

5 Random connected components

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5a Connected sets among closed sets

Recall some topological notions and facts.

A metrizable space X is *connected*, if $\text{Clopen}(X) = \{\emptyset, X\}$.

A subset $Y \subset X$ is connected if it is itself a connected space. Note that

$$(5a1) \quad \text{Clopen}(Y) \supset \{A \cap Y : A \in \text{Clopen}(X)\};$$

in general, these are not equal (indeed, it happens routinely that X is connected while Y is not). For a connected Y ,

$$(5a2) \quad \forall A \in \text{Clopen}(X) \quad (Y \subset A \vee Y \subset X \setminus A).$$

For arbitrary Y , choosing a compatible metric ρ on X and denoting as before $\text{dist}(x, A) = \inf\{\rho(x, a) : a \in A\}$, we have

$$\begin{aligned} \text{Clopen}(Y) &= \{A \subset Y : \overline{A} \cap (Y \setminus A) = \emptyset \wedge \overline{(Y \setminus A)} \cap A = \emptyset\} = \\ &= \{A \subset Y : (\forall a \in A \text{ dist}(a, Y \setminus A) > 0) \wedge (\forall b \in Y \setminus A \text{ dist}(b, A) > 0)\}. \end{aligned}$$

For every $A \in \text{Clopen}(Y)$ there exist open sets $U, V \subset X$ such that $U \cap V = \emptyset$, $U \cap Y = A$ and $V \cap Y = Y \setminus A$; namely, we may take

$$U = \bigcup_{a \in A} B^\circ(a, 0.5\rho(a, Y \setminus A)), \quad V = \bigcup_{b \in Y \setminus A} B^\circ(b, 0.5\rho(b, A)).$$

(Here $B^\circ(a, r) = \{x : \rho(x, a) < r\}$.) Thus,

$$(5a3) \quad \text{Clopen}(Y) = \{Y \cap U : U, V \text{ are open in } X \wedge U \cap V = \emptyset \wedge U \cup V \supset Y\}.$$

5a4 Proposition. For every compact metrizable space X the set

$$\{F \in \mathbf{F}(X) : F \text{ is connected}\}$$

is Borel measurable.

From now on (till 5a9) X is compact, and $(U_n)_n$ is a sequence¹ of open subsets of X such that for all $E, F \in \mathbf{F}(X)$,

$$(5a5) \quad E \cap F = \emptyset \implies \exists m, n (U_m \cap U_n = \emptyset \wedge E \subset U_m \wedge F \subset U_n).$$

5a6 Core exercise. Prove existence of such $(U_n)_n$.

Note that $\text{Clopen}(X) \subset \{U_1, U_2, \dots\}$; and by the way, it shows that $\text{Clopen}(X)$ is at most countable (provided that X is compact).²

5a7 Core exercise. For every $F \in \mathbf{F}(X)$,

$$\text{Clopen}(F) = \{F \cap U_m : U_m \cap U_n = \emptyset \wedge U_m \cup U_n \supset F\}.$$

Prove it.

5a8 Core exercise. A closed set $F \subset X$ is connected if and only if

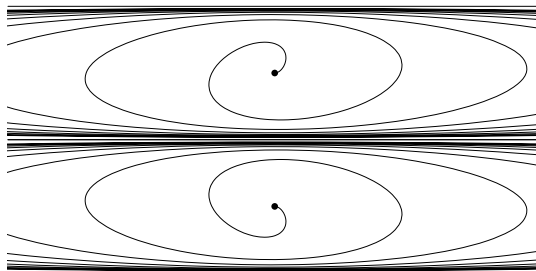
$$\forall m, n ((U_m \cap U_n = \emptyset \wedge U_m \cup U_n \supset F) \implies (U_m \cap F = \emptyset \vee U_n \cap F = \emptyset)).$$

Prove it.

5a9 Core exercise. Prove Prop. 5a4.

Prop. 5a4 fails for Polish (not just compact) spaces. In particular, it fails if X is an infinite-dimensional separable Hilbert space.³

Does 5a4 hold for $X = \mathbb{R}^d$? I do not know!⁴ If you feel enthusiastic to reduce connectedness of a closed set to some property of its compact subsets, take into account the following instructive example of a connected closed subset of \mathbb{R}^2 :



¹“Countable superbase”, if you like.

²For Polish X , $\text{Clopen}(X)$ need not be countable (try a discrete space).

³This is basically the phenomenon mentioned before 4d10.

