11 Typical functions via Banach spaces

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11a Divergence of Fourier series[1]

A normed space X may be defined as a vector[2] space over $\mathbb{R}$ or $\mathbb{C}$, endowed with a metric ρ such that

$$\rho(x,y) = \rho(x+z, y+z) \quad \text{for all } x,y,z \in X\,,$$
$$\rho(cx,cy) = |c|\rho(x,y) \quad \text{for all } x,y \in X \text{ and } c \in \mathbb{R} \text{ or } \mathbb{C}\,.$$

The norm is then $\|x\| = \rho(x,0)$, of course.

A Banach space is a complete normed space.

A linear functional $\alpha : X \to \mathbb{R}$ or $\mathbb{C}$ is continuous if and only if it is bounded, that is, of finite norm

$$\|\alpha\| = \sup_{\|x\|\le 1} |\alpha(x)| < \infty\,.$$

Here is the uniform boundedness principle.

11a1 Theorem. (Banach-Steinhaus).

If $\alpha_1, \alpha_2, \dots$ are linear[3] functionals on a Banach space X such that $\sup_n \|\alpha_n\| = \infty$ then $\sup_n |\alpha_n(x)| = \infty$ for quasi all $x \in X$.

Proof. Follows easily from 5b9. □

We consider the unit circle $\mathbb{T} = \{z \in \mathbb{C} : |z| = 1\} = \{\mathrm{e}^{\mathrm{i}\varphi} : \varphi \in \mathbb{R}\}$ with a probability measure $\mu : A \mapsto \frac{1}{2\pi} m\{\varphi \in [-\pi,\pi] : \mathrm{e}^{\mathrm{i}\varphi} \in A\}$, the Hilbert space $L_2(\mathbb{T}) = L_2(\mathbb{T},\mu)$, and the Banach space $C(\mathbb{T})$ of all continuous functions $f : \mathbb{T} \to \mathbb{C}$ with the norm $\|f\| = \max_{z\in\mathbb{T}} |f(z)|$.

[1] See Sect. 4.2.1 in book: E.M. Stein and R. Shakarchi, "Functional analysis", Princeton 2011. Also, Sect. 1 in: J.-P. Kahane (2000) "Baire's category theorem and triginometric series", Journal d'Analyse Mathématique **80** 143–182.

[2] In other words, linear.

[3] Continuous, I mean.

11a2 Exercise. Given $g \in L_2(\mathbb{T})$, define a linear functional α on $L_2(\mathbb{T})$ by $\alpha(f) = \int_{\mathbb{T}} fg \, \mathrm{d}\mu$ and a linear functional β on $C(\mathbb{T})$ by the same formula: $\beta(f) = \int_{\mathbb{T}} fg \, \mathrm{d}\mu$. Prove that

(a) $\|\alpha\| = \sqrt{\int_{\mathbb{T}} |g|^2 \, \mathrm{d}\mu}$;

(b) $\|\beta\| = \int_{\mathbb{T}} |g| \, \mathrm{d}\mu$.

The functions $z \mapsto z^k$ for $k \in \mathbb{Z}$ are a well-known orthonormal basis in $L_2(\mathbb{T})$;

$$f = \sum_{k \in \mathbb{Z}} \alpha_k(f)(z \mapsto z^k) \, , \quad \alpha_k(f) = \langle f, z \mapsto z^k \rangle = \int_{\mathbb{T}} f(z) \overline{z}^k \, \mu(\mathrm{d}z)$$

for $f \in L_2(\mathbb{T})$; the series converges in L_2.

Can we say that $f(z) = \sum_{k \in \mathbb{Z}} \alpha_k(f) z^k$?

First of all, $f(z)$ is ill-defined for $f \in L_2$, but well-defined for $f \in C(\mathbb{T})$; and $\alpha_k(f)$ are well-defined for $f \in C(\mathbb{T})$.

11a3 Proposition. $\forall^* f \in C(\mathbb{T}) \, \forall^* z \in \mathbb{T} \;\; \sup_n \bigl| \sum_{k=-n}^n \alpha_k(f) z^k \bigr| = \infty$.

