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QUANTUM AND QUASI-CLASSICAL ANALOGS OF BELL INEQUALITIES*

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1. INTRODUCTION

Thanks to J.S.Bell we know that the quantum theory predicts the possibility of violation of some probabilistic inequalities holding in all local (that is, Einstein-causal) classical theories. On violating classical restrictions, the quantum theory itself establishes new non-trivial restrictions - quantum analogs of the Bell inequalities. Like the classical Bell inequalities, they are model-independent, that is, do not depend on physical mechanisms and physical parameters, except the space-time parameters connected with the local causality. Accordingly, they do not contain the Plank constant. By introducing some model-dependent features, we obtain inequalities though not so general but allowing a quasi-classical passage to the limit; they are quasi-classical analogs of the Bell inequalities.

Another turn of development of investigations related to the Bell inequalities deals with space-time configurations more general than in the original case of two space-like separated domains. First of all, we give a mathematical definition formalizing the conception of a local classical (hidden variables),theory. Then we discuss the problem of what the quantum analog of this definition is, that is, how to formalize the conception of a local quantum theory, and give the obtained results.

The mathematical results of this paper are due to B.S.Tsirelson. General approaches were formed in the process of author's collaboration.

2. QUANTUM ANALOGS OF BELL INEQUALITIES

It is known that the Bell inequalities can be represented in the form of inequalities linear in observables and holding for all states. Thus they can be treated as inequalities for observables, while states can be eliminated. So the algebraic nature of the Bell inequalities becomes clear. In the classical case the observables are functions of parameters, and the states are probability distributions of the same parameters. The commutativity of the algebra of the observables is vital for the Bell inequalities.

In particular, the well-known Bell-CHSH inequality can be considered as averaging the following inequality for observables:

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \leq 2 \cdot \mathbf{1}; \tag{1}$$

here A_1, A_2, B_1, B_2 are arbitrary commuting observables such that $\|A_k\| \leq 1$, $\|B_l\| \leq 1$. The quantum analogs of the Bell inequalities, introduced in [1], can be treated as inequalities for observables holding in the non-commutative case. In particular, the following inequality was obtained in [1]:

$$A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \leq 2\sqrt{2} \cdot \mathbf{1}; \tag{2}$$

here A_1, A_2, B_1, B_2 are arbitrary hermitean elements of any C^*-algebra such that $\|A_k\| \leq 1$, $\|B_l\| \leq 1$, $[A_k, B_l] = 0$ for $k = 1,2$ and $l = 1,2$. The inequality for quantum correlations follows immediately:

$$f(A_1B_1) + f(A_1B_2) + f(A_2B_1) - f(A_2B_2) \leq 2\sqrt{2};$$

here f is an arbitrary state on the above C^*-algebra. It is known that the value $2\sqrt{2}$ occurs in the spin correlation gedankenexperiment - Bohm's version of the EPR one. This is practically the only case when $2\sqrt{2}$ is attained. See [3] for the exact and more general result that characterizes the canonical anticommutation relations or the Clifford algebras in terms of extremal properties connected with the quantum analogs of the Bell inequalities. Note that [1] and [3] investigate the case $k = 1, \dots, m$; $l = 1, \dots, n$, not only $m = n = 2$.

The class of correlation functions (or rather of "behaviors" in the sense of Section 4 below) allowed by quantum analogs of the Bell inequalities is essentially smaller than that allowed by general probabilistic axioms together with the widely interpreted local causality (see Section 4 below). In this sense the quantum analogs of the Bell inequalities are non-trivial. Therefore their violation can be revealed in principle in an experiment. In this case the conception of a local quantum theory would be rejected in such generality as a violation of the Bell inequalities themselves rejects the conception of a local classical theory. Some possible experiments for this purpose in the high-energy physics are discussed in [2].

Both (1) and (2) can be treated as a consequences of the following inequality:

$$(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)^2 \leq 4 \cdot \mathbf{1} - [A_1, A_2] \cdot [B_1, B_2], \tag{3}$$

holding under the same assumptions as (2). From (3) it follows immediately

$$\|A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2\| \leq \sqrt{4 + \|[A_1, A_2]\| \cdot \|[B_1, B_2]\|}. \tag{4}$$

In the classical case the commutators vanish and the right-hand side equals 2 in accordance with (1). In the quantum case the norms of the commutators are at most 2 and the right-hand side is at most $\sqrt{4+2\cdot 2} = 2\sqrt{2}$ in accordance with (2).

The proof of (3) is quite elementary. Firstly, the following equality can be verified by opening the brackets and using the relation $[A_k, B_l] = 0$:

$$(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)^2 + [A_1, A_2]\cdot[B_1, B_2] =$$
$$= ((A_1+A_2)B_1)^2 + ((A_1-A_2)B_2)^2 + (A_1(B_1+B_2))^2 + (A_2(B_1-B_2))^2 -$$
$$-(A_1^2+A_2^2)(B_1^2+B_2^2).$$

Secondly, the right-hand side of this equality can be rewritten as

$$4\cdot\mathbf{1} - (2\cdot\mathbf{1} - A_1^2 - A_2^2)(2\cdot\mathbf{1} - B_1^2 - B_2^2) - (A_1+A_2)^2(\mathbf{1} - B_1^2) -$$
$$-(A_1-A_2)^2(\mathbf{1}-B_2^2) - (\mathbf{1}-A_1^2)(B_1+B_2)^2 - (\mathbf{1}-A_2^2)(B_1-B_2)^2;$$

this does not exceed $4\cdot\mathbf{1}$ since $A_k^2 \leqslant \mathbf{1}$ and $B_l^2 \leqslant \mathbf{1}$.

3. QUASI-CLASSICAL ANALOGS OF BELL INEQUALITIES

It is natural to believe that a violation of the classical Bell inequalities by quantum objects must be small in quasi-classical situations. In this connection we want to obtain inequalities which, holding true for quantum objects, approximate the classical Bell inequalities in quasi-classical situations. Such inequalities will be called here quasi-classical analogs of the Bell inequalities. At first sight, it is enough to use the inequality (4) together with the well-known quasi-classical passage to the limit. The commutators in (4) "must be" proportional to the Plank constant h, and for $\hbar \to 0$ the right-hand side tends to its classical value 2. However there is some surprise here.

Let us consider the quantum mechanics of one-dimensional spinless particles. This case is attractive due to both its connection with the EPR gedankenexperiment and the fact that it was investigated from the view point of the quasi-classical passage to the limit. We are mainly interested in observables with two values ± 1. Typical examples of such observables are the sign of the coordinate sign Q and the sign of the momentum sign P. It turns out that

$$\|[\operatorname{sign} Q, \operatorname{sign} P]\| = 2; \tag{5}$$

the right-hand side does not depend on h despite of the fact that $\|[Q,P]\| = \hbar$.