⁴I only know that $\{F \in \mathbf{F}(\mathbb{R}^d) : F \text{ is connected}\}$ is coanalytic, since its complement is the image of the Borel set $\{(F_1, F_2) : F_1 \cap F_2 = \emptyset, F_1 \neq \emptyset, F_2 \neq \emptyset\}$ (recall 4d15) under the Borel map $(F_1, F_2) \mapsto F_1 \cup F_2$.

5a10 Remark. As a palliative we may treat a random closed subset of \mathbb{R}^d via the one-point compactification $\mathbb{R}^d \cup \{\infty\}$ (including ∞ into each unbounded closed set). Then all unbounded connected components (if any) are glued together.

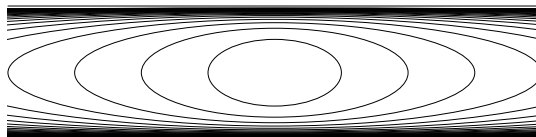
5b Connected components

Consider two equivalence relations on a metrizable space X : points $x, y \in X$ are equivalent, when

$$(5b1) \quad x, y \in Y \text{ for some connected } Y \subset X;$$

$$(5b2) \quad \forall A \in \text{Clopen}(X) \quad (x \in A \iff y \in A).$$

Clearly, (5b1) implies (5b2) (recall (5a2)). In general they are not equivalent; an example:



Equivalence classes for (5b1) are called *connected components* (of X); for (5b2) — *quasicomponents*. In general, every quasicomponent decomposes into connected components. But a compact X is simpler.

5b3 Lemma. For a compact X , (5b1) and (5b2) are equivalent.

Proof. Let Y be an equivalence class for (5b2); we'll prove that Y is connected. By (5a3) it is sufficient to prove $Y \subset U$ or $Y \subset V$ whenever open $U, V \subset X$ satisfy $U \cap V = \emptyset$ and $U \cup V \supset Y$. Compactness gives us $A \in \text{Clopen}(X)$ such that $Y \subset A \subset U \cup V$. Thus, $A \cap U = A \setminus V \in \text{Clopen}(X)$. All points of Y being (5b2)-equivalent, we get $Y \subset A \cap U$ or $Y \subset X \setminus (A \cap U)$; accordingly, $Y \subset U$ or $Y \subset V$. \square

Note that a compact X can have uncountably many connected components (try the Cantor set).

5b4 Proposition. The following subset of $\mathbf{F}(X) \times \mathbf{F}(X)$ is Borel measurable, provided that X is compact:

$$\{(E, F) : E \text{ is a connected component of } F\}.$$

Choosing a compatible metric ρ on X we define for $E, F \in \mathbf{F}(X)$

$$d_{\mathcal{C}}(E, F) = \sup_{x \in E} \text{dist}(x, F) = \sup_{x \in E} \inf_{y \in F} \rho(x, y) = \inf\{r > 0 : E \subset F_{+r}\},$$

$$d_H(E, F) = \max(d_{\mathcal{C}}(E, F), d_{\mathcal{C}}(F, E)) = \inf\{r > 0 : E \subset F_{+r} \wedge F \subset E_{+r}\}$$

(as usual, $\inf \emptyset = +\infty$); d_H is a metric on $\mathbf{F}(X) \setminus \{\emptyset\}$, — the well-known Hausdorff metric.¹

5b5 Lemma. $d_{\mathcal{C}} : \mathbf{F}(X) \times \mathbf{F}(X) \rightarrow [0, \infty]$ is a Borel function.

Proof. We take a sequence $(x_n)_n$ dense in X and note that

$$d_{\mathcal{C}}(E, F) = \inf_{r>0} \sup_{n: \text{dist}(x_n, E) \leq \varepsilon} \text{dist}(x_n, F)$$

and $\text{dist}(x_n, \cdot)$ is a Borel function by 4d8(a). □

Proof of Prop. 5b4. For $E, F \in \mathbf{F}(X)$, by 5b3, E is a connected component of F if and only if

- (a) E is connected, and
- (b) $E \subset F$, and E is the intersection of all $A \in \text{Clopen}(F)$ such that $A \supset E$.

Condition (a) leads to a Borel set by Prop. 5a4; we'll prove the same for (b). By compactness, (b) is equivalent to

- (b1) for every $\varepsilon > 0$ there exists $A \in \text{Clopen}(F)$ such that $A \supset E$ and $d_{\mathcal{C}}(A, E) \leq \varepsilon$.