Proof. Functions $(f, z) \mapsto \alpha_k(f) z^k$ are continuous on $C(\mathbb{T}) \times \mathbb{T}$, therefore $\{(f, z) : \sup_n | \dots | = \infty\} \in \mathrm{BP}(C(\mathbb{T}) \times \mathbb{T})$. By the Kuratowski-Ulam theorem it is sufficient to prove that $\forall^* z \in \mathbb{T} \, \forall^* f \in C(\mathbb{T}) \;\; \sup_n | \dots | = \infty$.

We reduce the general z to $z = 1$ using rotational symmetry, as follows.

Rotations $R_w : L_2(\mathbb{T}) \to L_2(\mathbb{T})$ defined for $w \in \mathbb{T}$ by $R_w(f) : z \mapsto f(wz)$ are unitary operators, and $R_w(z \mapsto z^n) = w^n (z \mapsto z^n)$, thus, $\alpha_n(R_w f) = w^n \alpha_n(f)$. Taking into account that

$$\sum_{k=-n}^n \alpha_k(f) z^k = \sum_{k=-n}^n \alpha_k(R_z f)$$

we see that it is sufficient to prove the following:

$$\forall^* f \in C(\mathbb{T}) \;\; \sup_n \biggl| \sum_{k=-n}^n \alpha_k(f) \biggr| = \infty \, .$$

We have

$$\sum_{k=-n}^n \alpha_k(f) = \sum_{k=-n}^n \langle f, z \mapsto z^k \rangle = \biggl\langle f, \sum_{k=-n}^n (z \mapsto z^k) \biggr\rangle \, ,$$

and

$$\sum_{k=-n}^n z^k = \frac{z^{n+0.5} - z^{-n-0.5}}{z^{0.5} - z^{-0.5}} = \frac{\sin(n+0.5)\varphi}{\sin 0.5\varphi} \quad \text{for } z = \mathrm{e}^{\varphi} \, ;$$

using 11a2(b),

$$\|\alpha_{-n}+\dots+\alpha_n\| = \int \Big| \frac{z^{n+0.5}-z^{-n-0.5}}{z^{0.5}-z^{-0.5}} \Big| \mu(\mathrm{d}z) = \\ = \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \frac{\sin(n+0.5)\varphi}{\sin 0.5\varphi} \Big| \mathrm{d}\varphi = \frac{1}{\pi} \int_0^{\pi} |\dots| \,\mathrm{d}\varphi \,.$$

Due to 11a1 it remains to prove that $\int_0^\pi |\dots| \mathrm{d}\varphi \to \infty$ as $n \to \infty$. We have

$$\int_0^{\pi} \Big| \frac{\sin(n+0.5)\varphi}{\sin 0.5\varphi} \Big| \mathrm{d}\varphi \ge \int_0^{\pi} \Big| \frac{\sin(n+0.5)\varphi}{0.5\varphi} \Big| \mathrm{d}\varphi = \\ = 2 \int_0^{(n+0.5)\pi} \frac{|\sin t|}{t} \mathrm{d}t \xrightarrow[n\to\infty]{} 2 \int_0^{\infty} \frac{|\sin t|}{t} \mathrm{d}t \ge 2 \sum_{n=1}^{\infty} \frac{1}{\pi n} \int_0^{\pi} \sin t \,\mathrm{d}t = \infty \,.$$

□

We see that the Fourier series fails to converge pointwise. However, Cesàro summation helps.

11a4 Proposition. $\forall f \in C(\mathbb{T}) \; \forall z \in \mathbb{T} \;\; \frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} \alpha_k(f) z^k \to f(z)$ as $N \to \infty$.

