb{R}$; we need $x \in \mathbb{R}$ such that $\forall n \; x \ne a_n$. To this end we take $\varepsilon_1, \varepsilon_2, \dots > 0$ such that $\sum_n \varepsilon_n < 1/2$ and consider open intervals $(a_n - \varepsilon_n, a_n + \varepsilon_n)$. A finite number of these intervals cannot cover $[0, 1]$ since their total length is less than 1. (Take the Riemann integral of the sum of indicators...) By the Heine-Borel theorem, the infinite sequence of these intervals still does not cover $[0, 1]$.

1b1 Definition. A set $A \subset \mathbb{R}$ is a *null set* if for every $\varepsilon > 0$ there exist $\varepsilon_1, \varepsilon_2, \dots > 0$ and $a_1, a_2, \dots \in \mathbb{R}$ such that $A \subset \cup_n (a_n - \varepsilon_n, a_n + \varepsilon_n)$ and $2 \sum_n \varepsilon_n \le \varepsilon$.

1b2 Theorem. If $A_1, A_2, \dots \subset \mathbb{R}$ are null sets then $\operatorname{Int}(\cup_n A_n) = \emptyset$.

1b3 Exercise. (a) Prove that $\cup_n A_n$ is also a null set.
(b) Prove the theorem.

1c Two approaches to small sets and typical objects

1c1 Definition. Given a set X, a set $\mathcal{N}$ of subsets of X is called
(a) an *ideal*[1] (on X), if

$$(A \subset B \wedge B \in \mathcal{N}) \implies A \in \mathcal{N}\,;$$
$$A, B \in \mathcal{N} \implies A \cup B \in \mathcal{N}\,;$$
$$\emptyset \in \mathcal{N}\,.$$

(b) a *σ-ideal* (on X), if it is an ideal and

$$A_1, A_2, \dots \in \mathcal{N} \implies \cup_n A_n \in \mathcal{N}\,.$$

An ideal (or σ-ideal) $\mathcal{N}$ on X is *proper* if $X \notin \mathcal{N}$.

Clearly, null sets are a proper σ-ideal on $\mathbb{R}$.
The complement of a null set is called a set of *full measure.*

1c2 Definition. A set $A \subset \mathbb{R}$ is *meager*[2] if $A \subset \cup_n A_n$ for some nowhere dense sets $A_1, A_2, \dots \subset \mathbb{R}$.

[1] This notion of set theory is different from (but related to) ideals in ring theory, order theory etc.
[2] Or *of the first category.*

Clearly, meager sets are a proper σ-ideal on $\mathbb{R}$.

The complement of a meager set is called *comeager.*[1]

When a property holds off a null set (in other words, on a set of full measure), one says that it holds *almost everywhere* or *for almost all* elements. Dealing with a probability measure one also says *almost sure(ly).*

When a property holds off a meager set (in other words, on a comeager set), one says that it holds *quasi-everywhere* or *for quasi all* elements.[2] One also says that this property holds *generically,* for a *generic* element, or for *most* of elements. Sometimes the word "typical" is used rather than "generic".

1d Compact metrizable spaces; sequence spaces

1d1 Definition. (a) A *metric space* is a pair (X, ρ) of a set X and a *metric* ρ on X, that is, a function $\rho : X \times X \to [0, \infty)$ such that $\rho(x, y) = 0 \iff x = y$, $\rho(x, y) = \rho(y, x)$, $\rho(x, z) \le \rho(x, y) + \rho(y, z)$ for all $x, y, z \in X$.

(b) Let ρ_1, ρ_2 be two metrics on X; ρ_2 is *stronger* than ρ_1 if $\rho_2(x_n, x) \to 0 \implies \rho_1(x_n, x) \to 0$ for all $x, x_1, x_2, \dots \in X$;[3] further, ρ_1, ρ_2 are *equivalent,* if $\rho_1(x_n, x) \to 0 \iff \rho_2(x_n, x) \to 0$ for all $x, x_1, x_2, \dots \in X$.

(c) A *metrizable space*[4] is a pair (X, R) where X is a set and R is an equivalence class of metrics on X (*metrizable topology*; metrics of R are called *compatible*).

(d) A metrizable space (as well as its metrizable topology) is *compact*[5] if every sequence has a convergent subsequence.

Every subset of $\mathbb{R}$ is a metric space with the metric $\rho(x, y) = |x - y|$. This space is compact if and only if the set is closed and bounded.

The Cantor set $C \subset [0, 1]$ may be defined as consisting of all numbers of the form

$$\varphi(x) = \sum_{k=1}^{\infty} \frac{2x(k)}{3^k}$$

for $x \in \{0, 1\}^\infty$, that is $x : \{1, 2, \dots\} \to \{0, 1\}$.

1d2 Exercise. (a) $\varphi : \{0, 1\}^\infty \to C$ is a bijection;

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[1] Or *residual.*

[2] However, "quasi" is also used in potential theory (in relation to capacity).

[3] However, a Cauchy sequence in (X, ρ_2) need not be Cauchy in (X, ρ_1).

[4] Equivalently, and usually, a metrizable space is defined as a special case of a topological space; but here we do not need the notion of general (not just metrizable) topological space.