Let us prove (5). The Fourier transform $\mathcal{F}$ connects the operators P, Q in such a way that in the coordinate representation $P = \mathcal{F}^{-1} \cdot hQ \cdot \mathcal{F}$; hence sign P = $= \mathcal{F}^{-1} \cdot (\mathrm{sign}(hQ)) \cdot \mathcal{F} = \mathcal{F}^{-1} \cdot (\mathrm{sign}\, Q) \cdot \mathcal{F}$, where h has already disappeared. Let us consider the wave functions $\psi_1(q) = |q|^{-1/2}$, $\psi_2(q) = |q|^{-1/2}\, \mathrm{sign}\, q$; then $\mathcal{F}\psi_1 = \psi_1$, $\mathcal{F}\psi_2 = -i\psi_2$, $(\mathrm{sign}\, Q)\,\psi_1 = \psi_2$, $(\mathrm{sign}\, Q)\,\psi_2 = \psi_1$ and hence $(\mathrm{sign}\, P)\,\psi_1 = \mathcal{F}^{-1} \cdot (\mathrm{sign}\, Q) \cdot \mathcal{F}\psi_1 = i\psi_2$, $(\mathrm{sign}\, P)\,\psi_2 = -i\psi_1$. **We see that the Pauli matrices represent the operators** sign Q, sign P on the two-dimensional subspace spanned by ψ_1 and ψ_2 . However, ψ_1, ψ_2 do not belong to L_2; this can be corrected by means of a regularization. We can put $\psi_{1,\varepsilon}(q) = |2q \ln \varepsilon|^{-1/2}$ for $\varepsilon < |q| < \varepsilon^{-1}$ and $=0$ otherwise; then $\|\psi_{1,\varepsilon}\| = 1$ and it can be shown that $\|\mathcal{F}\psi_{1,\varepsilon} - \psi_{1,\varepsilon}\| \longrightarrow 0$ for $\varepsilon \longrightarrow 0$. The same is true for $\psi_{2,\varepsilon}$. Arguing as above we see now that the matrix elements of the operators sign Q, sign P on the vectors $\psi_{1,\varepsilon}, \psi_{2,\varepsilon}$ tend to the corresponding elements of the Pauli matrices when $\varepsilon \longrightarrow 0$. This implies (5).

The appearance of the Pauli matrices reveals a possibility to replace the spin correlation gedankenexperiment of Bohm by a spinless coordinate-momentum gedankenexperiment, connected more closely with the EPR one. Indeed, let us consider two one-dimensional spinless particles described by the coordinate operators Q_1 , Q_2 and momentum operators P_1, P_2. Let

$$A_1 = \mathrm{sign}\, Q_1,\ A_2 = \mathrm{sign}\, P_1,\ B_1 = \mathrm{sign}\, Q_2,\ B_2 = \mathrm{sign}\, P_2 . \tag{6}$$

The correlated state ψ is formed by analogy with the singlet spin state: $\psi = e^{-\pi i/8}\psi_1 \otimes \psi_1 + e^{\pi i/8}\psi_2 \otimes \psi_2$ (the regularization is left to the reader); then $(A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2)\psi = 2\sqrt{2}\,\psi$, as in the scheme of Bohm. Thus,

$$\| A_1B_1 + A_1B_2 + A_2B_1 - A_2B_2 \| = 2\sqrt{2} . \tag{7}$$

The wave function for ψ in the coordinate representation is

$$\psi(q_1, q_2) = \begin{cases} 2\cos(\pi/8) \cdot |q_1q_2|^{-1/2} & \text{for } q_1q_2 > 0, \\ -2i\sin(\pi/8) \cdot |q_1q_2|^{-1/2} & \text{for } q_1q_2 < 0. \end{cases}$$

We see that the Bell-CHSH inequality is essentially (and actually to the greatest extent) violated in the coordinate-momentum gedankenexperiment. And the particles' masses are of no importance. Each particle can be replaced by a multi-particle system. In the latter case we use the total momentum and the barycentre coordinate.

Does is mean that the quantum theory predicts the possibility to observe an essential (that is, not small) violation of the Bell inequalities in experiments with macroscopic bodies? Formally the answer is "yes", if we insist on the postulate that each operator represents some observable and each vector - some state. However, it is not new that making use of this postulate together with the assumption on a possibility of quantum description of macroscopic bodies can lead to highly strange conclusions with a questionable possibility of experimental testing, even in principle. By the way, the gedankenexperiment discussed in [6] belongs just to this type. However, [5] has pretensions to suggest a feasible experiment showing an essential violation of the Bell inequalities in such a physical system whose being macroscopic is a controversial problem. We concern here with very interesting problems connected with Everett's programme. It is important to know, whether macroscopic bodies can essentially violate the Bell inequalities or not. If yes, one must perform such an experiment. If not, one must understand, how to agree this fact with the quantum theory. We do not pretend to decide, what is the true answer, yes or no. We pretend only to make some contributions to both cases; to the "yes" case - the gedankenexperiment described above, to the "no" case - the arguments we now proceed on.

We can point out two circumstances preventing from an essential violation of the Bell inequalities by macroscopic bodies. Firstly, restricted resolution power of instruments prevents from measuring observables like sign Q, sign P. This is some version of the long-known argument that the length of de Broglie's wave for the macroscopic body is so small that we cannot make sure that it exists. Secondly, the connection of mechanical degrees of freedom with thermal ones prevents from the preparation of a state described by an essentially (macroscopically) delocalized wave function for the macroscopic body. However, there are some attempts to overcome this obstacle [5], [13].

Both circumstances can be described in the first approximation by one simple and elementary model. Namely, let us introduce a classical noise affecting coordinates and velocities of our particles or, what is essentially the same, of our instruments. It is convenient for us to introduce the noise in our observables and to continue working without states. Thus, we are about to replace Q by $Q-\sigma_1\xi_1\cdot\mathbb{1}$ and P by $P-\sigma_2\xi_2\cdot\mathbb{1}$, where ξ_1,ξ_2 are random variables and σ_1,σ_2 are scale parameters. In accordance with the elementary nature of our model the random variables ξ_K are assumed independent and normally distributed with zero mean and variance 1. For any $p,q\in(-\infty,+\infty)$ we introduce the transform $\alpha_K(p,q)$ of the C^*-algebra of observables:

$$\alpha_K(p,q)A=\exp(i\hbar^{-1}(pQ_K-qP_K))\cdot A\cdot\exp(-i\hbar^{-1}(pQ_K-qP_K))$$

for any observable A; k is the particle number. This transform is an automorphism such that

$$\alpha_K(p,q)f(Q_K)=f(Q_K-q\cdot\mathbb{1}),\quad \alpha_K(p,q)f(P_K)=f(P_K-p\cdot\mathbb{1})$$

for any bounded function f; of course, $\alpha_K(p,q)$ does not affect Q_l, P_l with $l\neq k$. So, the above mentioned replacements of P_K, Q_K are expressed by transforms $\alpha_K(\sigma_{K,1}\xi_{K,1};\ \sigma_{K,2}\xi_{K,2})$; all random variables $\xi_{K,l}$ are assumed independent. The specified transform will be denoted for brevity by α_K (but it depend on the random event), and we put $\alpha=\alpha_1\dots\alpha_N$.

