# 5 Many points of continuity



## 5a Discontinuous derivatives

Let me start this section with a quote from Wikipedia:[1]

> An example of a differentiable function whose derivative is not continuous (at $x = 0$) is the function equal to $x^2 \sin(1/x)$ when $x \neq 0$, and 0 when $x = 0$. An infinite sum of similar functions (scaled and displaced by rational numbers) can even give a differentiable function whose derivative is not continuous anywhere.

So, we consider the function $\alpha(x) = x^2 \sin \frac{1}{x}$, $\alpha(0) = 0$, note that $\alpha'(x) = 2x \sin \frac{1}{x} - \cos \frac{1}{x}$, $\alpha'(0) = 0$, and

$$\liminf_{x \to 0} \alpha'(x) = -1 \, , \quad \limsup_{x \to 0} \alpha'(x) = 1 \, .$$

We choose some $a_n, c_n \in \mathbb{R}$ such that $a_n$ are pairwise distinct (possibly dense) and $\sum_n (1 + |a_n|)|c_n| < \infty$. The series

$$\beta(x) = \sum_{n=1}^{\infty} c_n \alpha(x - a_n)$$

converges locally uniformly,[2] since $|\alpha(x)| \le |x|$. That is, $\beta_n = \sum_{k=1}^n c_k \alpha(\cdot - a_k) \to \beta$ locally uniformly. Clearly, $\beta'_n$ is discontinuous at $a_1, \dots, a_n$ and continuous at every other point. Does it mean that $\beta'$ is discontinuous at $a_1, a_2, \dots$ and continuous at every other point? But first, is $\beta$ differentiable?

[1] Article "Baire function", old version of 1 June, 2013.

[2] That is, uniformly on $[-M, M]$ for every $M > 0$.

Yes, $\beta$ is differentiable, and $\beta'(x) = \sum_{n=1}^\infty c_n \alpha'(x-a_n)$, since $\alpha'(x) \le 3$ for all $x$ (therefore $|\alpha(x)-\alpha(y)| \le 3|x-y|$) and, denoting $\gamma_n(x) = \beta(x)-\beta_n(x) = \sum_{k=n+1}^\infty c_k \alpha(x-a_k)$, we have

$$\frac{\beta(x+h)-\beta(x)}{h} = \frac{\beta_n(x+h)-\beta_n(x)}{h} + \frac{\gamma_n(x+h)-\gamma_n(x)}{h};$$

$$\underbrace{\beta'_n(x)}_{\xrightarrow[n]{}\sum_{k=1}^\infty c_k\alpha'(x-a_k)} - 3 \underbrace{\sum_{k=n+1}^\infty |c_k|}_{\xrightarrow[n]{}0} \le \liminf_{h\to0} \frac{\beta(x+h)-\beta(x)}{h} \le \\ \le \limsup_{h\to0} \frac{\beta(x+h)-\beta(x)}{h} \le \beta'_n(x) + 3\sum_{k=n+1}^\infty |c_k| \,.$$

In order to examine (dis)continuity of $\beta'$ we introduce *oscillation function* $\omega_f$ of an arbitrary locally bounded[1] $f : \mathbb{R} \to \mathbb{R}$ by

$$\omega_f(x) = \inf_{\delta>0} \underbrace{\sup_{s,t\in(x-\delta,x+\delta)} |f(s)-f(t)|}_{\operatorname{diam} f((x-\delta,x+\delta))} \,.$$

Clearly, $f$ is continuous at $x$ if and only if $\omega_f(x) = 0$.

**5a1 Exercise.** The set $\{x : \omega_f(x) < \varepsilon\}$ is open (for arbitrary $f$ and $\varepsilon$). Prove it.

The oscillation is a bit similar to a (semi)norm:

**5a2 Exercise.** Prove that
(a) $\omega_{cf} = |c|\omega_f$ for $c \in \mathbb{R}$, $f : \mathbb{R} \to \mathbb{R}$;
(b) $\omega_{f+g} \le \omega_f + \omega_g$;
(c) $|\omega_f - \omega_g| \le \omega_{f-g}$;
(d) $\sup_{(-M.M)} \omega_f \le 2 \sup_{(-M.M)} |f|$;
(e) if $f_n \to f$ locally uniformly then $\omega_{f_n} \to \omega_f$ locally uniformly.