Proof. As before, by rotation we reduce the general z to $z=1$; it is sufficient to prove that

$$\frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} \alpha_k(f) \to f(0) \,.$$

In terms of functions

$$S_n(z) = \sum_{k=-n}^{n} z^k$$

we have

$$\frac{1}{N+1} \sum_{n=0}^{N} \sum_{k=-n}^{n} \alpha_k(f) = \frac{1}{N+1} \langle f, S_0 + \dots + S_N \rangle = \frac{1}{N+1} \int_{\mathbb{T}} (S_0 + \dots + S_N) f \,\mathrm{d}\mu \,.$$

Luckily,

$$S_0(z) + \dots + S_{2n}(z) = S_n^2(z) \quad \text{for all } z \in \mathbb{T} \text{ and } n = 0,1,2,\dots$$

(think, why).[1] We may restrict ourselves to $N \in 2\mathbb{Z}$, since

$$\Big| \frac{1}{N+1} \langle f, S_N \rangle \Big| \le \frac{1}{N+1} \|f\|_{C(\mathbb{T})} \frac{1}{2\pi} \int_{-\pi}^{\pi} \Big| \frac{\sin(N+0.5)\varphi}{\sin 0.5\varphi} \Big| \mathrm{d}\varphi \,,$$

[1] For $n=1$ it means, $z^0 + (z^{-1}+z^0+z^1) + (z^{-2}+z^{-1}+z^0+z^1+z^2) = (z^{-1}+z^0+z^1)^2$.

and

$$\int_{-\pi}^{\pi} \Big| \frac{\sin(N+0.5)\varphi}{\sin 0.5\varphi} \Big| \, \mathrm{d}\varphi \le 2 \int_0^{\pi} \frac{1}{\sin 0.5\varphi} \min\big(1, (N+0.5)\varphi\big) \, \mathrm{d}\varphi \le$$
$$\le 2 \int_0^{1/(N+0.5)} \frac{(N+0.5)\varphi}{\sin 0.5\varphi} \, \mathrm{d}\varphi + 2 \int_{1/(N+0.5)}^{\pi} \frac{1}{\sin 0.5\varphi} \, \mathrm{d}\varphi = O(1) + O(\log N) = o(N) \, .$$

We turn to $N = 2n \in 2\mathbb{Z}$:

$$\frac{1}{2n+1} \int_{\mathbb{T}} (S_0 + \dots + S_{2n}) f \, \mathrm{d}\mu = \int_{\mathbb{T}} \frac{S_n^2}{2n+1} f \, \mathrm{d}\mu \, .$$

Noting that $\int_{\mathbb{T}} \frac{S_n^2}{2n+1} \, \mathrm{d}\mu = 1$ (for two reasons…), we introduce probability measures μ_n on $\mathbb{T}$ by

$$\text{(11a5)} \qquad \frac{\mathrm{d}\mu_n}{\mathrm{d}\mu} = \frac{S_n^2}{2n+1} \, , \quad \text{that is, } \mu_n(A) = \int_A \frac{S_n^2}{2n+1} \, \mathrm{d}\mu \, ,$$

and note that $\mu_n(\mathbb{T} \setminus U) \to 0$ as $n \to \infty$ whenever U is a neighborhood of 1, since

$$\frac{1}{2n+1} S_n^2(\mathrm{e}^{\mathrm{i}\varphi}) = \frac{1}{2n+1} \Big(\frac{\sin(n+0.5)\varphi}{\sin 0.5\varphi} \Big)^2 \le \frac{1}{2n+1} \frac{1}{\sin^2 0.5\varphi} \, .$$

It follows easily that $\int f \, \mathrm{d}\mu_n \to f(0)$ for all $f \in C(\mathbb{T})$. □

11a6 Exercise. The convergence in Prop. 11a4 is uniform in $z \in \mathbb{T}$.
Prove it.

11b Decay of Fourier coefficients[1]

On the unit circle $\mathbb{T}$ (with μ as in 11a) we consider the Banach space $L_1(\mathbb{T}) = L_1(\mathbb{T}, \mu)$ of all (equivalence classes of) Lebesgue integrable functions, with the norm $\|f\| = \int_{\mathbb{T}} |f| \, \mathrm{d}\mu$. We note that $L_2(\mathbb{T})$ is embedded into $L_1(\mathbb{T})$ as a dense subset, and the linear functionals α_k (the Fourier coefficients, as in 11a) extend by continuity to $L_1(\mathbb{T})$; still,

$$\alpha_k(f) = \int_{\mathbb{T}} f(z) \overline{z}^k \, \mu(\mathrm{d}z) \quad \text{for } f \in L_1(\mathbb{T}) \, .$$

The linear operator

$$\alpha : f \mapsto \big(\alpha_k(f)\big)_{k \in \mathbb{Z}}$$

[1] See Sect. 4.3.1 in Stein and Shakarchi.

maps $L_1(\mathbb{T})$ to the (nonseparable) Banach space $l_\infty(\mathbb{Z})$ of all bounded functions on $\mathbb{Z}$ (with the supremum norm), and is continuous; moreover, $\|\alpha f\| \le \|f\|$.