Now we need two types of averaging: the classical averaging with respect to the random variables ξ_K and the quantum averaging with respect to a quantum state. The latter remains implicit, and the former is expressed by the completely positive mapping β defined by $\beta(A)=\mathbb{E}\,\alpha(A)$ for any observable A; here $\mathbb{E}$ denotes the mathematical expectation with respect to the random variables ξ_K ; and $\beta(A)$ is an observable again. It is clear that $\beta=\beta_1\dots\beta_n$ where $\beta_K(A)=\mathbb{E}\,\alpha_K(A)$. One should not conclude that we observe $\beta(A)$ instead of A if there is a noise; we observe various $\alpha(A)$ only, and for unknown ξ_K . So it is quite possible that $\beta(A)$ has a continuous spectrum whereas each $\alpha(A)$ has only two possible values according to the two-valued nature of the used instrument. In contrast with (7) we have

$$\|\beta(A_1B_1+A_1B_2+A_2B_1-A_2B_2)\|\leqslant 2\sqrt{1+\pi^{-2}h^2(\sigma_{1,1}\sigma_{1,2}\sigma_{2,1}\sigma_{2,2})^{-1}},\quad(8)$$

where A_K, B_l are defined by (6). The situation is quasi-classical when $\sigma_{1,1}\sigma_{1,2}\gg h$ and $\sigma_{2,1}\sigma_{2,2}\gg h$; in this case the right-hand side is nearly 2. Thus, (8) or rather its immediate consequence (9) falls under the category of the quasi-classical analogs of the Bell inequalities:

$$f(\beta(A_1B_1))+f(\beta(A_1B_2))+f(\beta(A_2B_1))-f(\beta(A_2B_2))\leqslant$$
$$\leqslant 2\sqrt{1+\pi^{-2}h^2(\sigma_{1,1}\sigma_{1,2}\sigma_{2,1}\sigma_{2,2})^{-1}}\ ;\quad(9)$$

here f is an arbitrary state, but the observables A_K, B_l are fixed by (6). It turns out, however, that they may be unfixed; there is an absolute constant C such that

$$\|\beta(A_1B_1+A_1B_2+A_2B_1-A_2B_2)\|\leqslant 2+C\,h^2(\sigma_{1,1}\sigma_{1,2}\sigma_{2,1}\sigma_{2,2})^{-1}\quad(10)$$

for any observables $A_K\in\mathcal{A}_1$ and $B_l\in\mathcal{A}_2$ with $\|A_K\|\leqslant 1$, $\|B_l\|\leqslant 1$; here and in what follows $\mathcal{A}_K$ is the von Neumann algebra generated by P_K, Q_K (to be more exact, by bounded functions of them). So (10) is a quasi-classical analog of the Bell inequalities more general than (9).

The proof of (10) consists in applying (4) together with the following result: there is an absolute constant C such that

$$\sup_{A_1, A_2 \in \mathcal{A}_1} \frac{\| [\beta_1(A_1), \beta_1(A_2)] \|}{\| A_1 \| \cdot \| A_2 \|} < C h (\sigma_{1,1} \sigma_{1,2})^{-1}. \qquad (11)$$

This is a non-trivial theorem; its proof is too long to be given here and will be published elsewhere. The proof consists of the following three steps: (1) any operator of $\mathcal{A}_1$ can be represented as the Weyl quantization of some generalized function of p, q; (2) the transform β can be represented by some "smoothing" of the corresponding functions; (3) the commutator for the quantized smooth functions can be estimated in a necessary way.

However, the proof of (8) is rather short and is given below. It is enough to show that

$$\| [\beta_1 (\mathrm{sign}\, Q_1), \beta_1 (\mathrm{sign}\, P_1)] \| \leq 2 \pi^{-1} h (\sigma_{1,1} \sigma_{1,2})^{-1}. \qquad (12)$$

But $\beta_1(\mathrm{sign}\, Q_1) = \mathbb{E}\, \alpha_1(\mathrm{sign}\, Q_1) = \mathbb{E}\, \mathrm{sign}(Q_1 - \sigma_{1,2} \xi_{1,2}) = f(\sigma_{1,2}^{-1} Q_1)$, where the function f is determined by the identity $\mathbb{E}\, \mathrm{sign}(q - \sigma \xi) = f(\sigma^{-1} q)$ and hence $f(x) = (2\pi)^{-1/2} \int_{-x}^{x} e^{-u^2/2} du$. Similarly, $\beta_1(\mathrm{sign}\, P_1) = f(\sigma_{1,1}^{-1} P_1)$. In the coordinate representation $P_1 = \mathcal{F}^{-1} \cdot h Q_1 \cdot \mathcal{F}$, where $\mathcal{F}$ is the Fourier transform, and $\beta_1(\mathrm{sign}\, P_1) = f(\mathcal{F}^{-1} \cdot \sigma_{1,1}^{-1} h Q_1 \cdot \mathcal{F}) = \mathcal{F}^{-1} \cdot f(\sigma_{1,1}^{-1} h Q_1) \cdot \mathcal{F}$. But $f(\sigma_{1,1}^{-1} h Q_1)$ is a multiplication operator, hence $\beta_1(\mathrm{sign}\, P_1)$ is a convolution operator; it is easy to calculate its kernel: $i\pi^{-1}(q-q')^{-1} \exp\left(-\frac{1}{2} \sigma_{1,1}^2 h^{-2} (q-q')^2\right)$. Therefore the operator $[\beta_1(\mathrm{sign}\, Q_1), \beta_1(\mathrm{sign}\, P_1)]$ transforms an arbitrary function $\psi(q)$ into the function $\int dq' \psi(q') i\pi^{-1}(q-q')^{-1}(f(\sigma_{1,2}^{-1} q) - f(\sigma_{1,2}^{-1} q')) \exp\left(-\frac{1}{2} \sigma_{1,1}^2 h^{-2} (q-q')^2\right)$. But $|f(\sigma_{1,2}^{-1} q) - f(\sigma_{1,2}^{-1} q')| \leq 2 (2\pi)^{-1/2} \sigma_{1,2}^{-1} |q - q'|$ and hence the absolute value of this function does not exceed $\int dq' |\psi(q')| \pi^{-1} \cdot 2 (2\pi)^{-1/2} \exp\left(-\frac{1}{2} \sigma_{1,1}^2 h^{-2} (q-q')^2\right)$. It only remains to calculate the norm of this convolution.

Completely positive mappings for which the supremum from the left-hand side of (11) is small, may be of some interest for Everett's programme. As any completely positive mappings they can arise from unitary operators in tensor products [4]. Such an approach to the quantum description of macroscopic objects can be more realistic than the "interpretation basis" approach [6]; and these bases are useful as a more elementary model; cf. [15]. But at the same time the approach we have just sketched removes us from the solution of the interpretation uniqueness problem, while the paper [6] brought us nearer to it.

4. FORMAL DESCRIPTION OF LOCAL CLASSICAL CONCEPTION

The Bell inequalities are considered proved for all local classical theories. However, to the best of our knowledge there is no definition of the relevant class of theories. One has a clear but informal idea of it and formal but "ad hoc" formulations used in proofs. So the local classical conception is not jet formalized. We suggest here such a formalization. It will be used in the next section as a base for constructing its quantum analogs. We use the word "conception" following[7] (p. 204). It would be useful to compare in detail with[7] the present text before its publication, but the time was pressing.

First of all we determine a necessary class of domains in the Minkowski space-time. Let T_{Mink} be the class of all open sets t in the space-time that are "past layers" or "decreasing sets" in the following sense: if $x \in t$ and y chronologically precedes x then $y \in t$. Also both the empty set and the whole space-time belong to T_{Mink}. However it is more convenient to base a formal theory on a suitable class of partially ordered sets T rather than on $T=T_{Mink}$ exclusively. We only assume here that T is a complete distributive lattice[8] and we call such T an admissible lattice, but we reserve the right to add some further technical conditions in publications to follow. Of course, T_{Mink} is an admissible lattice. The maximal element of T will be denoted by $+\infty$, the minimal by $-\infty$.

We treat the classical observables as measurable functions on a measurable (Borel) space Ω. A point $\omega \in \Omega$ represents a whole history, and the space-time structure will be represented by a family of σ-fields.