We have $\omega_{\alpha'} = 2 \cdot 1\!\!1_{\{0\}}$. Thus, $\omega_{\beta'_n} = \sum_{k=1}^n |c_k| \cdot 1\!\!1_{\{a_k\}}$ (since $\omega_{f-g}(x) = 0$ implies $\omega_f(x) = \omega_g(x)$). Also, $\beta'_n \to \beta'$ uniformly. Thus, $\omega_{\beta'} = \sum_{k=1}^\infty |c_k| \cdot 1\!\!1_{\{a_k\}}$.

We see that (not unexpectedly) $\beta'$ is discontinuous at $a_1, a_2, \dots$ and only at these points. Accordingly, the quoted phrase "whose derivative is not

[1] That is, bounded on $[-M, M]$ for all $M > 0$.

continuous anywhere" should be interpreted as "whose derivative is not continuous on any interval".

Similarly one can construct a continuous function

$$x \mapsto \sum_{n=1}^{\infty} c_n |x - a_n|$$

not differentiable at $a_1, a_2, \dots$ (possibly a dense set) and differentiable at every other point. However, this example is much less monstrous than the famous monster of Weierstrass, a continuous function differentiable nowhere, — not even at a single point.

A question arises naturally: can a derivative (of a differentiable function) be discontinuous at every point?

## 5b Baire class 1 (classical)

**5b1 Definition.** A function $f : X \to \mathbb{R}$ on a separable metrizable space $X$ is *of Baire class 1* if there exist continuous $f_1, f_2, \dots : X \to \mathbb{R}$ such that $f_n(x) \to f(x)$ for all $x \in X$.

**5b2 Exercise.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then $f'$ is of Baire class 1.

Prove it.

**5b3 Theorem** (Baire)**.** A Baire class 1 function on a Polish space is continuous quasi-everywhere.

**5b4 Corollary.** If $f : \mathbb{R} \to \mathbb{R}$ is differentiable then $f'$ is continuous quasi-everywhere.

**5b5 Corollary.** The Dirichlet function $1\!\!1_{\mathbb{Q}}$ is not of Baire class 1, therefore, not a derivative.

Before proving the theorem let us think about the interior of the union and the union of interiors. Trivially, $\operatorname{Int}(A \cup B) \supset (\operatorname{Int} A) \cup (\operatorname{Int} B)$. It happens readily that $(\operatorname{Int} A) \cup (\operatorname{Int} B) = \emptyset$ but $\operatorname{Int}(A \cup B) = \mathbb{R}$. On the other hand, $(\operatorname{Int} F_1) \cup (\operatorname{Int} F_2) = \emptyset$ implies $\operatorname{Int}(F_1 \cup F_2) = \emptyset$ for closed sets $F_1, F_2$ by 1d4(a). The simple example $F_1 = [a, b]$, $F_2 = [b, c]$ shows that $\operatorname{Int}(F_1 \cup F_2)$ can exceed $(\operatorname{Int} F_1) \cup (\operatorname{Int} F_2)$. Indeed, $(a, c) \neq (a, b) \cup (b, c)$, but the tiny distinction disappears if we take the closure. It appears that $\operatorname{Cl}\big(\operatorname{Int}(F_1 \cup F_2)\big) = \operatorname{Cl}\big((\operatorname{Int} F_1) \cup (\operatorname{Int} F_2)\big)$, and moreover…

**5b6 Exercise.**

$$\operatorname{Cl}\bigg(\operatorname{Int}\Big(\bigcup_n F_n\Big)\bigg) = \operatorname{Cl}\bigg(\bigcup_n \operatorname{Int} F_n\bigg)$$

whenever $F_1, F_2, \dots$ are closed sets in a completely metrizable space $X$.

(a) Prove it.