[5] Equivalently (for metrizable spaces), and usually, a compact space is defined by the Heine-Borel property: every open cover has a finite subcover.

(b) if $x, x_1, x_2, \dots \in \{0,1\}^\infty$ then

$$\varphi(x_n) \xrightarrow[n\to\infty]{} \varphi(x) \quad \Longleftrightarrow \quad \forall k \, \Big(x_n(k) \xrightarrow[n\to\infty]{} x(k) \Big) \, .$$

Prove it.

The metric $\rho(x,y) = |\varphi(x) - \varphi(y)|$ is not invariant under permutations of coordinates on $\{0,1\}^\infty$, but its equivalence class R is (see 1d2(b)). Thus, we have a compact metrizable space $\{0,1\}^\infty$, and moreover, the compact metrizable space $\{0,1\}^S$ is well-defined for an arbitrary countable set S (irrespective of its enumeration). The space $\{0,1\}^S$ may also be thought of as the space of all subsets of S.

1d3 Definition. A set A in a metrizable space X is *nowhere dense* if every nonempty open set contains some nonempty open subset that does not intersect A.

Still, A is nowhere dense if and only if $\operatorname{Int}(\operatorname{Cl}(A)) = \emptyset$.

1d4 Exercise. (a) Prove that nowhere dense sets are an ideal (on a metrizable space).

(b) On $\mathbb{R}$, prove that they are not a σ-ideal.

1d5 Exercise. A set $A \subset \{0,1\}^\infty$ is nowhere dense if and only if for all m and $t_1, \dots, t_m \in \{0,1\}$ there exist $n > m$ and $t_{m+1}, \dots, t_n \in \{0,1\}$ such that all sequences that start with $t_1, \dots, t_n$ do not belong to A.

Prove it.

1d6 Theorem (Baire)**.** Let X be a compact metrizable space. If $A_1, A_2, \dots \subset X$ are nowhere dense then $\operatorname{Int}(\cup_n A_n) = \emptyset$.

1d7 Exercise. (a) Prove the theorem.

(b) Find an example of a non-compact metrizable space such that the σ-ideal of meager sets is not proper.

Thus, the proper σ-ideal of meager sets is well-defined on every compact metrizable space, in particular, on $\{0,1\}^\infty$, and we may speak about generic elements, quasi-everywhere etc. Now, what about null sets? Can we transfer Lebesgue measure from $\mathbb{R}$ to $\{0,1\}^\infty$ by φ^{-1}? No, we cannot, since the Cantor set is itself a null set. But on the other hand, endless coin tossing should provide a useful probability measure on $\{0,1\}^\infty$; and binary digits can be thought of as endless coin tossing over Lebesgue measure!

We consider the map $\psi : [0,1) \to \{0,1\}^\infty$,

$$\psi(u) = \big(b_1(u), b_2(u), \dots \big) \, ,$$

where $b_k(u)$ are the binary digits of u, that is,

$$b_k(u) \in \{0,1\}\,, \quad \sum_{k=1}^\infty \frac{b_k(u)}{2^k} = u\,, \quad \liminf_k b_k(u) = 0\,.$$

True, ψ is not a bijection, but do not bother: the countable set $\{x : \liminf_k x(k) = 1\}$ is anyway a null set, and outside it ψ is a bijection,

$$\psi^{-1}(x) = \sum_{k=1}^\infty \frac{x(k)}{2^k}\,.$$

We transfer Lebesgue measure to $\{0,1\}^\infty$ by ψ. That is, a set $A \subset \{0,1\}^\infty$ is measurable if $\psi^{-1}(A)$ is Lebesgue measurable, and then $\mu(A)$ is equal to the Lebesgue measure of $\psi^{-1}(A)$. This probability measure μ is sometimes called Lebesgue measure on $\{0,1\}^\infty$.[1] It is invariant under permutations of coordinates on $\{0,1\}^\infty$. Thus, we have a probability space $\{0,1\}^\infty$, and moreover, the probability space $\{0,1\}^S$ is well-defined for an arbitrary countable set S (irrespective of its enumeration). It gives us the proper σ-ideal of null sets on such space, and we may speak about almost all elements etc.

1e "Almost all" versus "quasi all": first examples

1e1 Example. The famous strong law of large numbers (SLLN) states that

(a) $$\lim_n \frac{1}{n} \sum_{k=1}^n x(k) = \frac{1}{2} \quad \text{for almost all } x \in \{0,1\}^\infty\,.$$

In contrast,
(b)
$$\liminf_n \frac{1}{n} \sum_{k=1}^n x(k) = 0\,, \; \limsup_n \frac{1}{n} \sum_{k=1}^n x(k) = 1 \quad \text{for quasi all } x \in \{0,1\}^\infty\,,$$

as we will see soon.

1e2 Example. Consider sets

$$A_n = \big\{ x : x(1) = x(n+1), x(2) = x(n+2), \dots, x(n) = x(2n) \big\} \subset \{0,1\}^\infty\,.$$

Clearly, $\mu(A_n) = 2^{-n}$, thus $\sum_n \mu(A_n) < \infty$; by the first Borel-Cantelli lemma,

(a1) $$\mu\big(\limsup_n A_n\big) = 0\,.$$

[1] It is in fact the Haar measure on the topological group $(\mathbb{Z}_2)^\infty$.