Seeking the inverse operator we introduce operators $\beta_n, \gamma_n : l_\infty(\mathbb{Z}) \to L_1(\mathbb{T})$ by

$$\beta_n(a) : z \mapsto \sum_{k=-n}^{n} a(k) z^k \, ; \quad \gamma_n(a) = \frac{\beta_0(a) + \dots + \beta_n(a)}{n+1} \, .$$

Clearly, $\beta_n(l_\infty(\mathbb{Z})) \subset C(\mathbb{T}) \subset L_1(\mathbb{T})$, and the same holds for γ_n. By 11a3,

$$\forall^* f \in C(\mathbb{T}) \;\; \forall^* z \in \mathbb{T} \;\; \sup_n |\beta_n(\alpha(f))| = \infty \, .$$

By 11a6,

$$\forall f \in C(\mathbb{T}) \;\; \gamma_n(\alpha(f)) \to f \text{ in } C(\mathbb{T}) \, .$$

Also,

$$\alpha(L_2(\mathbb{T})) \subset l_2(\mathbb{Z}) \, ,$$

and $l_2(\mathbb{Z})$ is contained in the (closed) subspace $c_0(\mathbb{Z}) \subset l_\infty(\mathbb{Z})$ of all a such that $a(k) \to 0$ as $k \to \pm\infty$. It follows by continuity that $\alpha(L_1(\mathbb{T}))$ is contained in $c_0(\mathbb{Z})$, which is a well-known Riemann-Lebesgue lemma:

$$\forall f \in L_1(\mathbb{T}) \quad \alpha_k(f) \to 0 \quad \text{as } k \to \pm\infty \, .$$

11b1 Proposition. (a) $\forall f \in L_1(\mathbb{T}) \;\; \gamma_{2n}(\alpha(f)) \xrightarrow[n]{} f$ (in $L_1(\mathbb{T})$);

(b) For every $a \in c_0(\mathbb{Z})$, a belongs to $\alpha(L_1(\mathbb{T}))$ if and only if $\gamma_{2n}(a)$ converge (as $n \to \infty$) in $L_1(\mathbb{T})$.

In this sense, $\lim_n \gamma_{2n} = \alpha^{-1} : \alpha(L_1(\mathbb{T})) \to L_1(\mathbb{T})$.

11b2 Proposition. $\alpha(L_1(\mathbb{T}))$ is a meager subset of $c_0(\mathbb{Z})$.

11b3 Lemma. $\gamma_{2n}(\alpha(f)) : z \mapsto \int_{\mathbb{T}} f(zw) \mu_n(\mathrm{d}w)$.

Proof.

$$\gamma_{2n}(\alpha(f)) : z \mapsto \frac{1}{2N+1} \sum_{n=0}^{2N} \sum_{k=-n}^{n} \alpha_k(f) z^k = \frac{1}{2N+1} \sum_{n=0}^{2N} \sum_{k=-n}^{n} \alpha_k(R_z f) =$$

$$= \left\langle R_z f, \frac{S_N^2}{2N+1} \right\rangle = \int_{\mathbb{T}} (R_z f)(w) \, \mu_N(\mathrm{d}w) = \int_{\mathbb{T}} f(zw) \mu_N(\mathrm{d}w) \, .$$

□

11b4 Lemma. $\|\gamma_{2n}(\alpha(f))\| \le \|f\|$. (Here and below norms are taken in L_1.)