(b) If $X$ is just metrizable (not completely) then it can happen that $\operatorname{Cl}\big(\operatorname{Int}(\cup_n F_n)\big) = X$ but $\operatorname{Cl}\big(\cup_n \operatorname{Int} F_n\big) = \emptyset$; find an example.

(c) For an uncountable family $(F_i)_{i\in I}$ of closed subsets of $\mathbb{R}$ it can happen that $\operatorname{Cl}\big(\operatorname{Int}(\cup_{i\in I} F_i)\big) = \mathbb{R}$ but $\operatorname{Cl}\big(\cup_{i\in I} \operatorname{Int} F_i\big) = \emptyset$; find an example.

In every case, $\operatorname{Int}\big(\cup_{i\in I} F_i\big) \supset \cup_{i\in I} \operatorname{Int} F_i$ trivially. The point is that

$$\text{(5b7)} \qquad \bigcup_n \operatorname{Int} F_n \text{ is dense in } \operatorname{Int}\Big(\bigcup_n F_n\Big)$$

due to completeness.

*Proof of Theorem 5b3.* We have continuous $f_n : X \to \mathbb{R}$, $f_n \to f$ pointwise, and want to prove that $\{x : \omega_f(x) = 0\}$ is comeager. It is sufficient to prove for every $\varepsilon > 0$ that the set $\{x : \omega_f(x) \ge 3\varepsilon\}$ is nowhere dense.

We consider sets

$$F_n = \big\{x : \operatorname{diam}\{f_n(x), f_{n+1}(x), \dots\} \le \varepsilon\big\} = \bigcap_{k,l} \{x : |f_{n+k}(x) - f_{n+l}(x)| \le \varepsilon\}\,;$$

they are closed, $F_1 \subset F_2 \subset \dots$, and $\cup_n F_n = X$ (since $f_n$ converge pointwise). By (5b7), $\cup_n \operatorname{Int} F_n$ is dense (in $X$). It remains to prove that this dense open set does not intersect $\{x : \omega_f(x) \ge 3\varepsilon\}$.

Let $x \in \cup_n \operatorname{Int} F_n$, that is, $x \in \operatorname{Int} F_n$ for some $n$. We note that $|f_n(\cdot) - f(\cdot)| \le \varepsilon$ on $F_n$. Thus, $\omega_{f-f_n}(\cdot) \le 2\varepsilon$ on $\operatorname{Int} F_n$, and finally, $\omega_f = \omega_{f-f_n} \le 2\varepsilon < 3\varepsilon$ on $\operatorname{Int} F_n$ (by continuity of $f_n$; recall 5a2 and the paragraph after it). $\square$

The converse (to 5b3) fails. For example, consider the Cantor set $C \subset \mathbb{R}$ and a countable $A \subset C$ dense in $C$. Introduce the indicator function $\mathbb{1}_A$ (a bit like the Dirichlet function on $C$). Its discontinuity points are exactly the points of $C$. Thus, $\mathbb{1}_A$ is continuous quasi-everywhere on $\mathbb{R}$. Nevertheless it is not of Baire class 1, since its restriction to $C$ is not!

**5b8 Corollary** (to 5b3)**.** If $f$ is a Baire class 1 function on a Polish space $X$ then for every closed $F \subset X$ the restriction $f|_F$ is continuous quasi-everywhere on $F$.

Here is a generalization of Lemma 2b1 (based on Th. 5b3).

**5b9 Proposition.** Let $X$ be a Polish space and $f_n : X \to \mathbb{R}$ continuous functions. Then

$$\limsup_{n\to\infty, y\to x} f_n(y) = \limsup_{n\to\infty} f_n(x) \in (-\infty, +\infty]$$

for quasi all $x \in X$.

**5b10 Exercise.** It can happen for continuous $f_n : [0,1] \to [0,1]$ that $\limsup_{n\to\infty, y\to x} f_n(y) = 1$ for all $x \in [0,1]$, but $\limsup_{n\to\infty} f_n(x) = 0$ for almost all $x \in [0,1]$.

Find an example.