Let T be an admissible lattice. We define an admissible T-family on Ω as a family $\mathcal{F} = \{\mathcal{F}(t)\}_{t\in T}$ of σ-fields (in other words, σ-algebras) $\mathcal{F}(t)$ of subsets of Ω such that $\mathcal{F}(\sup t_k) = \sup \mathcal{F}(t_k)$ for any $t_1, t_2, \ldots \in T$. The supremum in the left-hand side is taken in T, in the right-hand side - in the complete lattice of all σ-fields. This definition is also not final; to avoid mathematical pathologies we are about to add and use some technical conditions in the future.

To express the local causality we use the idea of a local intervention in a space-time distributed physical system. The response of the system must be non-anticipative, that is, must be localized in the future with respect to the intervention.

4.1. Definition. A behavior scheme is a collection $(T, \Omega, \mathcal{F}_0, \mathcal{F}_1)$, where T is an admissible lattice, Ω

is an arbitrary set, $\mathcal{F}_0$ and $\mathcal{F}_1$ are admissible T-families on Ω such that $\mathcal{F}_0(t) \leqslant \mathcal{F}_1(t)$ (that is, $\mathcal{F}_0(t) \subset \mathcal{F}_1(t)$) for all $t \in T$.

A behavior scheme provides a formal description of kinematics (but not dynamics) of the system under consideration. A point $\omega \in \Omega$ represents a space-time history of the system (involving a history of interventions). A function on Ω , measurable with respect to $\mathcal{F}_1(t)$, represents a classical observable which is localized in the t-past (recall that an element of T_{Mink} is a "past layer" domain in the space-time). Such observables depending only on an interventions history are represented by $\mathcal{F}_0(t)$ -measurable functions.

4.2 Example. Let $T=T_{Mink}$; Ω consists of the pairs (u,g) formed by a real-valued functions u and g on the space-time (we leave to the reader the choice of suitable classes of functions); $\mathcal{F}_k(t)$ is generated by all measurable functionals of g (for k=0) or of u,g (for k=1) that are localized in the t-past, that is, depends on the restrictions of g (and u) to the given "past layer" domain only.

4.3 Example. Let $T=T_{Mink}$; Ω consists of the pairs (f,w) formed by a union w of a finite set of non-intersecting time-like curves in the space-time (each curve being unbounded both in the past and in the future) and a vector field f on w which is orthogonal to the tangent vectors; $\mathcal{F}_k(t)$ is generated by all measurable functionals of w (for k=0) or of f,w (for k=1) that are localized in the t-past.

4.4 Example. Let T consists of four elements denoted by $-\infty, t_1, t_2, +\infty$ with the lattice structure determined by $t_1 \wedge t_2 = -\infty$, $t_1 \vee t_2 = +\infty$; let Ω consists of $2^4=16$ elements, and let four functions denoted by a,b,A,B be given on Ω , each taking two values ± 1 only, with each combination of this values occuring once; σ-fields are generated by the functions listed below:

t :	$-\infty$	t_1	t_2	$+\infty$
functions generating $\mathcal{F}_0(t)$:	none	a	b	a,b
functions generating $\mathcal{F}_1(t)$:	none	a,A	b,B	a,b,A,B

The example 4.2 provides a formal description of the kinematics of a classical field u generated by sources which are continuously distributed in the space-time with a density function g; g is considered as an interventions history, u as a responses history. The example 4.3 deals with a finite system of classical relativistic particles; their forced trajectories w are considered as an interventions history, and the resulting forces f - as a responses history. In the example 4.4 the interventions are described by two parameters $a=\pm1$, $b=\pm1$, the responses also by two parameters $A=\pm1$, $B=\pm1$, and the space-time structure is such that the intervention a chronologically precedes the response A but not the response B, and b precedes B but not A. We shall see that this scheme is immediately connected with the Bell-CHSH inequality, and we call it the Bell-CHSH scheme.

4.5 Definition. A stochastic behavior (in a given behavior scheme) is a function $p : \Omega \times \mathcal{F}_1(+\infty) \to [0,1]$ which is a regular conditional probability [9] on $\mathcal{F}_1(t)$ with respect to $\mathcal{F}_0(t)$ for all $t \in T$. That is, $p(\omega,X)$ as a function of $\omega \in \Omega$ is $\mathcal{F}_0(t)$-measurable for any fixed $\mathcal{F}_1(t)$-measurable $X \subset \Omega$, and $p(\omega,X)$ as a function of $X \in \mathcal{F}_1(+\infty)$ is a probability measure on Ω for any fixed $\omega \in \Omega$, and $p(\omega,X) = 1$ for any $X \in \mathcal{F}_0(+\infty)$, $\omega \in X$.

4.6 Definition. A deterministic behavior is such a stochastic behavior that takes only two values, 0 and 1.

4.7 Example. We use the scheme 4.2 (that is, the scheme introduced in example 4.2); u_g denotes the solution of the wave equation $\Box u = g$ vanishing in the infinite past. We assume both the classes of functions u, g and the meaning the equation and the "initial" condition are understood to be such that the above solution always exists and is unique. Let ξ be a random field on the space-time. Then we put $p(\omega,X) = \mathbf{P}\{(u_g+\xi, g) \in X\}$ for any $\omega = (u,g) \in \Omega$, $X \in \mathcal{F}_1(+\infty)$. It is easy to see that p is a stochastic behavior.

4.8 Example. All as in 4.7 except for ξ that equals zero (identically and always); thus $p(\omega,X) = 1$ when $(u_g, g) \in X$, otherwise $=0$. It is clear that p is a deterministic behavior.

4.9 Example. We use the scheme 4.4. Let $p(\omega,X) = = \frac{1}{4} \sum (1 + \tau A(\omega_1) B(\omega_1))$, where $\tau = +1$ when $a(\omega) = b(\omega) = 1$, otherwise $\tau = -1$, and the sum is taken over all $\omega_1 \in X$ such that $a(\omega_1) = a(\omega)$, $b(\omega_1) = b(\omega)$. It is easy to see that p is a stochastic behavior. In fact, it is trivial that $p(\omega, \{A = 1\})$ as a function of ω is $\mathcal{F}_0(t_1)$-measurable, since it is equal to 1/2 for all ω.

The example 4.8 deals with the dynamics determined by the wave equation; naturally the behavior is deterministic. The random addition in the example 4.7 leads to a non-deterministic behavior. The behavior 4.9 is noteworthy as it violates the Bell-CHSH inequality. Indeed, let us consider the correlation between A and B as a function of the parameters a, b: $E(a,b) = = \int A(\omega_1) B(\omega_1) p(\omega, d\omega_1)$, where ω is such that $a(\omega) = a$, $b(\omega) = b$; of course, the integral is reduced to a finite sum. It is easy to calculate that $E(a,b) = \tau = +1$ when $a = b = 1$, otherwise $= -1$. If we substitute this values in the Bell-CHSH inequality $|E(a,b)-E(a,b')| + |E(a',b)+E(a',b')| \leq 2$ for $a=b=+1$, $a'=b'=-1$, then the left-hand side attains the value 4. However, any deterministic behavior in this scheme satisfies this inequality since the corresponding $E(a,b)$ is factorizable: $E(a,b)=E_1(a)E_2(b)$.