Proof. Using 11b3 and Tonelli's theorem,

$$\|\gamma_{2n}(\alpha(f))\| = \int_{\mathbb{T}} \mu(\mathrm{d}z) \biggl| \int_{\mathbb{T}} \mu_n(\mathrm{d}w)\, f(zw) \biggr| \le \int_{\mathbb{T}} \mu(\mathrm{d}z) \int_{\mathbb{T}} \mu_n(\mathrm{d}w)\, |f(zw)| = \\ = \int_{\mathbb{T}} \mu_n(\mathrm{d}w) \int_{\mathbb{T}} \mu(\mathrm{d}z)\, |f(zw)| = \int_{\mathbb{T}} \mu_n(\mathrm{d}w)\, \|R_w f\| = \|f\| \, .$$

□

Proof of Prop. 11b1. Item (a).

For every $\varepsilon > 0$ there exists $g \in C(\mathbb{T})$ such that $\|g - f\|_{L_1(\mathbb{T})} \le \varepsilon$. We know that $\gamma_{2n}(\alpha(g)) \to g$ in $C(\mathbb{T})$, the more so, in $L_1(\mathbb{T})$. Using 11b4,

$$\|\gamma_{2n}(\alpha(f)) - f\| \le \|\gamma_{2n}(\alpha(f)) - \gamma_{2n}(\alpha(g))\| + \|\gamma_{2n}(\alpha(g)) - g\| + \|g - f\| \le \\ \le \|\gamma_{2n}(\alpha(g)) - g\| + 2\varepsilon \, .$$

Thus, $\limsup_n \|\gamma_{2n}(\alpha(f)) - f\| \le 2\varepsilon$ for all ε.

Item (b).

First, by Item (a), $a = \alpha(f)$ implies $\gamma_{2n}(a) = \gamma_{2n}(\alpha(f)) \to f$.

Second, if $\gamma_{2n}(a) \to f$ then

$$\alpha_k(f) = \lim_n \alpha_k(\gamma_{2n}(a)) = \lim_n \frac{\alpha_k(\beta_0(a)) + \dots + \alpha_k(\beta_{2n}(a))}{2n+1} = \\ = \lim_n \frac{(2n - |k| + 1)a(k)}{2n+1} = a(k)$$

for all k; thus, $a = \alpha(f) \in \alpha(L_1(\mathbb{T}))$. □

For proving 11b2 we need a bit stronger form of the Banach-Steinhaus theorem.

11b5 Exercise. Let X, Y be Banach spaces, and $T_n : X \to Y$ linear[1] operators such that $\sup_n \|T_n\| = \infty$. Then $\sup_n \|T_n x\| = \infty$ for quasi all $x \in X$.

Prove it.

Proof of Prop. 11b2. By 11b1(b) it is sufficient to prove that $\forall^* a \in c_0(\mathbb{Z})$ $\sup_n \|\gamma_{2n}(a)\| = \infty$. By 11b5 it is sufficient to prove that $\sup_n \|\gamma_{2n}\| = \infty$. We introduce $a_n \in c_0(\mathbb{Z})$ by

$$a_n(k) = \begin{cases} 1 & \text{if } |k| \le n, \\ 0 & \text{otherwise,} \end{cases}$$

[1] Continuous, I mean.

note that $\beta_n(a_m) = S_{\min(m,n)}$ (here $S_n(z) = \sum_{k=-n}^{n} z^k$, as in Sect. 11a) and get

$$\|\gamma_{m+k}(a_m)\| = \left\| \frac{\beta_0(a_m) + \dots + \beta_{m+k}(a_m)}{m+k+1} \right\| = \left\| \frac{S_0 + \dots + S_m + kS_m}{m+k+1} \right\| \xrightarrow[k]{} \|S_m\| \, ,$$

thus,

$$\sup_n \|\gamma_{2n}(a_m)\| \ge \|S_m\| \, ;$$

$$\sup_n \|\gamma_{2n}\| \ge \sup_n \frac{\|\gamma_{2n}(a_m)\|}{\|a_m\|} = \sup_n \|\gamma_{2n}(a_m)\| \ge \|S_m\|$$

for all m. Finally,

$$\|S_m\| = \frac{1}{\pi} \int_0^\pi \left| \frac{\sin(n+0.5)\varphi}{\sin 0.5\varphi} \right| \mathrm{d}\varphi \to \infty \quad \text{as } m \to \infty \, ,$$

as was shown in the end of the proof of 11a3. □

Instead of the "uniform boundedness principle" we could use the "bounded inverse theorem" or a more general "open mapping theorem" when proving 11b2.