*Proof of 5b9.* For every $n$ the function $g_n : x \mapsto \sup_k f_{n+k}(x)$ is of Baire class 1. By Theorem 5b3 quasi all $x$ are points of continuity of all $g_n$. It is sufficient to check the equality for such $x$. Assume the contrary:

$$\limsup_{n\to\infty, y\to x} f_n(y) > a > \limsup_{n\to\infty} f_n(x)$$

for some $a \in \mathbb{R}$. We note that $g_n(x) \downarrow \limsup_n f_n(x)$ and take $n$ such that $g_n(x) < a$. Using continuity of $g_n$ at $x$ we take a neighborhood $U$ of $x$ such that $\sup_{y\in U} g_n(y) < a$. We get a contradiction: $\limsup_{n;\, y\to x} f_n(y) > \sup_{y\in U;\, k} f_{n+k}(y)$. $\square$

Lemma 2b1 follows immediately: if $\limsup_{n;\, y\to x} f_n(y) \ge c$ for all $x$ then $\limsup_n f_n(x) \ge c$ for quasi all $x$.

## 5c Baire class 1 (modern)

Given two separable metrizable spaces $X, Y$, we consider maps $f : X \to Y$ and wonder, what could "Baire class 1" mean in this case? An example $\mathbb{R} \to \{0,1\}$ shows that we should not follow 5b1.

Recall that an $F_\sigma$ set in a separable metrizable space is, by definition, a countable union of closed sets. Thus, a closed set is an $F_\sigma$ set, and a countable union of $F_\sigma$ sets is an $F_\sigma$ set.

**5c1 Exercise.** (a) The intersection of two (or finitely many) $F_\sigma$ sets is an $F_\sigma$ set;

(b) Every open set is an $F_\sigma$ set;

(c) if $F_1, F_2$ are closed then $F_1 \setminus F_2$ is an $F_\sigma$ set.

Prove it.

The complement of an $F_\sigma$ set is a $G_\delta$ set, that is, a countable intersection of open sets; and every $G_\delta$ set is an $F_{\sigma\delta}$ set (countable intersection of $F_\sigma$ sets) due to 5c1(b).

Some sets are both $F_\sigma$ and $G_\delta$; in particular, all closed sets are, and all open sets are. This class of sets (that are both $F_\sigma$ and $G_\delta$) is closed under finite unions, finite intersections, and complement (it is an algebra of sets).

A dense $G_\delta$ set is comeager.

**5c2 Exercise.** If $A \subset \mathbb{R}$ is both $F_\sigma$ and $G_\delta$ then $(\operatorname{Int} A) \cup \operatorname{Int}(\mathbb{R} \setminus A)$ is dense.

Prove it.

In particular, for such $A$ it cannot happen that $A$ is dense and $\mathbb{R} \setminus A$ is also dense. For example, $\mathbb{Q}$ (the rationals) is $F_\sigma$ and not $G_\delta$. Also, $\mathbb{R} \setminus \mathbb{Q}$ is $G_\delta$ (and therefore $F_{\sigma\delta}$) but not $F_\sigma$.

**5c3 Exercise.** (a) For every $f : \mathbb{R} \to \mathbb{R}$ the set of all continuity points of $f$ is a $G_\delta$ set.[1]

(b) The same holds for $f : X \to Y$.

Prove it.

**5c4 Definition.** A map $f : X \to Y$ between separable metrizable spaces is of Baire class 1 if $f^{-1}(V)$ is an $F_\sigma$ set for every open $V \subset Y$.

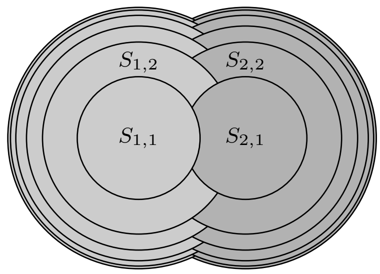
**5c5 Exercise.** If $f : X \to \mathbb{R}$ is of Baire class 1 according to Definition 5b1 then it is of Baire class 1 according to Definition 5c4.

Prove it.

The converse needs more effort.