We choose $(U_n)_n$ satisfying (5a5). By (5a7), (b1) is equivalent to

- (b2) for every $\varepsilon > 0$ there exist m, n such that $U_m \cap U_n = \emptyset$, $U_m \cup U_n \supset F$, $F \cap U_m \supset E$ and $d_{\mathcal{C}}(F \cap U_m, E) \leq \varepsilon$.

It remains to check, for given m, n satisfying $U_m \cap U_n = \emptyset$, that each of the following three conditions leads to a Borel set:

- (c) $F \subset U_m \cup U_n$,
- (d) $E \subset F \setminus U_n$,
- (e) $d_{\mathcal{C}}(F \setminus U_n, E) \leq \varepsilon$.

We rewrite (c), (d) as

- (c1) $F \cap (X \setminus (U_m \cup U_n)) = \emptyset$,
- (d1) $d_{\mathcal{C}}(E, F \setminus U_n) = 0$.

For (c1) we use 4d12(b). For (d1) and (e) we use 5b5, taking into account that the map $F \mapsto F \setminus U_n = F \cap (X \setminus U_n)$ is Borel measurable by 4d14. □

We conclude.

¹In fact, the Hausdorff metric on a compact X turns $\mathbf{F}(X)$ into a compact metric space whose Borel σ -algebra coincides with that of (4d2). The set of connected components of X need not be closed in $(\mathbf{F}(X), d_H)$; an example: $[-1, 0] \cup \{1, \frac{1}{2}, \frac{1}{3}, \dots\}$. For a Polish X , the metric space $(\mathbf{F}(X), d_H)$ is complete but generally nonseparable (try a discrete X). See also Sect. 2 in Beer's article (cited in footnote 5 on page 71).

5b6 Theorem. Let X be a compact metrizable space and S a random closed subset of X . Then the set of all connected components of S^1 is a random Borel subset of the standard² Borel space $\mathbf{F}(X)$.

5c Counting the connected components

5c1 Core exercise. Let S be a random Borel subset of a standard Borel space. Then $\{\omega : |S(\omega)| \leq n\}$ is measurable for every n . (Here $|S(\omega)|$ is the number of points in $S(\omega)$.)

Prove it.

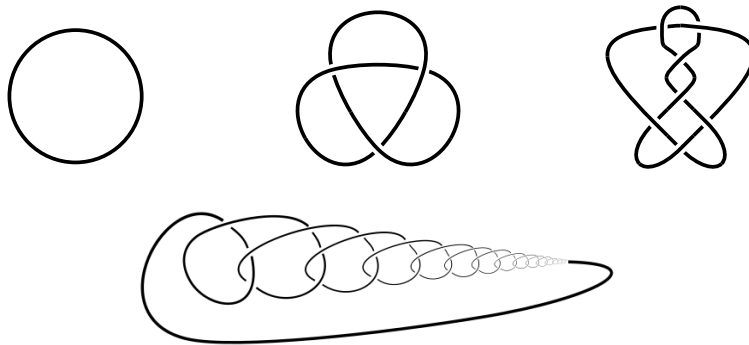
By 5c1 (and Theorem 5b6), given a random closed set in a compact metrizable X , the number of its connected components is a random variable (with values in $\{0, 1, 2, \dots\} \cup \{\infty\}$). In this sense we may count the connected components.³ Given a Borel set $B \subset \mathbf{F}(X)$, we may count the connected components belonging to B (think, why). For example we may count the connected components contained in a given ball, or intersecting a given ball, etc.

5c2 Core exercise. Let S be a random closed subset of \mathbb{R}^d . Then the set of all bounded connected components of S is a random Borel subset of $\mathbf{F}(\mathbb{R}^d)$.

Prove it.

5d Classifying the connected components: random knots

A knot is a subset of \mathbb{R}^3 homeomorphic to a circle.



¹I mean the set of all connected components of $S(\omega)$ as a function of ω .

²Recall 4d5.

³The set of ω such that $S(\omega)$ is uncountable (and therefore of cardinality continuum, see Kechris, Th. (13.6) or Srivastava, Th. 4.3.5) is also measurable, see Kechris, Th. (29.19).

Knots K_1, K_2 are of the same *type* (in other words, equivalent) if $h(K_1) = K_2$ for some homeomorphism $h : \mathbb{R}^3 \rightarrow \mathbb{R}^3$.¹ A knot is *tame* if it is equivalent to a polygonal knot. Or, equivalently, to a smooth knot (continuously differentiable). Otherwise it is a *wild* knot.² All knots that lie in a plane are of the same type (the trivial type; the unknot).³ This is a deep theorem in general, but relatively simple for tame knots.