4.10 Definition. A hiddenly deterministic behavior is a stochastic behavior which can be represented in the form $p = \int_0^1 p_\xi \, d\xi$, that is, $p(\omega,X) = \int_0^1 p_\xi(\omega,X)\, d\xi$, where each p_ξ for $\xi \in [0,1]$ is a deterministic behavior (in the same scheme), and $p_\xi(\omega,X)$ as a function of ξ and ω is jointly measurable for any fixed X. (Of course, one can replace $[0,1]$ by another suitable measure space)

It is clear that the behavior 4.7 is hiddenly deterministic. On the contrary, the behavior 4.9 is not hiddenly deterministic, since otherwise it would satisfy the Bell-CHSH inequality.

The Bell-CHSH scheme (see 4.4) is solved to the end. The set of all stochastic behaviors in this scheme is a polytope, as well as that of hiddenly deterministic ones. All tops and faces can be written out explicitly for both polytopes. In fact, any scheme with finite Ω can be solved to the end in the same sense, as it was noted [10] in another form. Indeed, all the tops are given for the polytope of hiddenly deterministic behaviors by their definition; applying known algorithms we can find the faces in a finite number of steps. Conversely, the faces are obviously known for the polytope of stochastic behaviors, and we can find the tops. By the way, the behavior 4.9 is a top of the polytope of stochastic behaviors, and it is the only non-deterministic top up to obvious symmetries.

However, finite schemes are interesting not on their own but due to their connections with infinite schemes, and such connections should be pointed out. We sketch here several modes (ways) of reduction of schemes which allow to pass from a given scheme $(T, \Omega, \mathcal{F}_0, \mathcal{F}_1)$ to a smaller scheme $(T', \Omega', \mathcal{F}_0', \mathcal{F}_1')$.

4.11 Mode. Let an admissible lattice T' be a complete sub-lattice of T. Then one can restrict the families $\mathcal{F}_0$, $\mathcal{F}_1$ to T'.

4.12 Mode. Let Θ be a congruence on T such that the factor-lattice $T' = T/\Theta$ is admissible and $t_1 \equiv t_2(\Theta)$ implies $\mathcal{F}_k(t_1) = \mathcal{F}_k(t_2)$ for k=0,1. Then one can transfer $\mathcal{F}_0, \mathcal{F}_1$ to T'.

4.13 Mode. Let Ω' be a $\mathcal{F}_0(+\infty)$-measurable subset of Ω. Then one can restrict all given σ-fields to Ω'.

4.14 Mode. Let $\mathcal{F}_1'$ be an admissible T-family on Ω such that $\mathcal{F}_0(t) \leq \mathcal{F}_1'(t) < \mathcal{F}_1(t)$ for all $t \in T$. Then one can replace $\mathcal{F}_1$ by $\mathcal{F}_1'$.

4.15 Mode. If the σ-field $\mathcal{F}_1(+\infty)$ does not separate points of Ω , then one can identify the equivalent points obtaining the factor-set Ω' .

It is easy to see for all listed modes that any stochastic behavior in the initial scheme induces a corresponding stochastic behavior in the reduced scheme, and that a deterministic/hiddenly deterministic behavior induces a behavior of the same type. It is possible to pass from the scheme 4.2 or 4.3 to the Bell-CHSH scheme, applying in sequence the listed modes. Thus, we may apply the Bell-CHSH inequality to all hiddenly deterministic behaviors in various schemes.

We introduce now a composition of behaviors, and (as a preliminary) that of schemes. Let $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3$ be admissible T-families on the same Ω and $\mathcal{F}_1(t) < \mathcal{F}_2(t) < \mathcal{F}_3(t)$ for any $t \in T$. Then we have the three schemes: $(T, \Omega, \mathcal{F}_1, \mathcal{F}_2)$; $(T, \Omega, \mathcal{F}_2, \mathcal{F}_3)$; $(T, \Omega, \mathcal{F}_1, \mathcal{F}_3)$. We call the third scheme the composition of the first and the second ones.

4.16 Definition. The composition p of stochastic behaviors p_1 and p_2 given in the schemes $(T, \Omega, \mathcal{F}_1, \mathcal{F}_2)$ and $(T, \Omega, \mathcal{F}_2, \mathcal{F}_3)$ respectively is a stochastic behavior in the scheme $(T, \Omega, \mathcal{F}_1, \mathcal{F}_3)$ defined by the equation

$$p(\omega, X) = \int p_1(\omega, d\omega')\, p_2(\omega', X) \quad \text{for all } \omega \in \Omega \ , \quad X \in \mathcal{F}_3(+\infty).$$

It is easy to see that this equation gives in fact a stochastic behavior. If both p_1 and p_2 are deterministic/ /hiddenly deterministic then p is of the same type.

The composition of behaviors describes a composition of two systems which are distributed in the same space-time and locally interact in such a way that the output of the first system enters the input of the second one.

5. ON FORMAL DESCRIPTION OF LOCAL QUANTUM CONCEPTION

The quantum theory predicts the possibility of a realization of some behaviors violating the Bell inequalities and therefore not hiddenly deterministic. At the same time the quantum theory does not require going out of the class of stochastic behaviors. Furthermore, the quantum theory prohibits some stochastic behaviors, as it obviously results from quantum analogs of the Bell inequalities. It is natural to believe that there is a new class intermediate between the class of hiddenly deterministic behaviors and that of stochastic behaviors, namely the class of quantum behaviors. If so, a mathematical definition of this class should be given.

This problem turns out to be non-trivial. To shed some light on the peculiar nature of the problem, we draw an analogy with the old problem of formalization of the notion of the effective computability. According to the well-known Church thesis (see for example [11), 12)]), any of the known equivalent definitions of the general recursive function (definitions due to Turing, Kleene, Church, Post, Markov and the others) should be taken for the mathematical definition of the notion of effective computable function. We do not touch here on the subjective aspect, that is the execution of an algorithm by consciousness; we touch only on the objective aspect, namely the execution of an algorithm by a machine. On the one hand, the Church thesis cannot be proved mathematically, as it connects a formal notion with an informal one. On the other hand, this thesis was not deduced from any physical theory. Its acceptance by mathematicians is based on informal arguments appealing to both the common sense and the physical intuition. Some formal arguments also exist, but they are not crucial. As a matter of fact the Church thesis consists of two statements that we shall consider separately. By the way, recently more profound discussion [13), 14)] has been promoted about physical aspects of this problem.

Let us start with the first statement that each recursive function is effective computable. To be convinced in this, it is sufficient to show that the Turing machine is physically realizable. This follows evidently from the present know-how on practical realization of automatic devices, if one slightly idealizes empirical facts. It is nothing else but some gedankenexperiment indeed. It is worth to stress that such gedankenexperiment needs only a very restricted list of various physical phenomena. In addition, one can use various lists on his own choice obtaining the same final result.

Turn now to the second statement that each effective computable function is recursive. Here the arguments are less precise. The question has been discussed of what kind of restrictions our conception of physical reality imposes on computing processes, resulting in the conclusion that only such processes are realizable that can be described by recursive functions. Various gedankenexperiments were considered leading to the corresponding well-defined classes of computing processes.

Additional arguments of formal nature are, firstly, the presence of proper general properties of the class of all recursive functions (for example, the composition of any such functions is again a such function), secondly, the presence of many equivalent definitions of this class.