**5c6 Lemma.** ("Reduction property for $F_\sigma$") For arbitrary $F_\sigma$ sets $A_1, \dots, A_n$ there exist pairwise disjoint $F_\sigma$ sets $B_1, \dots, B_n$ such that $B_1 \subset A_1, \dots, B_n \subset A_n$ and $B_1 \cup \dots \cup B_n = A_1 \cup \dots \cup A_n$.

*Proof (sketch).* $B_k = \cup_i S_{k,i}$ where $S_{k,i}$ are differences of closed sets, therefore $F_\sigma$ sets. $\square$




If in addition $A_1 \cup \dots \cup A_n = X$ then each $B_k$ must be both $F_\sigma$ and $G_\delta$.

**5c7 Lemma.** If $B \subset X$ is both $F_\sigma$ and $G_\delta$ then $\mathbb{1}_B$ is of Baire class 1 according to Definition 5b1.

[1] This set may be empty, of course.

*Proof (sketch).* We take closed $F'_n \uparrow B$, $F''_n \uparrow X \setminus B$ and define

$$f_n(x) = \frac{\operatorname{dist}(x, F''_n)}{\operatorname{dist}(x, F'_n) + \operatorname{dist}(x, F''_n)} \,,$$

then $f_n$ are continuous, and $f_n \to \mathbb{1}_B$ pointwise. $\square$

**5c8 Lemma.** Let $\varepsilon_n > 0$, $\sum_n \varepsilon_n < \infty$, and $f_n : X \to [-\varepsilon_n, \varepsilon_n]$ are of Baire class 1 according to Definition 5b1. Then $f = f_1 + f_2 + \dots$ also is of Baire class 1 according to Definition 5b1.

*Proof.* Having continuous $g_{n,i} : X \to [-\varepsilon_n, \varepsilon_n]$, $g_{n,i} \xrightarrow[i]{} g_n$ we introduce $g_i = g_{1,i} + g_{2,i} + \dots + g_{i,i}$, then

$$\underbrace{\liminf_i (g_{1,i} + g_{2,i} + \dots + g_{n,i})}_{= f_1 + \dots + f_n \xrightarrow[n]{} f} - \underbrace{\sum_k \varepsilon_{n+k}}_{\xrightarrow[n]{} 0} \le \liminf_i g_i \le$$

$$\le \limsup_i g_i \le \underbrace{\limsup_i (g_{1,i} + g_{2,i} + \dots + g_{n,i})}_{= f_1 + \dots + f_n \xrightarrow[n]{} f} + \underbrace{\sum_k \varepsilon_{n+k}}_{\xrightarrow[n]{} 0} \,,$$

therefore $g_i \xrightarrow[i]{} f$. $\square$

**5c9 Corollary.** If $f, f_n : X \to \mathbb{R}$, $f_n \to f$ uniformly, and each $f_n$ is of Baire class 1 according to Definition 5b1 then $f$ also is.

**5c10 Proposition.** Definition 5c4 for $Y = \mathbb{R}$ is equivalent to Definition 5b1.

*Proof (sketch).* By 5c5, 5b1 $\Longrightarrow$5c4.

Assume 5c4 for $f : X \to \mathbb{R}$. We may assume in addition that $f$ is bounded (otherwise we turn to, say, $\arctan f$ and force $f_n$ into $(-\pi/2, \pi/2)$). We cover $f(X) \subset \mathbb{R}$ by small open intervals $V_1, \dots, V_n$ (they overlap, of course), consider $F_\sigma$ sets $A_k = f^{-1}(V_k)$ and note that $A_1 \cup \dots \cup A_n = X$. Lemma 5c6 given $B_1, \dots, B_n$, and we approximate $f$ (uniformly) by a function $f_n$ constant on each $B_k$. Due to 5c7, $f_n$ is of Baire class 1 according to Definition 5b1. It remains to apply 5c9. $\square$

So, Definition 5b1 is generalized. Now we'll generalize Theorem 5b3.

**5c11 Theorem** (Baire)**.** A Baire class 1 map from a Polish space to a separable metrizable space is continuous quasi-everywhere.