All tame knot types are a countable set.⁴

5d1 Proposition. Each knot type is a universally measurable⁵ subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, all tame knots are also a universally measurable set.

By 5c2 and 5d1, given a random closed set in \mathbb{R}^3 , we may count its connected components that are (a) tame knots; (b) tame knots of a given type; (c) tame knots of a given type that are contained in a given ball; etc.

You may think about a random smooth map $f : \mathbb{R}^3 \rightarrow \mathbb{R}^2$ (or $\mathbb{R}^3 \rightarrow \mathbb{C}$) such that almost surely, f is regular (that is, $\text{rank } df(\cdot) = 2$) at all x satisfying $f(x) = 0$. Then all bounded connected components of the random closed set $f^{-1}(0)$ are tame knots (almost surely).

The group $\text{Homeo}(\mathbb{R}^3)$ of all homeomorphisms $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ acts on the standard Borel space $\mathbf{F}(\mathbb{R}^3)$.

Here is a generalization of Prop. 5d1.

5d2 Proposition. For every $F \in \mathbf{F}(\mathbb{R}^3)$ its orbit $\{h(F) : h \in \text{Homeo}(\mathbb{R}^3)\}$ is a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

5d3 Core exercise. The set $C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$ of all continuous maps $\mathbb{R}^3 \rightarrow \mathbb{R}^3$ endowed with the σ -algebra generated by evaluations $f \mapsto f(x)$ is a standard Borel space.

Prove it.

5d4 Core exercise. $f(x)$ is jointly measurable in $f \in C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$ and $x \in \mathbb{R}^3$.

Prove it.

¹See for instance Sect. 1.1 in: R.H. Crowell, R.H. Fox, "Introduction to knot theory", Springer 1963.

²Crowell and Fox, Sect. 1.2.

³Crowell and Fox, Sect. 1.2.

⁴Crowell and Fox, Chapter I, Exercise 5. In fact, they are a semigroup isomorphic to $\{1, 2, 3, \dots\}$ with multiplication. By the way, some wild knots are infinite products of tame knots.

⁵In fact, Borel measurable, but this is harder to prove.

5d5 Core exercise. The composition $g \circ f \in C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$ is jointly measurable in $f, g \in C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$.

Prove it.

We embed the set $\text{Homeo}(\mathbb{R}^3)$ into $C(\mathbb{R}^3 \rightarrow \mathbb{R}^3) \times C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$ as follows:

$$h \mapsto (h, h^{-1}).$$

By 5d5 the image $\{(f, g) : f \circ g = \text{id} \wedge g \circ f = \text{id}\}$ is a Borel set, therefore a standard Borel space. We endow $\text{Homeo}(\mathbb{R}^3)$ with the corresponding σ -algebra and observe that

(5d6) $\text{Homeo}(\mathbb{R}^3)$ is a standard Borel space.¹

5d7 Extra exercise. $\text{Homeo}(\mathbb{R}^3)$ is both a Borel subset and a measurable subspace of $C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$.

Prove it.²

(By the way, it follows that the map $f \mapsto f^{-1}$ is Borel measurable on $\text{Homeo}(\mathbb{R}^3)$ treated as a subset of $C(\mathbb{R}^3 \rightarrow \mathbb{R}^3)$.)

5d8 Core exercise. Let $F \subset \mathbb{R}^3$ be a compact set, then the map

$$C(\mathbb{R}^3 \rightarrow \mathbb{R}^3) \ni f \mapsto f(F) \in \mathbf{F}(\mathbb{R}^3)$$

is Borel measurable.

Prove it.

Thus, the orbit $\{h(F) : h \in \text{Homeo}(\mathbb{R}^3)\}$ is the image of the standard Borel space $\text{Homeo}(\mathbb{R}^3)$ under the Borel map $h \mapsto h(F)$.

Here is a topology-free counterpart of Def. 3e1.

5d9 Definition. A subset of a countably separated measurable space is *analytic* if it is the image of some standard Borel space under some measurable map.

Does 5d9 conflict with 3e1? No, it does not.

5d10 Lemma. For every subset A of a separable metrizable space, the following two conditions are equivalent:

- (a) A is the image of some Polish space under some continuous map;
- (b) A is the image of some standard Borel space under some Borel map.