In other words, almost every x belongs to A_n only for finitely many n. Equivalently,[1]

(a2) $$\sum_n \mathbb{1}_{A_n}(x) < \infty \quad \text{for almost all } x \in \{0,1\}^\infty$$

($\mathbb{1}_A$ being the indicator of A). In contrast,

(b) $$\sum_n \mathbb{1}_{A_n}(x) = \infty \quad \text{for quasi all } x \in \{0,1\}^\infty\,,$$

as we will see soon. That is, quasi every x belongs to A_n for infinitely many n. (Of course, the infinite set of n depends on x.)

1e3 Exercise. Denote by B_n the complement of A_n, and by C_n the set $B_n \cap B_{n+1} \cap \dots$ Prove that

(a) C_n is closed;

(b) C_n is nowhere dense.

Thus, $C = \cup_n C_n$ is meager, and its complement $\cap_n (A_n \cup A_{n+1} \cup \dots) = \limsup_n A_n$ is comeager, which proves 1e2(b).

1e4 Exercise. Now consider sets $A_n = \{x : x(n) = x(n+1) = \dots = x(n^2) = 0\}$. Prove that

(a) the set $\limsup_n A_n$ is comeager;

(b) $\liminf_n \frac{1}{n} \sum_{k=1}^n x(k) = 0$ for all $x \in \limsup_n A_n$.

A half of 1e1(b) is thus proved; the other half is similar.

1f Digits of a typical number

We return to the map $\psi : [0,1) \to \{0,1\}^\infty$, $\psi(u) = \big(b_1(u), b_2(u), \dots\big)$ where $b_k(u)$ are the binary digits of u. Of course, ψ is discontinuous; and nevertheless...

1f1 Exercise. Prove that

(a) If $A \subset \{0,1\}^\infty$ is nowhere dense then $\psi^{-1}(A) \subset [0,1)$ is nowhere dense.

(b) If $A \subset \{0,1\}^\infty$ is meager then $\psi^{-1}(A) \subset [0,1)$ is meager.

(c) If $A \subset \{0,1\}^\infty$ is comeager then $\psi^{-1}(A) \subset [0,1)$ is comeager.

[1] The sum of the indicators is integrable, therefore, finite almost everywhere. (This is the proof of the first Borel-Cantelli lemma.)

1f2 Exercise. Let $A \subset \{0,1\}^\infty$. Prove or disprove:
(a) If $\psi^{-1}(A)$ is nowhere dense then A is nowhere dense.
(b) If $\psi^{-1}(A)$ is meager then A is meager.

1f3 Remark. A map satisfying the equivalent conditions 1f1(b,c) (but not necessarily (a)) may be called *genericity preserving.*[1] Informally, such map transforms a generic element of the first space into a generic element of the second space.

Combining 1f1 with 1e1(b) and 1e2(b) we see that quasi all $u \in [0,1)$ satisfy

$$\liminf_n \frac{1}{n} \sum_{k=1}^n b_k(u) = 0 \, , \quad \limsup_n \frac{1}{n} \sum_{k=1}^n b_k(u) = 1 \, ,$$

and the relation

$$b_1(u) = b_{n+1}(u), \dots, b_n(u) = b_{2n}(u)$$

holds for infinitely many n.

All said about $\{0,1\}^\infty$ and binary digits generalizes readily to $\{0,1,\dots,9\}^\infty$ and decimal digits, as well as any other basis. Given comeager sets $A_p \subset \{0,\dots,p-1\}^\infty$, we observe for a generic number $u \in [0,1)$ the following property: for every basis $p = 2,3,\dots$ the corresponding digits of u are a sequence that belongs to A_p.

Hints to exercises

1a4: $[b_1,c_1] \supset [b_2,c_2] \supset \dots$

1d2: if $x(1) = y(1), \dots, x(n) = y(n)$ then $|\varphi(x) - \varphi(y)| \le \frac{2}{3^{n+1}} + \frac{2}{3^{n+2}} + \dots$; otherwise $|\varphi(x) - \varphi(y)| \ge \frac{2}{3^n} - \frac{2}{3^{n+1}} - \frac{2}{3^{n+2}} - \dots$

1d4: (a) $[b_1,c_1] \supset [b_2,c_2] \supset [b_3,c_3]$; (b) the union can be dense.

1d7: (a) similar to 1a4 with balls rather than intervals; (b) try a dense countable set.

1e3: (b) use 1d5.

1e4: (b) try $n \in \{1,4,9,16,\dots\}$

1f1: (a) by 1d5 every binary interval $[\frac{k}{2^n}, \frac{k+1}{2^n})$ contains a binary subinterval such that… (b), (c) follow from (a).

1f2: consider $\{0,1\}^\infty \setminus \psi([0,1))$.

[1] According to Melleray and Tsankov, a *continuous* map with this property is called *category-preserving;* see arXiv:1201.4447, Def. 2.7.

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