*Proof.* Given $f : X \to Y$, we choose a countable base $(V_n)_n$ of $Y$ and note that $f$ is continuous at $x$ if and only if

$$\forall n \ \bigl( f(x) \in V_n \implies f(\cdot) \in V_n \text{ near } x \bigr) ,$$

that is,

$$\forall n \ \bigl( x \in f^{-1}(V_n) \implies x \in \operatorname{Int}(f^{-1}(V_n)) \bigr) .$$

We write the set of all discontinuity points of $f$ as

$$\bigcup_n \bigl( f^{-1}(V_n) \setminus \operatorname{Int} f^{-1}(V_n) \bigr)$$

and note that each $f^{-1}(V_n) \setminus \operatorname{Int} f^{-1}(V_n)$ is an $F_\sigma$ set with no interior points, therefore, a meager set. $\square$

Finally, we generalize 5b8.

**5c12 Corollary.** Let $X$ be Polish. If $f : X \to Y$ is of Baire class 1 then for every closed $F \subset X$ the restriction $f|_F$ is continuous quasi-everywhere on $F$.

Guess, what about the converse?

## 5d The converse holds

Now we deal with a map $f : X \to Y$ between separable metrizable spaces $X, Y$ (and use Def. 5c4, not 5b1).

**5d1 Theorem** (Baire)**.** If for every nonempty closed set $F \subset X$ the restriction $f|_F$ has a point of continuity (at least one) then $f$ is of Baire class 1.

In combination with 5c12 it gives the following.

**5d2 Corollary.** If $X$ is a Polish space then the following three conditions on $f$ are equivalent:

(a) $f$ is of Baire class 1;

(b) $f|_F$ is continuous quasi-everywhere on $F$, for every nonempty closed set $F \subset X$;

(c) $f|_F$ has a point of continuity, for every nonempty closed set $F \subset X$.

**5d3 Lemma.** $f$ is of Baire class 1 if and only if for all open $V_0, V_1 \subset Y$ such that $V_0 \cup V_1 = Y$ there exist $F_\sigma$ sets $A_0, A_1 \subset X$ such that $A_0 \cup A_1 = X$ and $f(A_0) \subset V_0$, $f(A_1) \subset V_1$.

*Proof.* "Only if" ("$\Rightarrow$") is trivial: take $A_0 = f^{-1}(V_0)$, $A_1 = f^{-1}(V_1)$.

"If" ("$\Leftarrow$"): let $V \subset Y$ be open; we have to prove that $f^{-1}(V)$ is an $F_\sigma$ set. The closed set $Y \setminus V$ being $G_\delta$, we take open $V_n \subset Y$ such that $\cap_n V_n = Y \setminus V$. We note that $V \cup V_n \supset V \cup (Y \setminus V) = Y$ for each $n$, and choose[1] $F_\sigma$ sets $A_n, B_n \subset X$ such that $A_n \cup B_n = X$, $f(A_n) \subset V$, $f(B_n) \subset V_n$. It remains to check that $f^{-1}(V) = \cup_n A_n$. "$\supset$" is trivial; "$\subset$": if $x \in f^{-1}(V)$ then $f(x) \in V = Y \setminus \cap_n V_n = \cup_n (Y \setminus V_n)$, thus $f(x) \notin V_n$ for some $n$, therefore $x \notin B_n$, hence $x \in A_n$. $\square$

From now on, till the end of the proof of Theorem 5d1, we fix a map $f : X \to Y$, open sets $V_0, V_1 \subset Y$ such that $V_0 \cup V_1 = Y$, and call an open $U \subset X$ a "good" set, if there exist $F_\sigma$ sets $A_0, A_1 \subset X$ such that $A_0 \cup A_1 = U$ and $f(A_0) \subset V_0$, $f(A_1) \subset V_1$.

**5d4 Exercise.** If $U_1, U_2, \dots$ are good then $\cup_n U_n$ is good.

Prove it.

**5d5 Exercise.** For every family $(U_i)_{i \in I}$ of open sets $U_i$ in a separable metrizable space there exists a (finite or) countable $J \subset I$ such that[2] $\cup_{i \in I} U_i = \cup_{i \in J} U_i$.