Almost all said above on the formalization of the effective computability is applicable mutatis mutandis to the formalization of the quantum realizability of behaviors. This is why we dwell on the Church thesis. Its history suggests a possible approach to the problem of how to define quantum behaviors. We are not able to solve this problem now, that is, to formulate such a definition. In this connection we propose the following way: to suggest, to argue and to investigate, on the one hand, necessary quantum conditions, that is, such conditions for a stochastic behavior that are believed to be obligatory from the view point of fundamentals of a local quantum theory, and, on the other hand, sufficient quantum conditions, that is, such conditions for a stochastic behavior that are believed to ensure the availability of a physical realization of a given behavior within the framework of a local quantum theory. In both cases one must investigate general properties of suggested classes of behaviors and also find equivalent definitions. Making first steps on this way, we expect that one will achieve the coincidence of a necessary condition with a sufficient one.

The investigation of necessary quantum conditions, initiated in the paper [1] which is based on [16] and [17], is continued here. Note that the paper [15] may be considered as an attempt to suggest a sufficient quantum condition, however from another view point and in other terms.

5.1 Definition. First quantum construction in a given scheme of behavior $(T, \Omega, \mathcal{F}_0, \mathcal{F}_1)$ is a collection (H,W,P), where H is a Hilbert space (of a finite or the countable dimension), W is a density matrix in H, and P is a function on $\Omega \times \mathcal{F}_1(+\infty)$, whose values are orthogonal projectors in H. $P(\omega, X)$ as a function of $\omega \in \Omega$ is assumed $\mathcal{F}_0(t)$-measurable for any fixed $\mathcal{F}_1(t)$-measurable $X \subset \Omega$, and $P(\omega, X)$ as a function of $X \in \mathcal{F}_1(+\infty)$ is assumed to be a projector-valued measure on Ω for any fixed $\omega \in \Omega$, and $P(\omega, X) = 1$ for any $X \in \mathcal{F}_0(+\infty)$, $\omega \in X$.

It is important that projectors $P(\omega, X)$ for different ω are not necessarily commuting. First quantum construction is called finite-dimensional if H is finite-dimensional. Any first quantum construction (H,W,P) induces obviously the stochastic behavior p by the formula $p(\omega, X) = Tr(P(\omega, X)W)$.

5.2 Definition. A stochastic behavior p satisfies the first necessary quantum condition if it is induced by some first quantum construction.

The set of all stochastic behaviors in a scheme $(T, \Omega, \mathcal{F}_0, \mathcal{F}_1)$ satisfying the first necessary quantum con-

dition, will be denoted by $QB_1^+(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$; "QB" means "quantum behavior", "+" means "necessary" (because QB^+ approximates the unknown QB excessively), "1" is the version number. If p is induced by some finite-dimensional first quantum construction then we write
$p \in QB_{1,0}^+(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$.

5.3 <u>Theorem</u>. (a) Both sets $QB_1^+(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$, $QB_{1,0}^+(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$ are convex.
(b) If $p_1 \in QB_1^+(T,\Omega,\mathcal{F}_1,\mathcal{F}_2)$ and $p_2 \in QB_1^+(T,\Omega,\mathcal{F}_2,\mathcal{F}_3)$, then the composition p of p_1 and p_2 belongs to $QB_1^+(T,\Omega,\mathcal{F}_1,\mathcal{F}_3)$; the same holds for $QB_{1,0}^+$.
(c) All hiddenly deterministic behaviors belong to QB_1^+ ; all deterministic behaviors belong to $QB_{1,0}^+$.

The proof is easy. (a) It is enough to consider the direct sum $P(\omega,X) = P_1(\omega,X) \oplus P_2(\omega,X)$. (b) We use the tensor product with $P(\omega,X) = \int P_1(\omega,d\omega') \otimes P_2(\omega',X)$.
(c) For deterministic p it is enough to put $P(\omega,X) = p(\omega,X)\cdot \mathbf{1}$ with arbitrary H, W. For hiddenly deterministic p we use the continuous direct sum of Hilbert spaces ([18], Appendix A).

In the rest of this section we give an equivalent definition of the class $QB_{1,0}^+$ for a class of schemes defined below.

Let S be a finite partially ordered set, and let for any $s \in S$ be given two finite sets $\Omega_0(s)$, $\Omega_1(s)$. Then we can construct the scheme $(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$ as follows. The lattice T consists of all subsets $t \subset S$ which are "decreasing", that is, $(s_1 \leq s_2 \,\&\, s_2 \in t) \Rightarrow (s_1 \in t)$. The set Ω is the Cartesian product $\Omega = \Omega_0 \times \Omega_1$, where $\Omega_k = \times_s \Omega_k(s)$ for k=0,1. Thus, an element $\omega \in \Omega$ is a pair of functions ω_0, ω_1 on S with $\omega_k(s) \in \Omega_k(s)$ for all s. The σ-field $\mathcal{F}_k(t)$ is generated by "functionals" $\omega \longrightarrow \omega_l(s)$ with $s \in t$ and $l \leq k$. Such a scheme will be called finite factorizable.

Let $\mathcal{A} = \{\mathcal{A}_s\}_{s\in S}$ be a family of algebras of operators in H. We say that $\mathcal{A}$ decomposes H, if each $\mathcal{A}_s$ is a factor of the type 1 and these factors mutually commute. It is well-known (see [18], Appendix A), that it is possible in this case to identify H with a tensor product $H = (\bigotimes_s H_s) \otimes H_0$ in such a way that each $\mathcal{A}_s$ is identified with the algebra of all operators in H_s.

5.4 <u>Definition</u>. Let $(T,\Omega,\mathcal{F}_0,\mathcal{F}_1)$ be a finite factorizable scheme, constructed as above from S and $\{\Omega_k(s)\}$. We define second quantum construction in this scheme as a collection $(H,\mathcal{A}^-,\mathcal{A}^+,W,W_0,P)$ such that:
(a) H is a Hilbert space;
(b) both $\mathcal{A}^- = \{\mathcal{A}_s^-\}$ and $\mathcal{A}^+ = \{\mathcal{A}_s^+\}$ decompose H;
(c) for any $s', s'' \in S$ such that $s' \nleq s''$ (that is, $s' > s''$

or they are non-comparable) the algebras $\mathcal{A}^-_{s'}$ and $\mathcal{A}^+_{s''}$ commute;
(d) $W = \{W_s\}$, each W_s being a function on $\Omega_0(s)$ whose values are density matrices in $\mathcal{A}^-_s$;
(e) W_0 is a density matrix in $\mathcal{A}^-_0$, where $\mathcal{A}^-_0$ is the algebra of all operators in H which commute with all $\mathcal{A}^-_s$;
(f) $P = \{P_s\}$, each p_s being a projector-valued measure on $\Omega_1(s)$ with values in $\mathcal{A}^+_s$.

Thus we have two decompositions of one and the same space: $(\bigotimes_{s \in S} H^-_s) \otimes H^-_0 = H = (\bigotimes_{s \in S} H^+_s) \otimes H^+_0$.
One may consider $W_s(\omega)$ a density matrix in H^-_s for any $s \in S$ and $\omega \in \Omega_0(s)$, and W_0 a density matrix in H^-_0. Furthermore $P_s(\omega)$ may be considered an orthogonal projector in H^+_s for any $s \in S$ and $\omega \in \Omega_1(S)$, and $\sum_{\omega \in \Omega_1(s)} P_s(\omega) = \mathbb{1}(H^+_s)$ for all s. Here and in what follows $\mathbb{1}$ (H) denotes the identity operator in H. Second quantum construction is called finite-dimensional if H is finite-dimensional. Any second quantum construction induces a stochastic behavior. Namely, we put $W(\omega_0) = W_0 \cdot \prod_S W_s(\omega_0(s))$ for any $\omega_0 \in \Omega_0$; $P(\omega_1) = \prod_S P_s(\omega_1(s))$ for any $\omega_1 \in \Omega_1$; $p(\omega', \omega'') = Tr(P(\omega''_1) W(\omega'_0))$ for any $\omega' = (\omega'_0, \omega'_1)$ and $\omega'' = (\omega''_0, \omega''_1)$ such that $\omega'_0 = \omega''_0$, otherwise $p(\omega', \omega'') = 0$; finally $p(\omega', X) = \sum_{\omega'' \in X} p(\omega', \omega'')$.
It is easy to see that p is a stochastic behavior.