¹It is in fact a Polish group (and therefore, by 4d7, a standard Borel space).

²In fact, this follows from standardness by a general theorem; but you are asked to prove it explicitly.

Proof. (a) \implies (b): the Polish space is a standard Borel space by 4d7, and the continuous map is Borel measurable.

(b) \implies (a): we may assume that the standard Borel space is a Borel subset of the Cantor set, or even the whole Cantor set (extend the map by a constant...); it remains to use Prop. 3d1. \square

And here is a counterpart of Theorem 3f22.

5d11 Theorem. Analytic sets in countably separated measurable spaces are universally measurable.

Proof. Let (X, \mathcal{A}) be a standard Borel space, (Y, \mathcal{B}) a countably separated measurable space, $f : X \rightarrow Y$ a measurable map, and μ a probability measure on (Y, \mathcal{B}) ; we have to prove that $f(X)$ is μ -measurable.

Without loss of generality we may assume that (Y, \mathcal{B}) is a Borel space. Proof: first, by 1d35, (Y, \mathcal{B}_1) is a Borel space for some $\mathcal{B}_1 \subset \mathcal{B}$; second, f is measurable from (X, \mathcal{A}) to (Y, \mathcal{B}_1) ; third, every $(\mu|_{\mathcal{B}_1})$ -measurable set is μ -measurable.

By 1d40 we may assume that $Y \subset \mathbb{R}$. Moreover, we may assume that $Y = \mathbb{R}$. Proof: we define a probability measure ν on \mathbb{R} by $\nu(B) = \mu(B \cap Y)$ for $B \in \mathcal{B}(\mathbb{R})$ and observe that every ν -measurable subset of Y is μ -measurable.

It remains to use Lemma 5d10 and Theorem 3f22. \square

Propositions 5d2 and 5d1 follow.

Proposition 5d2 (and its proof) generalizes readily to arbitrary locally compact separable metrizable spaces (in place of \mathbb{R}^3).

So, we may count tame knots. What about wild knots?

5d12 Proposition. All knots are a universally measurable subset of $\mathbf{F}(\mathbb{R}^3)$.

Thus, wild knots are also a universally measurable set.

The circle is a compact metrizable space; knots are its homeomorphic images in \mathbb{R}^3 .

5d13 Core exercise. Let K be a compact metrizable space, and (similarly to 5d3) $C(K \rightarrow \mathbb{R}^3)$ the standard Borel space of continuous maps. Then the set of all homeomorphisms $f : K \rightarrow f(K) \subset \mathbb{R}^3$ is a Borel subset of $C(K \rightarrow \mathbb{R}^3)$.

Prove it.

Similarly to 5d8, the map

$$C(K \rightarrow \mathbb{R}^3) \ni f \mapsto f(K) \in \mathbf{F}(\mathbb{R}^3)$$

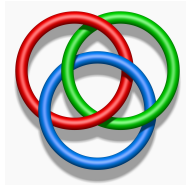
is Borel measurable. Its image

$$\{F : F \text{ is homeomorphic to } K\} \subset \mathbf{F}(\mathbb{R}^3)$$

is analytic, thus, universally measurable. Prop. 5d12 follows.

In fact, the set of all knots is Borel measurable.¹

You may think also about random links, and a lot of other random geometric objects.



¹The same holds for subsets of a given σ -compact separable metrizable space X , that are homeomorphic to a given compact metrizable space K ; see: C. Ryll-Nardzewski, “On a Freedman’s problem”, *Fund. Math.* **57** (1965), 273–274. The corresponding Borel complexity is not bounded in K even if $X = [0, 1] \times [0, 1]$; see Fact 3.12 in: A. Marcone, “Complexity of sets and binary relations in continuum theory: a survey” (2005). For $K = \mathbb{S}^1$ this Borel set is $F_{\sigma\delta}$ and not $G_{\delta\sigma}$; see Lemma 6.2 and Theorem 8.5 in: R. Camerlo, U.B. Darji, A. Marcone, “Classification problems in continuum theory”, *Trans. AMS* **357**:11, 4301–4328 (2005).

Hints to exercises

5a6: consider a countable base and all finite unions of its sets.

5c1: use 3f24, taking into account that the Borel space is countably separated.

5c2: recall 5a10.

5d3: recall 2c10(c).

5d4: similar to (4b4).

5d5: use 5d4.

5d8: use a dense sequence.

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