Prove it.

**5d6 Corollary.** The union of all good sets is a good set, — the greatest good set.

*Proof of Theorem 5d1.* Due to 5d3 and 5d6 it is sufficient to prove that the greatest good set $\tilde U$ is the whole $X$. Assume the contrary. Consider a nonempty closed set $F = X \setminus \tilde U$. The restriction $f|_F$ is continuous at some $x \in F$. Assume that $f(x) \in V_0$ (otherwise $f(x) \in V_1$, which is completely similar). We have $f(U \cap F) \subset V_0$ for some open neighborhood $U$ of $x$. In order to get a contradiction it remains to check that $\tilde U \cup U$ is a good set.

We have $\tilde A_0 \cup \tilde A_1 = \tilde U$, $f(\tilde A_0) \subset V_0$, $f(\tilde A_1) \subset V_1$ for some $F_\sigma$ sets $\tilde A_0, \tilde A_1$. The set $A_0 = \tilde A_0 \cup (U \cap F)$ is $F_\sigma$; $A_0 \cup \tilde A_1 = \tilde U \cup U$ (since $U \setminus \tilde U = U \cap F$); $f(A_0) \subset V_0$; thus, $\tilde U \cup U$ is a good set. $\square$

## 5e Riemann integrability

Functions that are continuous almost everywhere (rather than quasi-everywhere) appear in the well-known Lebesgue's criterion for Riemann integrability: a bounded function on $[0, 1]$ is Riemann integrable if and only if it is

[1] Some choice axiom is needed; the countable choice axiom, weaker than the dependent choice axiom, is sufficient.

[2] Every second-countable topological space is strongly Lindelöf.

continuous almost everywhere. This condition can be violated by Baire class 1 functions and moreover, by bounded derivatives, as we'll see.

An open set in $\mathbb{R}$ is a (finite or) countable union $G = \cup_n (a_n, b_n)$ of pairwise disjoint open intervals. It can be of finite (even small) measure and nevertheless dense in $\mathbb{R}$ (therefore comeager).

Consider the indicator of the Cantor set. Its set of discontinuity points is exactly the Cantor set, a meager null set of cardinality continuum. However, the "fat Cantor set" is meager but not null. Its indicator is of Baire class 1 but not Riemann integrable.

Here an interesting function related to $G$:

$$\begin{aligned} f(x) &= \alpha\biggl(\frac{(x-a_n)(b_n-x)}{b_n-a_n}\biggr) && \text{for } x \in (a_n,b_n)\,, \\ f(x) &= 0 && \text{for } x \in \mathbb{R}\setminus G\,; \end{aligned}$$

here $\alpha(x) = x^2 \sin \frac{1}{x}$, as in Sect. 5a. It is easy to see that

$$|f(x)| \le \bigl(\operatorname{dist}(x, \mathbb{R}\setminus G)\bigr)^2 \qquad \text{since } |\alpha(x)| \le x^2\,,$$

which implies differentiability of $f$; $f'(x) = 0$ for $x \in \mathbb{R}\setminus G$;

$$|f'(x)| \le 3 \text{ for all } x\,, \qquad \text{since } |\alpha'(\cdot)| \le 3\,;$$

$$\liminf_{x\to a_n+} f'(x) = \liminf_{x\to b_n-} f'(x) = -1\,, \quad \limsup_{x\to a_n+} f'(x) = \limsup_{x\to b_n-} f'(x) = 1\,,$$

which implies discontinuity of $f'$ on $\mathrm{Cl}(G)\setminus G$ (generally not a null set).

Countable sets (in $\mathbb{R}$) are both null and meager, of course. Every increasing (or piecewise monotone) function has (at most) countable set of discontinuities. The same holds for differences of increasing functions, the functions of locally bounded variation.