5.5 <u>Theorem</u>. A stochastic behavior in a finite factorizable scheme belongs to $QB^+_{1,0}$ if and only if it is induced by some finite-dimensional second quantum construction.

Thus, we have two equivalent constructions under some additional assumptions. The first construction resembles the Heisenberg picture: the quantum state is left fixed, and interventions affect observables. The second construction resembles the scattering: two decompositions of one Hilbert space are given, "in" and "out"; interventions affect density matrices in the "in" spaces H^-_s; the responses result from quantum measurements in the "out" spaces H^+_s. Unlike the real scattering theory, it is essential here that "particles" come and go away in the prescribed order. This order imposes specific restrictions on the corresponding stochastic behavior, which is here a peculiar analog of the S-matrix. It is interesting to note that two equivalent constructions have at the first sight different symmetry groups. In fact, the class $QB^+_{1,0}$ is evidently invariant under any invertible maps $\Omega \to \Omega$ preserving all σ-fields $\mathcal{F}_K(t)$. The obvious symmetry group for second quantum constructions is the direct product over $s \in S$ and k=0,1 of the permutation groups for $\Omega_K(s)$. The latter is essentially smaller than the former. Hence, the class of all second quantum constructions has non-evident "hidden" symmetries.

The notion of first quantum construction for a finite factorizable scheme appears in fact in [1] (the condition (2) of Theorem 4). An attempt was made there to reformulate this condition but it failed. Firstly, the condition (1) of Theorem 4 contains mistakes. The second author of the present paper (B. Ts.) makes use of this opportunity to offer his apologies. Secondly, the very idea of formulation such conditions in terms of operations seems to be worse than the "scattering" idea used here.

The rest of this section contains a part of the proof of Theorem 5.5. As above let us construct from S and $\{\Omega_K(S)\}$ the finite factorizable behavior scheme $(T, \Omega, \mathcal{F}_0, \mathcal{F}_1)$ which is fixed in what follows. Within the proof we use the classes B1, B2, B2w, B2s of stochastic behaviors and abbreviations C1, C2, C2w, C2s for the types of constructions introduced below. Each of these behavior classes is the class of all stochastic behaviors induced by constructions of the corresponding type described as

C1: first quantum construction;
C2: second quantum construction;
C2w: C2 weakened by eliminating the condition that each algebra $\mathcal{A}_s^+$ is a factor;
C2s: C2 strengthened by adding the following condition on the density matrices $W_s(\omega)$: $W_s(\omega') W_s(\omega'') = W_s(\omega')$ when $s' = s''$, otherwise $= 0$. In terms of the tensor product $H = (\otimes H_s^-) \otimes H_0^-$ we can say that $W_s(\omega)$ are one-dimensional density matrices in the spaces H_s^- and the corresponding one-dimensional subspaces in each H_s^- are mutually orthogonal.

Thus, $B1 = QB_1^+$. Each C2s is a C2, and each C2 is a C2w, hence $B2s \subset B2 \subset B2w$. The finite-dimensional versions will be denoted similarly with adding "f": Cf1, Bf2w, and so on. Thus, $Bf1 = QB_{1,0}^+$, $Bf2s \subset Bf2 \subset Bf2w$, and $Bf1 \subset B1, \ldots,$ $Bf2w \subset B2w$. Two constructions are called equivalent if they induce the same behavior. To distinguish objects related to different constructions we use left-side indices: ${}^1H$, ${}^2H$, ${}^sH$, ${}^wH$ and so on.

Theorem 5.5 asserts that $Bf1 = Bf2$. Its proof consists of the proofs of the following four inclusions: $Bf2 \subset Bf2s \subset Bf1 \subset Bf2w \subset Bf2$. The first three of them can be proved easily, the finite-dimensioness being of no importance; in fact we prove at the same time that $B2 \subset B2s \subset B1 \subset B2w$. The proof of the fourth inclusion $Bf2w \subset Bf2$ is more complicated, the finite-dimensioness being used essentially, and whether or not $B2w \subset B2$ is true is unknown.

5.6 Proof of inclusions $B1 \subset B2w$, $Bf1 \subset Bf2w$. Given ${}^1H$, ${}^1W$, ${}^1P$ form a C1; we have to find ${}^wH$, ${}^w\mathcal{A}^-$, ${}^w\mathcal{A}^+$, ${}^wW$, ${}^wW_0$, ${}^wP$ forming an equivalent C2w.

Put ${}^wH_0^- = {}^1H$, ${}^wW_0 = {}^1W$, ${}^wH_s^- = L_2(\Omega_0(s))$; here and in what follows $L_2(Z)$, Z being a finite set, denotes the finite-dimensional Hilbert space of all functions on S; e(z) denotes the unit vector in this space corresponding to $z \in Z$, and w(z) denotes the corresponding density matrix. Put ${}^wW_s(\omega) = \mathcal{W}(\omega)$ for $\omega \in \Omega_0(S)$;

$$ {}^wH = (\bigotimes_s {}^wH_s^-) \otimes {}^wH_0^- = (\bigotimes_s L_2(\Omega_0(s))) \otimes {}^1H = $$
$$ = L_2(\mathop{\times}_s \Omega_0(s)) \otimes {}^1H = L_2(\Omega_0) \otimes {}^1H. $$

This decomposition determines the algebras ${}^w\mathcal{A}_s^-$.

It is clear that ${}^1P(\omega, X) = \sum P(\omega_0, \omega_1)$, where $\omega_0 \in \Omega_0$ is the projection of the point $\omega \in \Omega = \Omega_0 \times \Omega_1$ into Ω_0 , and the sum is taken over all $\omega_1 \in \Omega_1$ such that $(\omega_0, \omega_1) \in X$. Further, P can be factorized: $P(\omega_0, \omega_1) = \prod_s P_s(\omega_0, \omega_1(s))$. Actually, putting $\Omega_s = \{\omega_1' \in \Omega_1 : \omega_1'(s) = \omega_1(s)\}$ for a fixed ω_1 we see that $\bigcap_s \Omega_s$ contains the point ω_1 only; and, in general, any projector measure Q has the property $Q(X \cap Y) = Q(X)Q(Y)$. Thus, $P_s(\omega_0, \omega_1)$ is defined for $\omega_0 \in \Omega_0$, $s \in S$, $\omega_1 \in \Omega_1(s)$; as a function of ω_0 it is $\mathcal{F}_0(t)$ -measurable when $s \in t$; as a function of ω_1 it determines a projector measure. Now we define the operator ${}^wP_s(\omega_1)$ in ${}^wH = L_2(\Omega_0) \otimes {}^1H$ for $s \in S$, $\omega_1 \in \Omega_1(s)$ by the equality

$$ {}^wP_s(\omega_1)(e(\omega_0) \otimes h) = e(\omega_0) \otimes (P_s(\omega_0, \omega_1) h) $$

for all $\omega_0 \in \Omega_0$, $h \in {}^1H$. As a function of ω_1 , it determines a projector measure. And it commutes with ${}^w\mathcal{A}_{s'}^-$, when $s' \not\leq s$, since $P_s(\omega_0, \omega_1)$ in fact does not depend on $\omega_0(s')$ for such s'. All the operators ${}^wP_s(\omega_1)$ mutually commute; in fact, they leave invariant each subspace $e(\omega_0) \otimes {}^1H$, and on such subspace they are reduced to $P_s(\omega_0, \omega_1)$.