11b6 Theorem. ("Bounded inverse theorem")
Let X, Y be Banach spaces and $T : X \to Y$ a linear[1] operator. If T is bijective then T^{-1} is continuous (in other words, T is a homeomorphism).

A continuous bijection between *compact* metrizable spaces is well-known to be a homeomorphism. An infinite-dimension Banach space cannot be even σ-compact. Amazingly, completeness and linearity (together) can sometimes replace compactness!

11b7 Theorem. (Banach-Schauder, "Open mapping theorem")
Let X, Y be Banach spaces and $T : X \to Y$ a linear operator. Then $T(X)$ is either a meager subset of Y, or the whole Y.[2] In the latter case T is open (in other words, gives a homeomorphism between Y and the quotient space $X/T^{-1}(0)$).

Clearly, 11b7 implies 11b6. And 11b7 follows easily from (the Baire category theorem and) a wonderful lemma.

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[1] Continuous, I mean.

[2] Thus, $T(X) \in \mathrm{BP}(Y)$ always. By the way, a continuous image of a Polish space (in another Polish space) has the Baire property (since it is analytic). However, the Banach space X need not be separable.

11b8 Lemma. Let X, Y be Banach spaces and $T : X \to Y$ a linear operator. If $T(B_X)$ is dense in B_Y then $T(B_X)$ contains B_Y. Here $B_X = \{x : \|x\| < 1\} \subset X$ and $B_Y = \{y : \|y\| < 1\} \subset Y$.

Proof. Let $y \in B_Y$ and $\varepsilon \in (0,1)$. We take $x_0 \in B_X$ such that $\|y_0 - y\| < \varepsilon$ where $y_0 = Tx_0$. Then we take $x_1 \in B_X$ such that

$$\left\| \frac{y_0 - y}{\varepsilon} - y_1 \right\| < \varepsilon \quad \text{where } y_1 = Tx_1 \,.$$

We get $\|y - (y_0 + \varepsilon y_1)\| < \varepsilon^2$. Now we take $x_2 \in B_X$ such that

$$\left\| \frac{y - (y_0 + \varepsilon y_1)}{\varepsilon^2} - y_2 \right\| < \varepsilon \quad \text{where } y_2 = Tx_2 \,,$$

get $\|y - (y_0 + \varepsilon y_1 + \varepsilon^2 y_2)\| < \varepsilon^3$, and so on. Finally, $x = x_0 + \varepsilon x_1 + \varepsilon^2 x_2 + \dots$ and $Tx = y_0 + \varepsilon y_1 + \varepsilon^2 y_2 + \dots = y$. We have $(1-\varepsilon)\|x\| < 1$, thus $(1-\varepsilon)y \in T(B_X)$ whenever $\|y\| < 1$ and $\varepsilon \in (0,1)$. Every point of B_Y is such $(1-\varepsilon)y$. □

11c Non-continuation of holomorphic functions[1]

Recall that a holomorphic[2] function on an open disk $U = \{z \in \mathbb{C} : |z - z_0| < r\}$ is (according to one of several equivalent definitions) a function $f : U \to \mathbb{C}$ of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where $a_n \in \mathbb{C}$ and the series converges for all $z \in U$. Then $a_n = \frac{1}{n!} f^{(n)}(z_0)$, of course. A holomorphic function on an open set $U \subset \mathbb{C}$ may be defined as a function $U \to \mathbb{C}$ holomorphic on every open disk contained in U. If $U, V \subset \mathbb{C}$ are open, $f : U \cup V \to \mathbb{C}$ and $f|_U, f|_V$ are holomorphic then f is holomorphic. The same holds for any union of open sets (finite or not).