Some seemingly bad functions have only countably many discontinuities. For example, functions $f_n$ defined by

$$f_n(x) = \sum_{k=1}^\infty 2^{-k} b_{(2k-1)2^n} \quad \text{for } x = \sum_{k=1}^\infty 2^{-k} b_k$$

(where $b_k$ are the binary digits of $x$) on $(0,1)$ treated as a probability space (with Lebesgue measure) are *independent* random variables distributed uniformly on $(0,1)$. They are a measure preserving map from the one-dimensional interval to an infinite-dimensional cube. Still, they are uniform limits of step functions, discontinuous only at points of the form $k/2^n$. However, they are not differences of increasing functions.

We may also ask about existence of Riemann-Stieltjes integrals $\int_0^1 f(x)\,\mathrm{d}g(x)$ for all increasing bijections $g : [0,1] \to [0,1]$ (they are homeomorphisms). By change of variable, $\int f(x)\,\mathrm{d}g(x) = \int f(g^{-1}(x))\,\mathrm{d}x$; thus, the Riemann-Stieltjes integrability of $f$ for all $g$ is equivalent to Riemann integrability of $f(g^{-1}(\cdot))$ for all $g$; equivalently, $g(A_f)$ must be a null set for all $g$, where $A_f$ is the discontinuity set of $f$. Evidently, countability of $A_f$ is sufficient. Less evidently, it is also necessary, as we'll see.

Recall that a nonempty closed set without isolated points is called *perfect.*

**5e1 Exercise.** Every uncountable closed set (in $\mathbb{R}$) contains a perfect subset. Prove it.

**5e2 Lemma.** For every perfect set $F \subset \mathbb{R}$ there exists a continuous one-to-one[1] map $\{0,1\}^\infty \to F$.

*Proof.* We choose in $F$ two disjoint perfect subsets, each of diameter $< 1/2$; then we choose in each of them two disjoint perfect subsets of diameter $< 1/4$; and so on... □

By the way, we see that every perfect set (and therefore every uncountable closed set) in $\mathbb{R}$ is of cardinality continuum.

**5e3 Corollary.** For every uncountable closed set $F \subset \mathbb{R}$ there exists a nonatomic probability measure on $F$. (Take the image of the "Lebesgue measure on $\{0,1\}^\infty$".)

It follows that the same holds for every uncountable $F_\sigma$ set, and therefore, due to 5c3, for uncountable set $A_f$ of discontinuities. Having such measure $\mu$ we take $g(x) = \frac{1}{2}\big(x + \mu((0,x))\big)$ and get $m(g(A_f)) = \frac{1}{2}\big(m(A_f) + \mu(A_f)\big) \ge \frac{1}{2} > 0$ ($m$ being Lebesgue measure). Thus, the Riemann-Stieltjes integral $\int f(x)\,\mathrm{d}g(x)$ does not exist.

[1] Bijective and therefore homeomorphic (by compactness) as a map from $\{0,1\}^\infty$ to a subset of $F$.

## Hints to exercises

5a2: (c) use (b); (e) use (c), (d).

5b2: $x \mapsto (f(x+h) - f(x))/h$.

5b6: (a) if $U = \operatorname{Int}(\cup_n F_n) \setminus \operatorname{Cl}(\cup_n \operatorname{Int} F_n) \neq \emptyset$ then $F_n$ are meager in $U$ but $\cup_n F_n$ is not;
(b) try $X = \mathbb{Q}$.

5b10: think about (say) $\cos^{2m} nx$.

5c1: (a) $(\cup_k F_k') \cap (\cup_l F_l'') = \dots$;
(b) think about $\operatorname{dist}(x, X \setminus U)$;
(c) $F_1 \cap (X \setminus F_2)$.

5c2: use (5b7).

5c3: (a) use 5a1; (b) generalize.

5c5: $f(x) \in (a,b)$ if and only if $\exists \varepsilon, n \ \forall k \ \ f_{n+k}(x) \in [a+\varepsilon, b-\varepsilon]$.

5d4: take the union of the $F_\sigma$ sets.

5d5: take a countable base $(V_n)_n$ and choose $i_n$ such that $U_{i_n} \supset V_n$ whenever possible.

5e1: consider all rational open intervals that contain only a countable portion of the given set.

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