Let ${}^w\mathcal{A}_s^+$ be the algebra of operators on ${}^wH$ generated by all ${}^wP_s(\omega_1)$, $\omega_1 \in \Omega_1(s)$. These algebras mutually commute, and ${}^w\mathcal{A}_s^+$ commutes with ${}^w\mathcal{A}_{s'}^-$ when $s' \not\leq s$. Now we have for $\omega_0 \in \Omega_0$, $\omega_1 \in \Omega_1$

$$ \mathrm{Tr}\,({}^wW(\omega_0) \cdot {}^wP(\omega_1)) = $$
$$ = \mathrm{Tr}\,(((\bigotimes_s {}^wW_s(\omega_0(s))) \otimes {}^wW_0) \cdot \prod_s {}^wP_s(\omega_1(s))) = $$
$$ = \mathrm{Tr}\,((\mathcal{W}(\omega_0) \otimes {}^1W) \cdot \prod_s {}^wP_s(\omega_1(s))). $$

The trace is taken in fact in the subspace $e(\omega_0) \otimes {}^1H$, and it can be rewritten as the trace in ${}^1H$:

$$\ldots = \mathrm{Tr}\,({}^{1}W \cdot \prod_{s} P_s(\omega_0, \omega_1(s))) = \mathrm{Tr}\,({}^{1}W \cdot P(\omega_0, \omega_1)),$$

which proves the claim.

The inclusion Bf2w $\subset$ Bf2 follows easily from Theorem 5.7 whose proof will be published elsewhere.

5.7 <u>Theorem</u>. Let H be a finite-dimensional Hilbert space and $\mathcal{A}_1, \ldots, \mathcal{A}_n$ be arbitrary algebras of operators on H. Then there are a finite-dimensional Hilbert space H_1, an unit vector $h_1 \in H_1$ and algebras $\tilde{\mathcal{A}}_1, \ldots, \tilde{\mathcal{A}}_n$, $\mathcal{B}_1, \ldots, \mathcal{B}_n$ of operators on $\tilde{H} = H \otimes H_1$ such that:

(a) for each $k=1,\ldots,n$ the algebra $\mathcal{B}_k$ is a factor and $\tilde{\mathcal{A}}_k \subset \mathcal{B}_k$;

(b) for each $k=1,\ldots,n$ there is an isomorphism between algebras $\mathcal{A}_k$ and $\tilde{\mathcal{A}}_k$ such that for corresponding operators $A \in \mathcal{A}_k$ and $\tilde{A} \in \tilde{\mathcal{A}}_k$ the equality $\tilde{A}(h \otimes h_1) = (Ah) \otimes h_1$ holds for all $h \in H$;

(c) if there are pairs (k,l) such that $1 \leqslant k < l \leqslant n$ and $\mathcal{A}_k$ commutes with $\mathcal{A}_l$, then $\mathcal{B}_k$ commutes with $\mathcal{B}_l$ for each such pair;

(d) if there are k such that $1 \leqslant k \leqslant n$ and the algebra $\mathcal{A}_k$ is a factor, then $\mathcal{B}_k = \tilde{\mathcal{A}}_k = \mathcal{A}_k \otimes \mathbb{1}(H_1)$ for each such k.

REFERENCES

1) Cirel'son, B.S., "Quantum generalizations of Bell's inequality", Lett. Math. Phys. <u>4</u>, 2, 93-100 (1980). ("Cirel'son" is another transliteration of "Tsirelson").

2) Khalfin, L.A., "Bell's inequalities, Tsirelson's inequalities and K^0-$\bar{K}^0$ mesons", report on the session of Nucl. Phys. Department USSR Acad. Sci., Moscow, April 1983.

3) Tsirelson, B.S., "Quantum analogs of Bell's inequalities. The case of two space-separated domains" (in Russian), In: "Problems of the theory of probability distributions, IX", Zap. Naučn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) <u>142</u>, 175-194 (1985).

4) Kraus, K., "Operations and effects in the Hilbert space formulation of quantum theory", Lecture Notes in Physics, Springer, <u>29</u>, 206-229 (1974).

5) Leggett, A.J., and Garg, A., "Quantum mechanics versus macroscopic realism: is the flux there when nobody looks?", Phys. Rev. Letters <u>54</u>, 9, 857-860 (1985).

6) Deutsch, D., "Quantum theory as a universal physical theory", Intern. J. Theor. Phys. <u>24</u>, 1, 1-41 (1985).

7) D'Espagnat, B., "Nonseparability and the tentative description of reality", Physics Reports 110, 4, 201-264 (1984).

8) Grätzer, G., "General lattice theory", Academic Press, New York, 1978, xiii+381 pp.

9) Loève, M., "Probability theory", 2nd ed., Van Nostrand, 1960.

10) Froissart, M., "Constructive generalization of Bell's inequalities", Il Nuovo Cimento 64 B, 2, 241-251 (1981).

11) Rogers, H., "Theory of recursive functions and effective computability", Mc-Graw-Hill, 1967.

12) Manin, Ju.I., "Computable and noncomputable" (in Russian), Sovet. Radio, Moscow, 1980, 128 pp.

13) Proc. Conf. "Physics of computation", held at MIT in May 1981, Int. J. Theor. Phys. 21, nos. 3/4, 6/7, 12 (1982).

14) Wolfram, S., "Undecidability and intractability in theoretical physics", Phys. Rev. Letters 54, 8, 735-738 (1985).

15) Tsirelson, B.S., "On formal description of quantum systems that are similar to systems of stochastic automata" (in Russian), In: "Взаимодействующие марковские процессы и их применение в биологии", Proc. second school-seminar on locally interacting systems and their application in biology, held at Pushchino, Moscow region, in May 1978; edited by R.L.Dobrushin, V.I.Kryukov and A.L.Toom; Sci. Centre for Biological Research USSR Acad. Sci., Pushchino, 1979, pp.100-138.

See also the expository paper:
Oseledets, V.I., "Completely positive linear mappings, non-Hamiltonian evolution and quantum stochastic processes" (in Russian), In: Probability Theory, Mathematical statistics, Theoretical Cybernetics, 20, 52-94, Итоги науки и техники, АН СССР и ВИНИТИ, Moscow, 1983.

16) Schlieder, S., "Einige Bemerkungen zur Zustandsänderung von relativistischen quantenmechanischen Systemen durch Messungen und zur Lokalitätsforderung", Commun. Math. Phys. 7, 4, 305-331 (1968).

17) Hellwig, K.-E., and Kraus, K., "Formal description of measurements in local quantum field theory", Phys. Rev. (D), 1,2,566-571 (1970).

18) Dixmier, J., "Les C*-algèbres et leurs représentations", Gauthier-Villars, Paris, 1969.