If the radius of convergence of the series $\sum_n \frac{1}{n!} f^{(n)}(0) z^n$ is equal to 1 then f cannot be extended from $\mathbb{D} = \{z : |z| < 1\}$ to (a holomorphic function on) any $(1+\varepsilon)\mathbb{D}$; but it does not mean that f cannot be extended to some $\mathbb{D} \cup (\mathrm{e}^{\mathrm{i}\varphi} + \varepsilon \mathbb{D})$. If for some $r \in (0,1)$ the radius of convergence of the series $\sum_n \frac{1}{n!} f^{(n)}(r\mathrm{e}^{\mathrm{i}\varphi})(z - r\mathrm{e}^{\mathrm{i}\varphi})^n$ exceeds $1 - r$ then f extends to some $\mathbb{D} \cup (r\mathrm{e}^{\mathrm{i}\varphi} + (1 - r + \varepsilon)\mathbb{D})$; the latter contains some neighborhood of $\mathrm{e}^{\mathrm{i}\varphi}$. If this never happens (for a given f), that is, the radius of convergence of the series $\sum_n \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$ equals $1 - |z_0|$ for all $z_0 \in \mathbb{D}$, one says that the

[1] See Sect. 3 in Kahane.
[2] In other words, complex analytic.

circle $\partial\mathbb{D} = \{z : |z| = 1\}$ is a *natural boundary* for f. In this case f cannot be extended to any $\mathbb{D} \cup (\mathrm{e}^{\mathrm{i}\varphi} + \varepsilon\mathbb{D})$.

The so-called *disk algebra* is the set $A(\mathbb{D})$ of all continuous functions $f : \overline{\mathbb{D}} \to \mathbb{C}$ such that $f|_{\mathbb{D}}$ is holomorphic. Endowed with the norm $\|f\| = \max_{z\in\overline{\mathbb{D}}} |f(z)|$, $A(\mathbb{D})$ is a Banach space (separable).

11c1 Theorem. For quasi all $f \in A(\mathbb{D})$ the circle $\partial\mathbb{D}$ is the natural boundary.

Proof. The radius of convergence of the series $\sum_n \frac{1}{n!} f^{(n)}(z_0)(z - z_0)^n$ is a continuous (moreover, Lip(1)) function of z_0, since $z_1 + r\mathbb{D} \supset z_2 + (r - \varepsilon)\mathbb{D}$ whenever $|z_1 - z_2| \le \varepsilon$. Thus, points $z_0 \in \mathbb{D}$ such that the radius exceeds $1 - |z_0|$ are an open set. In order to prove that it is empty it is sufficient to prove that it contains no point of a given dense countable set. Therefore it is sufficient to prove for a given z_0 that the radius does not exceed $1 - |z_0|$ for quasi all f.

If the radius exceeds $1 - |z_0|$ then

$$\sup_n \frac{1}{n!} |f^{(n)}(z_0)| (1 - |z_0| + \varepsilon)^n < \infty$$

for some $\varepsilon > 0$. We introduce linear functionals α_n on $A(\mathbb{D})$ by

$$\alpha_n(f) = \frac{1}{n!} f^{(n)}(z_0)(1 - |z_0| + \varepsilon)^n ;$$

by Theorem 11a1 it is sufficient to prove that $\sup_n \|\alpha_n\| = \infty$.

We take $\delta < \varepsilon$, define $f \in A(\mathbb{D})$ by

$$f(z) = \frac{1}{z - z_1} , \quad z_1 = (1 + \delta)\frac{z_0}{|z_0|}$$

and observe that

$$f^{(n)}(z) = \pm\frac{n!}{(z - z_1)^{n+1}} , \quad |\alpha_n(f)| = \frac{(1 - |z_0| + \varepsilon)^n}{|z_0 - z_1|^{n+1}} = \text{const}\cdot\Big(\frac{1 - |z_0| + \varepsilon}{1 - |z_0| + \delta}\Big)^n ;$$

clearly, $|\alpha_n(f)| \to \infty$, thus $\|\alpha_n\| \to \infty$. □

Hints to exercises

11a2: (b) $C(\mathbb{T})$ is dense in $L_1(\mathbb{T})$; try the function $\operatorname{sgn} g$.

11a6: f is uniformly continuous.

11b5: either generalize the proof of 11a1, or alternatively, choose linear functionals α_n on Y such that $\|\alpha_n\| \le 1$ and $\|\alpha_n \circ T_n\| \ge \frac{1}{2}\|T_n\|$.

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